G4G15 EXCHANGE BOOK

ATLANTA, GEORGIA

Wednesday, Feb 21st – Sunday, Feb 25th, 2024

Presented by G4G Foundation

G4G15 Exchange Book

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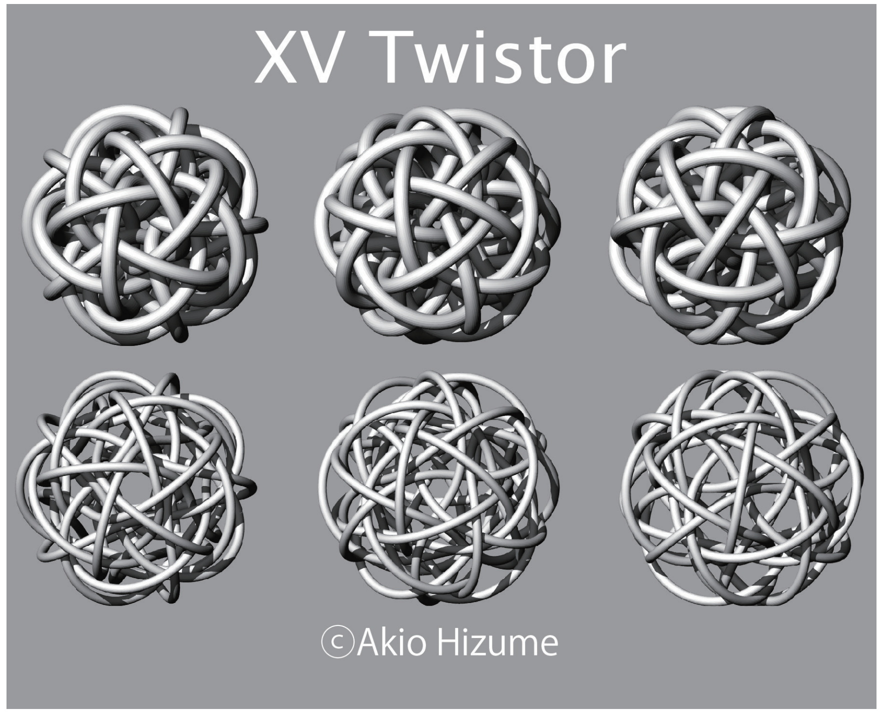
PUZZLES (continued)

LEGACY

SCIENCE

Unofficial Logos



Submitted by
Akio Hizume

GatherinG 4 Gardner 15

Submitted by
Matjuska Teja Krasek

Gathering 4 Gardner 15
Presentation Schedule

Thursday, February 22nd, 2024

FEATURED PRESENTATION

Morning Session: 8:30 AM - 12:00 PM

Time	Speaker	Title	Duration	Time
8:30 AM	Henry Segerman	Variants of the 15-Puzzle and the Effects of Holonomy	*8 min*	8:30 AM
	Tom Roby	Jeu de Taquin: The Fifteen Puzzle in Research Mathematics	*8 min*	
	Peter Knoppers	YA15PP — Yet Another 15-Piece Puzzle	*6 min*	
	Roger Manderscheid	Human Compatibility for Time Travel	*6 min*	
	Alexa Meade	Wonderland Dreams	*12 min*	
	Gioia De Cari	How I Terrified the MIT Math Department	*6 min*	
	Robert Becker	Soap Film Structures and the Invention of HyperTiles	*6 min*	
	Eric Harshbarger	The Latest Improvements With "Go First Dice"	*6 min*	
	Richard Stanley	A Fibonacci Array	*6 min*	
	Margaret Kepner	Tumbling Blocks, Doubling Cubes	*6 min*	
10:00 AM		*Break*	*30 min*	10:00 AM
10:30 AM	Roger Penrose	On the Escherization of David Smith's "SPECTRE" Einstein, and Related Topics	*20 min*	10:30 AM
	Tali Siegel, Lila Siegel, Aaron Siegel	Combinatorial Jenga	*6 min*	
	Josh Laison	Graphs of Rectangles Looking Diagonally	*6 min*	
	Alexandre Muñiz	Mixed Connections: Polyominoes, Piece by Piece	*6 min*	
	Timothy Snyder	Correspondence between Projective Geometries and Regular Polytopes	*6 min*	
	Stephen Macknik	Champions of Illusion	*6 min*	
	Douglas O'Roark	Recreational Math at Scale: Math Circles of Chicago	*6 min*	
	Ivo David	Paradoxical Boxes	*6 min*	
	Yossi Elran	Recreational Math as a Beacon of Hope: Empowering Youth and Healing Trauma Through Puzzles	*6 min*	
	Kenneth Brecher	Introducing the Mesmoid	*6 min*	
12:00 PM				12:00 PM

Afternoon Session: 1:30 PM - 5:15 PM

Time	Speaker	Title	Duration	Time
1:30 PM	James Gardner	Growing Up With Martin Gardner (2024)	*30 min*	1:30 PM
	Skona Brittain	A Collection from the Collection of Collections	*6 min*	
	Robert Bosch	Single-Line Drawings via Mathematical Optimization	*6 min*	
	Jim Weinrich	On Spinning and Toppling Coins — Did Martin Gardner Change His Mind?	*6 min*	
	David Richeson	How Much String to String a Cardioid?	*6 min*	
	Ivars Peterson	Patterns in a Botanical Garden	*6 min*	
	Vladimir Bulatov	SymSim - Creation of Natural Seamless Symmetric Patterns	*6 min*	
	Braden Ganetsky	"Bask3twork": Procedurally Generating Symmetric Celtic Knots	*6 min*	
	Robert Fathauer	Fractal-Limit Tilings	*6 min*	
	Douglas McKenna	Fibbinary Zippers and Self-Avoiding, Hilbert-style, Squace-Filling Curve Motifs	*6 min*	
3:15 PM		*Break*	*30 min*	3:15 PM
3:45 PM	Raymond Hall	The Magic Octagon	*6 min*	3:45 PM
	Haoran Chen	The Great Beetle Escape	*6 min*	
	The Amazing Kreskin	Interview with The Amazing Kreskin by Meir Yedid	*70 min*	
5:15 PM				5:15 PM

Presentation Abstracts Available Online: **www.gathering4gardner.org/g4g15-abstracts.pdf**

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Presentation Schedule

Friday, February 23rd, 2024

FEATURED PRESENTATION

Morning Session: 8:30 AM - 12:00 PM

Time	Speaker	Title	Length
8:30 AM	Tantan Dai	Gateway to World Sudoku Championship	*8 min*
	Alyssa Williams, Christian Scott	Learning to Solve and Set Variant Sudoku	*6 min*
	Eliza Gallagher	9x9 Project: Building Math Identity One Digit at a Time	*6 min*
	Eleftherios Pavlides	Polymorphic Elastegrity (PE): From Recreational Paper Folding to the Tree of Polymorphisms	*6 min*
	Inna Zakharevich	Computing with Paper	*6 min*
	Jeanette Shakalli	FUNDAPROMAT: A Free Source for Math Enthusiasts	*6 min*
	Bill Cheswick	Paul Erdos' Brain Was in Town.	*6 min*
	John Harris	Puzzle Variations From a Rod Bentley Text	*6 min*
	Jim Propp	Industrious Dice, or, Pip-Pip Hooray!	*6 min*
	Audrey Nasar	Symmetry Card Game	*6 min*
	Jorge Nuno Silva	Lewis Carroll's Game of Logic	*6 min*
10:00 AM		*Break*	*30 min*
10:30 AM	**Alvy Ray Smith**	**What Do the Theory of Computing and the Movies Have in Common?**	*45 min*
	David Michael Greene	How Many Gliders Do You Really Need?	*6 min*
	Tom Rokicki	The Speed of Light in the Game of Life	*6 min*
	Roy Leban	Markov Algorithms and Life	*8 min*
	John Winston Garth	Cellular Auto-mon	*6 min*
	Eric Bates	Passing the Spark	*6 min*
12:00 PM			

Afternoon Session: 1:30 PM - 5:30 PM

Time	Speaker	Title	Length
1:30 PM	Chaim Goodman-Strauss	The Hat Aperiodic Monotile	*30 min*
	Greg Whitehead	Nerd Sniped by the Hat Tile	*6 min*
	Bob Hearn	The Magnetic Fifty-Nine Icosahedra	*6 min*
	Scott Vorthmann	The Virtual Fifty-Nine Icosahedra	*6 min*
	Carolyn Yackel	Depicting Catalan Solids on Temari Balls	*6 min*
	Robin Houston	Polyhedra Whose Faces Meet at Right-Angles Except on One Edge	*6 min*
	George Hart	Abraham Sharp and the "Sharpohedron"	*6 min*
	Stella Grosser	The Post Office Metric	*6 min*
	Spandan Bandyopadhyay	The Egg Sandwich Problem	*6 min*
	Bjoern Muetzel	Apollonian Dream	*6 min*
	Tom Bessoir	Digits of Pi Film	*6 min*
	Tiago Hirth	Sphinx Revue Mensuelle Des Questions Récréatives: An Overview	*6 min*
3:30 PM		*Break*	*30 min*
4:00 PM	Neil Calkin	XV: The Ides Are on Our Side!	*6 min*
	Dana Richards	Beware the Ides of Math	*6 min*
	Amina Buhler-Allen	Hyperspace Heroes 1: Odd Bedfellows	*6 min*
	Paul Hildebrandt	Hyperspace Heroes 2: All You Need is Cubes	*6 min*
	Michael Stranahan	Hyperspace Heroes 3: Love is All You Need	*6 min*
	Richard Esterle	Still Amazing — The Windy Windy Twisty Route	*8 min*
	Tanya Khovanova	My Favorite Math Jokes	*6 min*
	Gilryong Song	Pickagram = Tangram x Golden Ratio	*6 min*
	Adam Rubin	An Informal Poll	*6 min*
	David Cohen	Scrabblegrams	*6 min*
	Aaron Siegel	Losing at Hackenbush	*6 min*
5:30 PM			

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Gathering 4 Gardner 15
Presentation Schedule

Saturday, February 24th, 2024

FEATURED PRESENTATION

Morning Session: 8:30 AM - 12:00 PM

Time	Speaker	Title	Length
8:30 AM	Andrew Rhoda	Update on the Slocum Mechanical Puzzle Collection at the Lilly Library	*6 min*
	George Bell	Coordinate Motion Puzzles	*6 min*
	Carl Hoff	Hazmat Cargo 2: An Even More Difficult 9-Piece Packing Puzzle	*6 min*
	Ryuhei Uehara	Common Shape Puzzles	*6 min*
	Rod Bogart	Jack in the Box Puzzle	*6 min*
	Matthew Hayden	My Degree in Enigmatology	*6 min*
	P. Justin Kalef	A Puzzle Course at Rutgers University	*6 min*
	Sabetta Matsumoto	Bending Seams — How to Create Couture Curves	*6 min*
	Dana S. Richards	The Works of Martin Gardner	*25 min*
10:00 AM		*Break*	*30 min*
10:30 AM	**Sarah Hart**	**The Mathematics of Musical Composition**	*45 min*
	Nancy Blachman	An Unexpected Surprise Followed by Disappointment	*6 min*
	Daniel Valente-Matias	Database of Common Nets of Polyhedra	*6 min*
	Stuart Moskowitz	Math Where You'd Least Expect It, but Then, It's From Lewis Carroll	*6 min*
	John Edward Miller	15 Geodesics Between Buckminster Fuller and Martin Gardner	*6 min*
	George Hart	Hands-On Activities Saturday Afternoon	*12 min*
12:00 PM			

Gathering 4 Gardner 15
Presentation Schedule

Sunday, February 25th, 2024

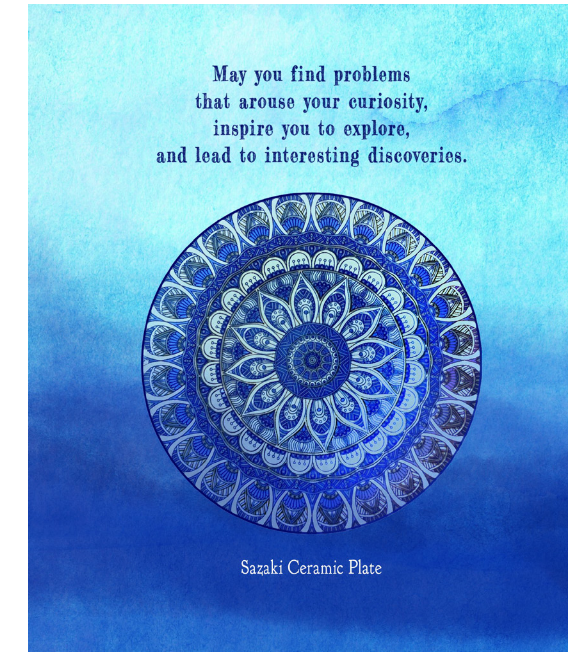
Morning Session: 8:30 AM - 12:00 PM

Time	Speaker	Title	Length
8:30 AM	Debbie Denise Reese	ROOT(Math Success) = Childhood + Concrete Analogs + Challenges	*6 min*
	Glen Whitney	Restoring Lost Digital Mathematical Treasures	*6 min*
	Louis Hirsch Kauffman	Rope Magic and Topology	*6 min*
	Matt Parker	The campaign to fix geometrically impossible UK traffic signs	*6 min*
	Rodi Steinig	Bringing G4G to Children	*6 min*
	Ann Schwartz	Guess What ... I'm a Flexagon!	*6 min*
	Akio Hizume	El Caja de Música de Arena (The Music Box of Sand)	*6 min*
	Robert Pope Vermillion	The Impossomatic 3000.1	*6 min*
	Barry Cipra	Color Contrast Contortions	*6 min*
	Akio Hizume	Demonstration of Cone-Pass	*6 min*
	Arthur Benjamin	The Bingo Paradox	*6 min*
10:00 AM		*Break*	*30 min*
10:30 AM	Jay Gilligan	The Future of Juggling	*10 min*
	Thomas Draper	Monoidal Categories for Modelling Physical Systems and Language	*6 min*
	Steven Landsburg	How To Organize a Waiting Line	*6 min*
	Gary Antonick	Stress Test Your Thinking	*7 min*
	Woody Aragón	Sleight of Math	*6 min*
	Susan Goldstine	Granny Likes Regular Maps	*6 min*
	Arezoo Islami	Applicability of Mathematics: Shortcomings of Philosophy	*6 min*
	Colin Wright	Incircles of Primitive Pythagorean Triangles	*6 min*
	Peter Kagey	Spinning Switches on a Wreath Product	*6 min*
	Michael Tanoff	A Selection of 15 Meter Stamps for G4G15	*6 min*
12:00 PM			

ART

Apollonian Dream | Bjoern Muetzel & Julian Muetzel | Page 31

About the Inspiration Cloth



This picture excludes the contact information included on this exchange gift.

My goal was to make something both inspirational and useful. Our gift can be used to clean glasses, screens, and other surfaces. I appreciate ideas from the following books:

- *How to Solve It* by George Polya
- *A Mathematician's Lament: How School Cheats Us Out of Our Most Fascinating and Imaginative Art Form* by Paul Lockhart

"[If the teacher] challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent Thinking." –*How to Solve It*, second paragraph from the preface to the First Printing, page v.

"A good problem is something you don't know how to solve. That's what makes it a good puzzle, and a good opportunity. A good problem does not just sit there in isolation, but serves as a springboard to other interesting questions." –*A Mathematician's Lament*, second to last paragraph, page 41.

"Give your students a good problem, let them struggle and get frustrated. See what they come up with. Wait until they are dying for an idea, then give them some technique. But not too much."
–*A Mathematician's Lament*, top of page 42.

Artwork

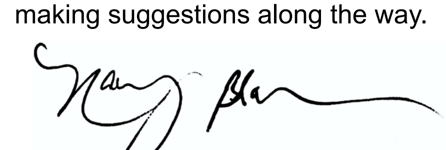
I really like the geometric pattern on a plate that I recently purchased so I took a photo of it and put it on the front of the cloth.

When searching for another image, I came across *Waiting for a Vaccine* from the 2021 Bridges Math Art. I'm impressed with how Krystyna Burczyk represents the coronavirus with colored paper. She gave me permission to include her image on the cloth.

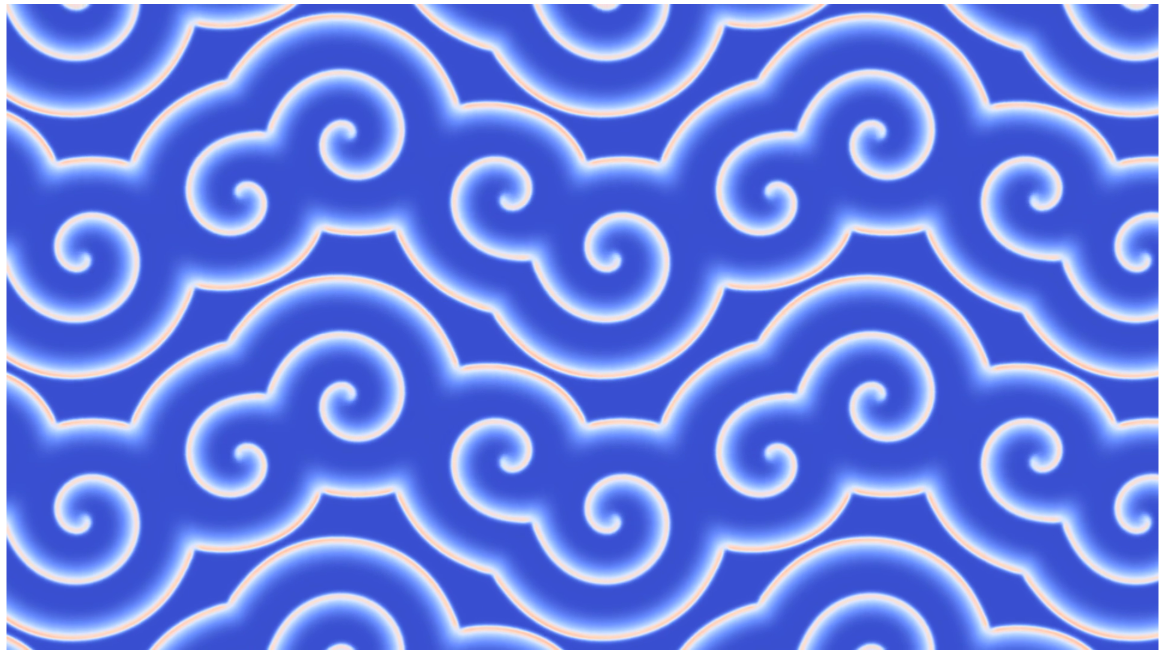
I wanted to come up with a quote that fit well with the quote on the front and Krystyna's *Waiting for a Vaccine*.

For the quotes, I selected the font **Fredericka the Great** because it is casual but classic.

Skona Brittain has been wonderful at reviewing ideas and making suggestions along the way.



Nancy Blachman



SymSim - creation of natural animated symmetric patterns using Symmetric Simulation

Vladimir Bulatov
http://bulatov.org
03/20/2024

Introduction

The motivation of this work is the development of algorithms to create animated seamless patterns with discrete symmetry in various geometries.

It is relatively easy to create a seamless pattern with symmetry generated by pure reflections. The basic kaleidoscope is constructed in such a way. Reflections are continuous functions and the resulting patterns are continuous. Animation of such a kaleidoscope keeps the pattern continuous. However discontinuity of derivatives along reflection lines generates obvious visible artifacts.

In case of more general symmetries it is rather difficult to make the pattern even continuous. A way to make symmetric patterns with arbitrary symmetry is the rubber stamp approach used by M.C.Escher in his tessellation work. This requires very difficult manual fitting of the tiles. No general way to animate such patterns exists.

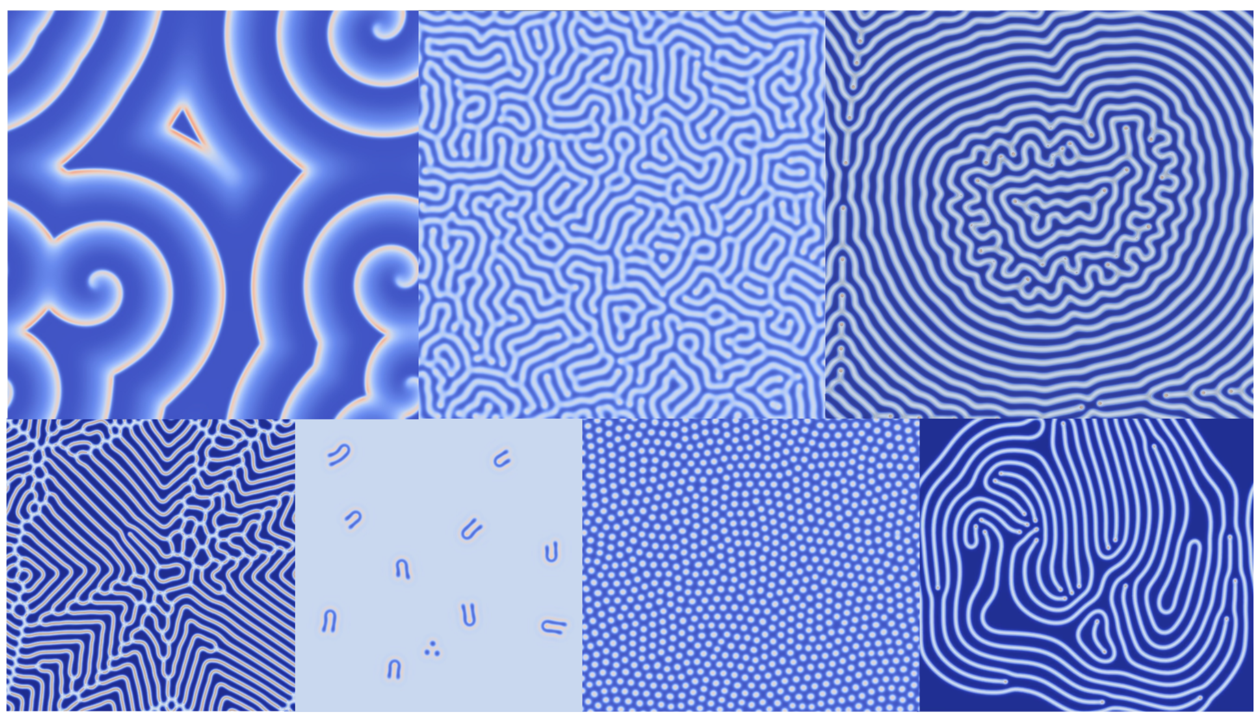
Another approach is to construct a function with fine tuned symmetric properties and use the function as a tool for domain coloring. It is difficult to construct such functions and it is hard to control its properties.

We are offering a general approach to make seamless patterns using dynamic (ordinary differential equations or partial differential equations) on an orbifold. An orbifold can be cut and flattened. The flattened orbifold will tile the whole space. If the pattern is seamless on the orbifold the tiled pattern will be seamless as well. So the problem of making a seamless symmetric pattern is reduced to creating a seamless pattern on the orbifold. This problem can be successfully solved using appropriate boundary conditions. Such patterns can be animated in real time in web browsers using javascript and WebGL.

Animated results of such an approach are presented in youtube videos.

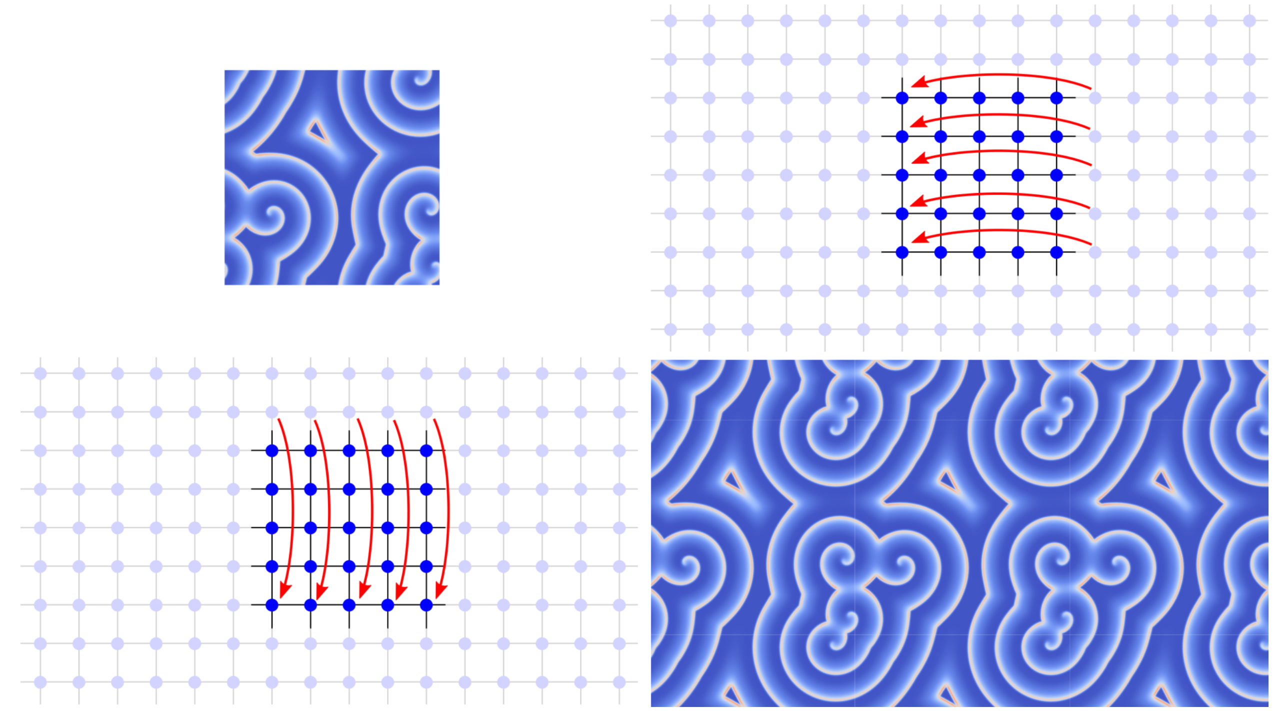
Gray-Scott Reaction diffusion

Gray-Scott reaction diffusion equation is a system of two nonlinear Partial Differential Equations (PDE) for a concentration of two different chemicals. The general behavior of its solution depends on two parameters: “kill” and “feed”. The equation is being solved numerically on a regular spatial grid patch. Starting from some initial values the solution evolves into a recognizable time-dependent pattern. The results of the simulation are visualized by mapping the concentration of one chemical to shades of color. The sequence of images for each time step can be displayed in real time during the simulation or can be exported into a video stream. Few static examples of generated patterns for different parameters are shown below.



Regular simulation on a 2-torus manifold

The solution of PDE depends not only on initial values but also on boundary conditions - constraints on the solution on the boundary of the system. In order to minimize the effect of the boundary the Periodic Boundary Conditions (PBC) are normally used. PBC requires that the values on opposite sides of the rectangular grid patch are equal. This effectively allows us to imitate the behavior of an infinite system using a grid of finite size. If the solution on the finite grid patch is periodically extended to the whole plane, the result will also be a solution of the PDE on the whole plane and the solution will be continuous and smooth across the boundaries of the finite grid patch. The simulation on the finite grid patch using Periodic Boundary Conditions is equivalent to simulation on the finite two dimensional manifold 2-torus.



Symmetric Simulation on Orbifold

The 2-torus is one of the 17 orbifolds which correspond to two dimensional wallpaper symmetries. Tiling the plane with images of the finite patch creates a pattern in the plane with symmetry **(O)** (in orbifold notations). Can we obtain other symmetries using appropriate replacement instead of Periodic Boundary Conditions? The cut and flattened orbifold is just some region of a plane with a boundary. In principle we can try to carefully fit a grid into the orbifold directly and perform the simulation on that grid and try to obey the corresponding boundary condition which reflects the way the orbifold was cut. This however would be a rather tedious procedure which requires dealing with details of a specific shape of the orbifold.

Instead we are using a simple rectangular grid which is larger than the flattened orbifold and completely covers it. The periodic boundary conditions are replaced with symmetrization. We call the rather universal and simple procedure **Sym**metric **Sim**ulation (**SymSim**). The idea of SymSim is to replace each regular simulation steps with three simple substeps:

a: symmetrization

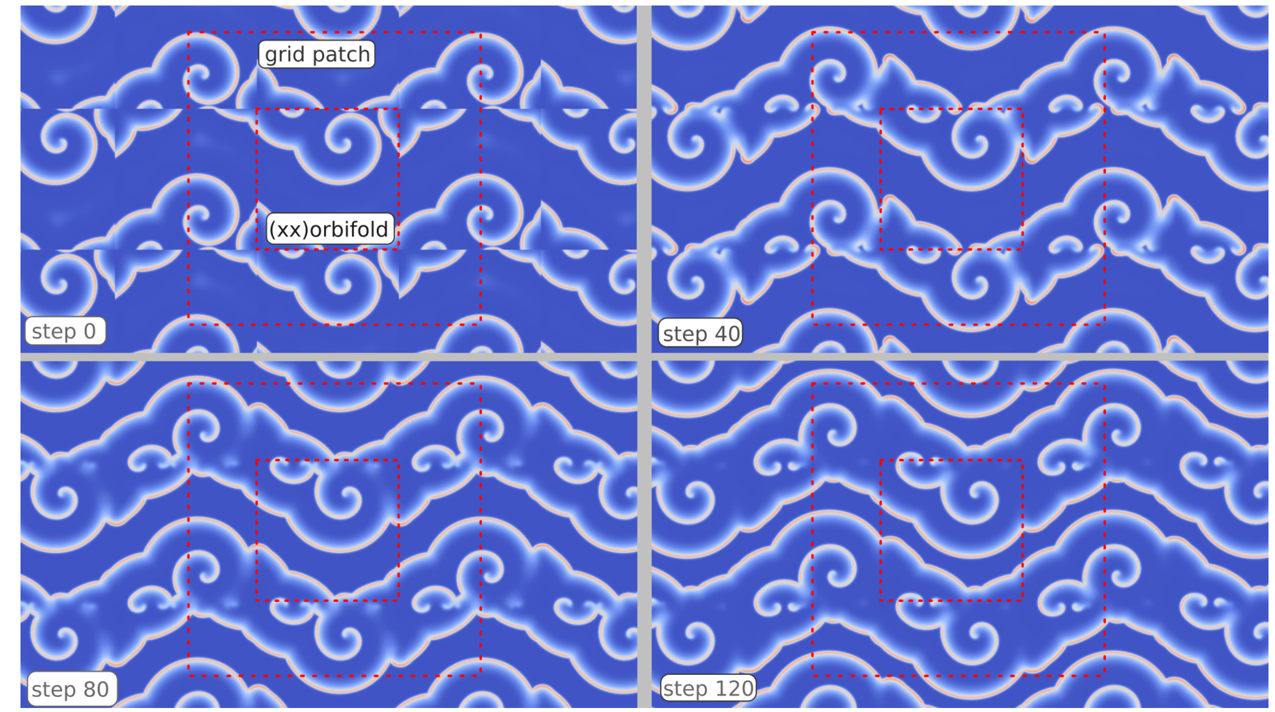
b: simulation

c: symmetric visualization

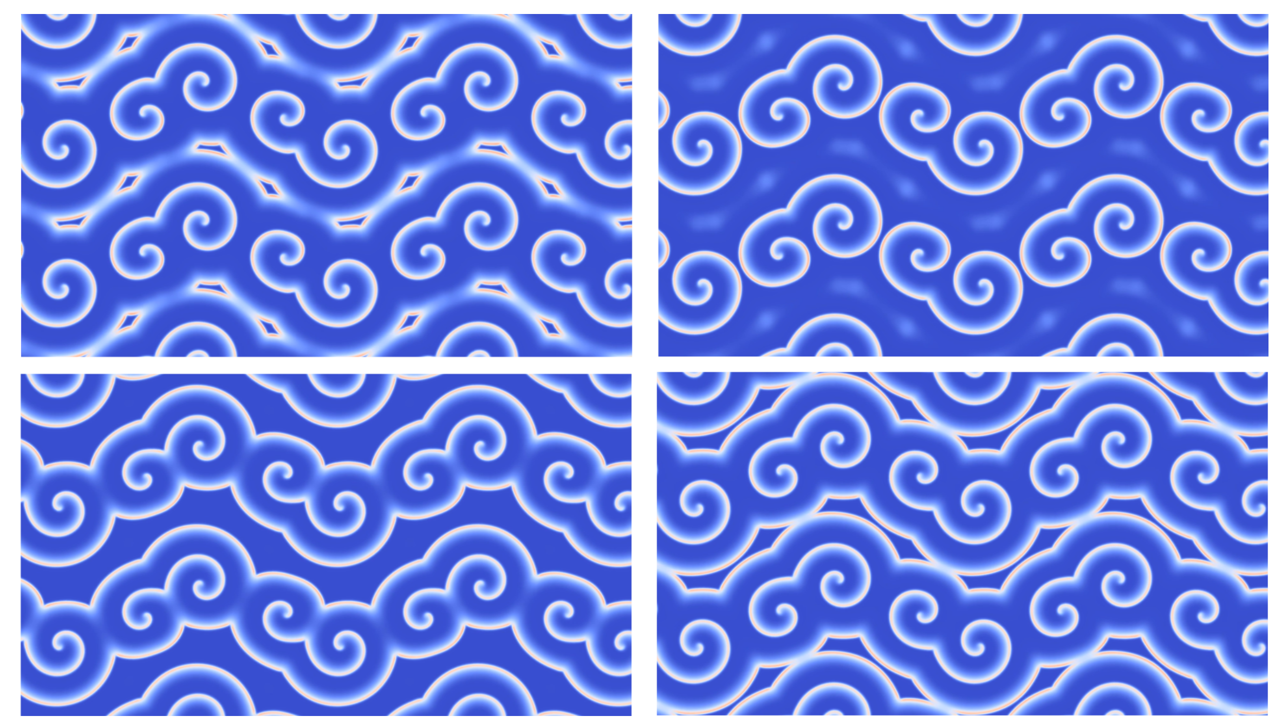
During the symmetrization step we replace the values at all the grid points which are outside of the flattened orbifold with the values from the corresponding points inside of the orbifold. It can be very efficiently performed using reverse pixel lookup procedure. The regular simulation step is performed as usual on the rectangular grid. The symmetric visualization step is once again using reverse pixel lookup procedure to map coordinates of each pixel of the plane into a point inside of the flattened orbifold and map its value into shade of color.

The symmetrization step effectively takes care of all the potentially complicated boundary conditions on the orbifold. The symmetric visualization does the extension of the finite image of the flattened orbifold into the whole plane.

Below are a few steps from simulation on orbifold **(xx)** . We start from initial values which were generated during simulation on a grid patch with periodic boundary conditions used in the previous image. The initial values were continuous functions on the 2-torus but are not continuous functions on the orbifold **(xx).** Therefore after the first symmetrization step we see strong discontinuities in the pattern. However after a few simulation steps those discontinuities disappear and the pattern of spiral waves becomes smooth.



Below are a few frames from the video of the previous spiral waves pattern taken after the waves become completely smooth and the initial discontinuities are completely forgotten.



SymSim properties

- SymSim can be used in a wide range of geometries: Euclidean, Hyperbolic, Spherical, Inversive. The simulation grid is simple. The symmetrization works.
- SymSim can be used in any number of dimensions.
- The solutions obtained using the SymSim procedure are true solutions of the original equations. The effect of the symmetrization procedure is to select a symmetrical solution from the wider class of arbitrary solutions.
- SymSim computations add very little cost to the original simulation. In practice the symmetrization step can be used only once per 1000 simulation steps.
- SymSim was initially implemented for the Gray-Scott Reaction Diffusion equation. At the moment it works also for the Complex Ginzburg-Landau equation (superconductivity phase transitions). The Navier–Stokes equations (fluid dynamics) support is under development.
- SymSim is implemented as an online application using HTML, JavaScript and WebGL2. It runs on any platform which supports these technologies (desktop, tablet, cellphone). It is available at http://bulatov.org/symsim

SymSim examples

There are numerous SymSim videos on youtube.

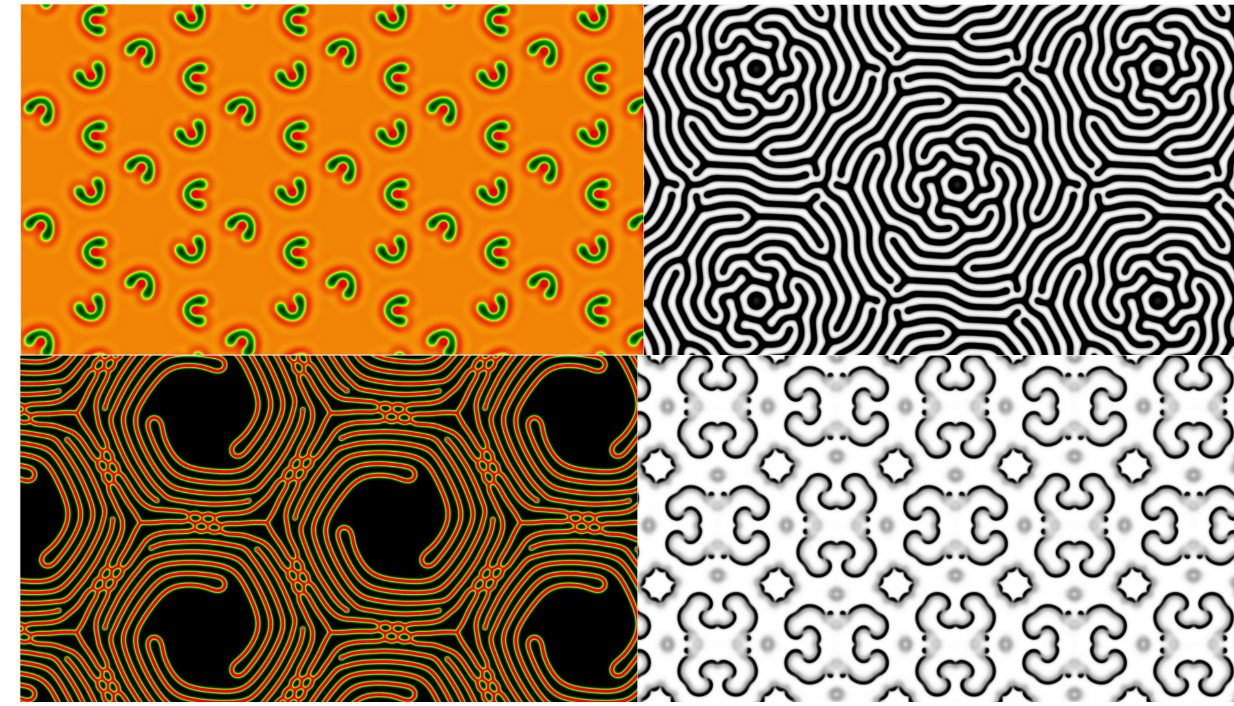
Gray-Scott reaction diffusion videos

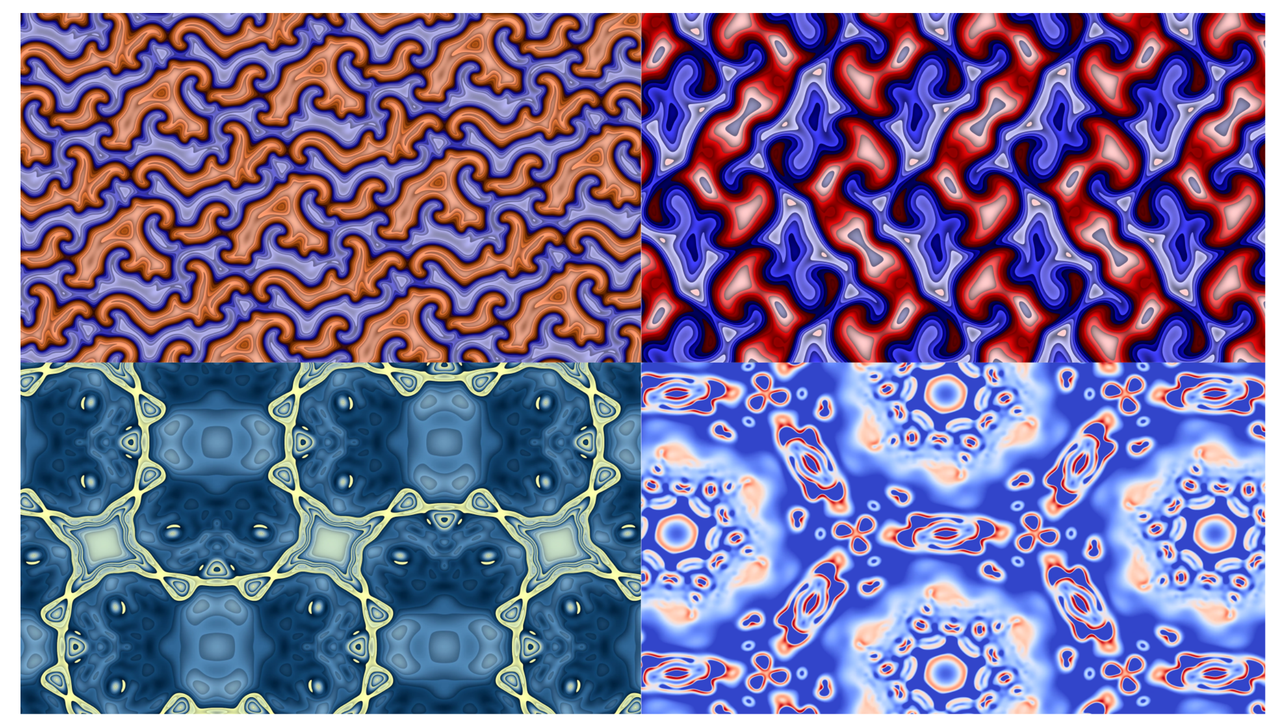
https://www.youtube.com/playlist?list=PLcayjHq4OzLYKBc8TSTKAEAAyPn6NGTUZ

Complex Ginzburg Landau equation videos

https://www.youtube.com/playlist?list=PLcayjHq4OzLY0GvQ4_uGUXA3YDrMPRKTo

More information is available at http://bulatov.org/symsim





Four Paper Projects

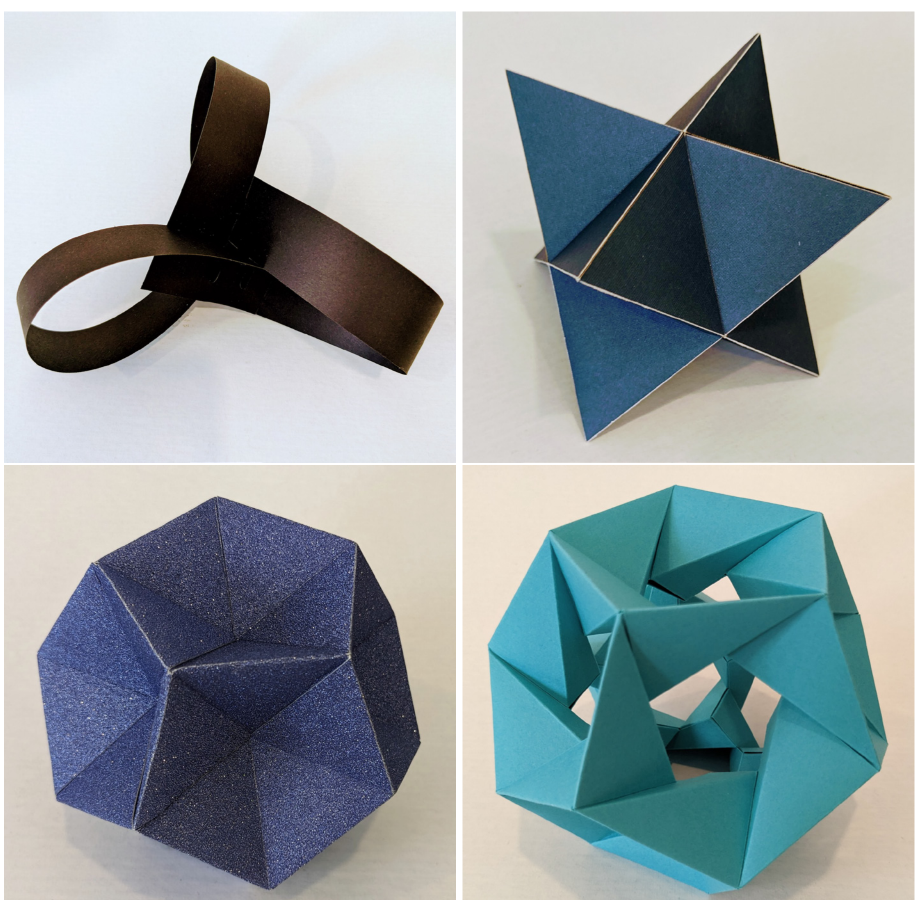
Shiying Dong

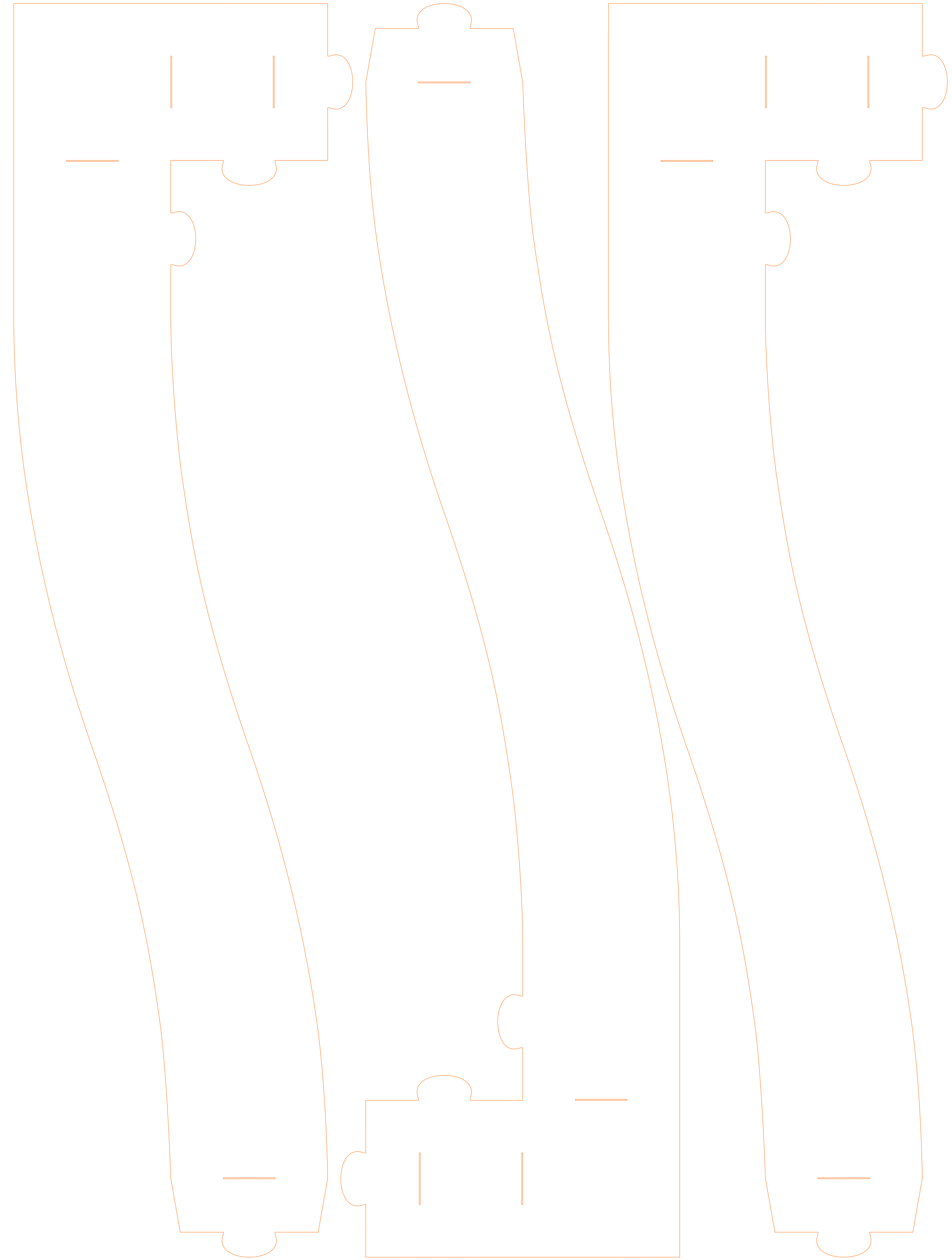
Greenwich, Connecticut, USA; shiyingdong@gmail.com

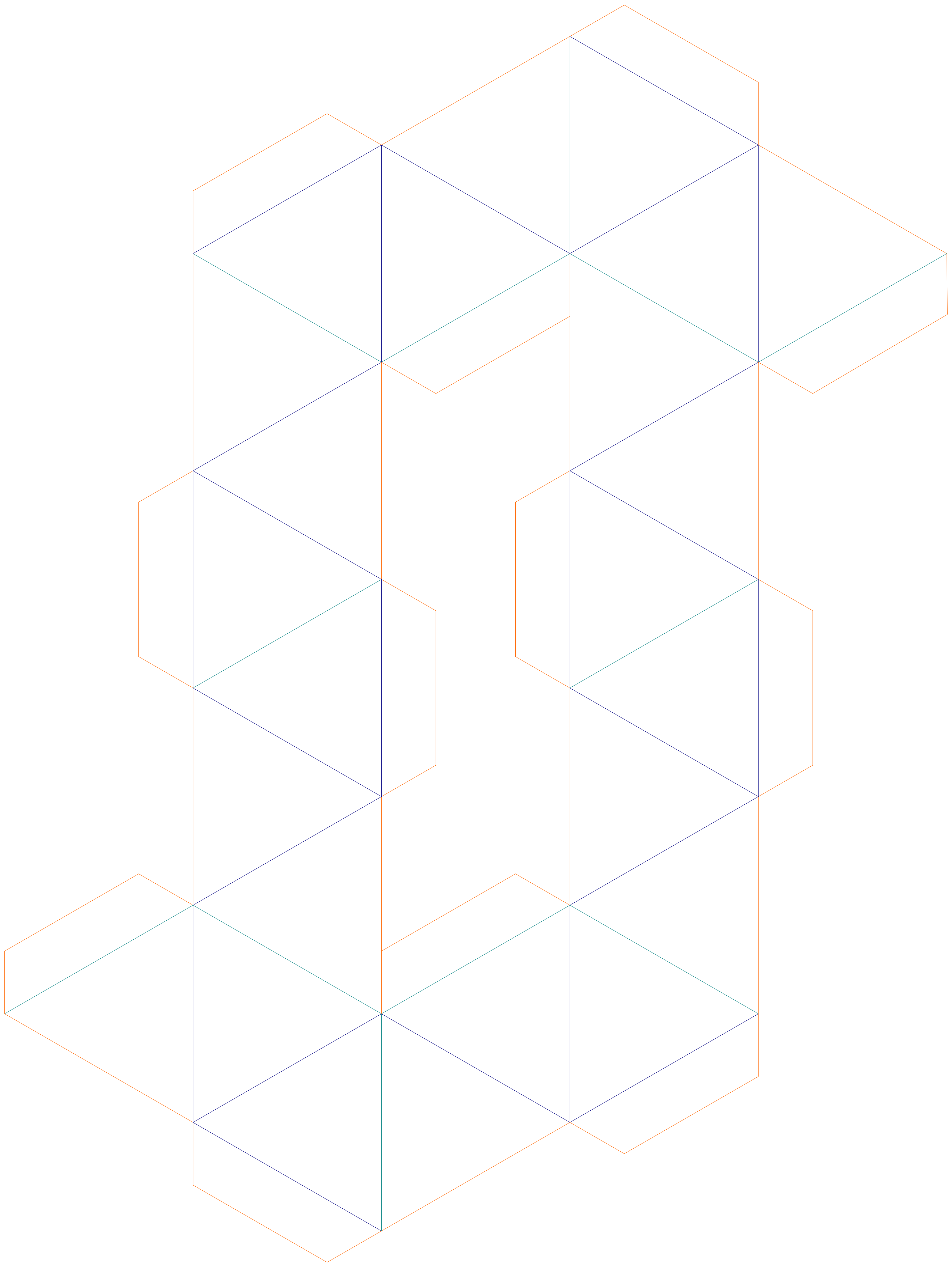
A lot of paper is available in letter size, and it is convenient to have a design that fits that size, ready to print or cut in devices like Cricut or laser cutter. In this paper, I've attached drawings of four projects I'm very fond of. I recommend light cardstock (150 - 200 gsm) for best results. All drawings other than this cover page are vector, therefore can be directly used in vector drawing softwares.

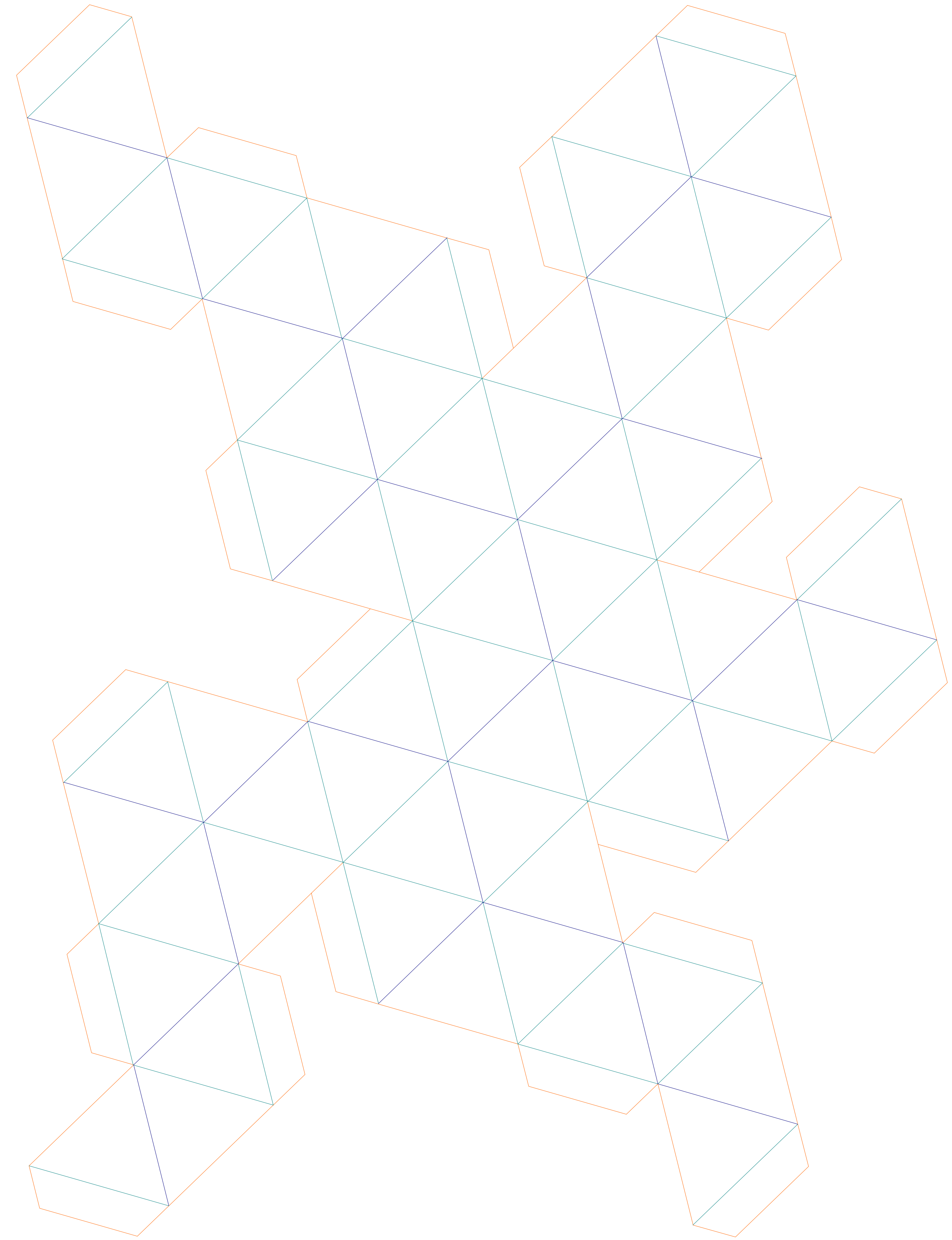
The projects, ordered by difficulty level, are the following:

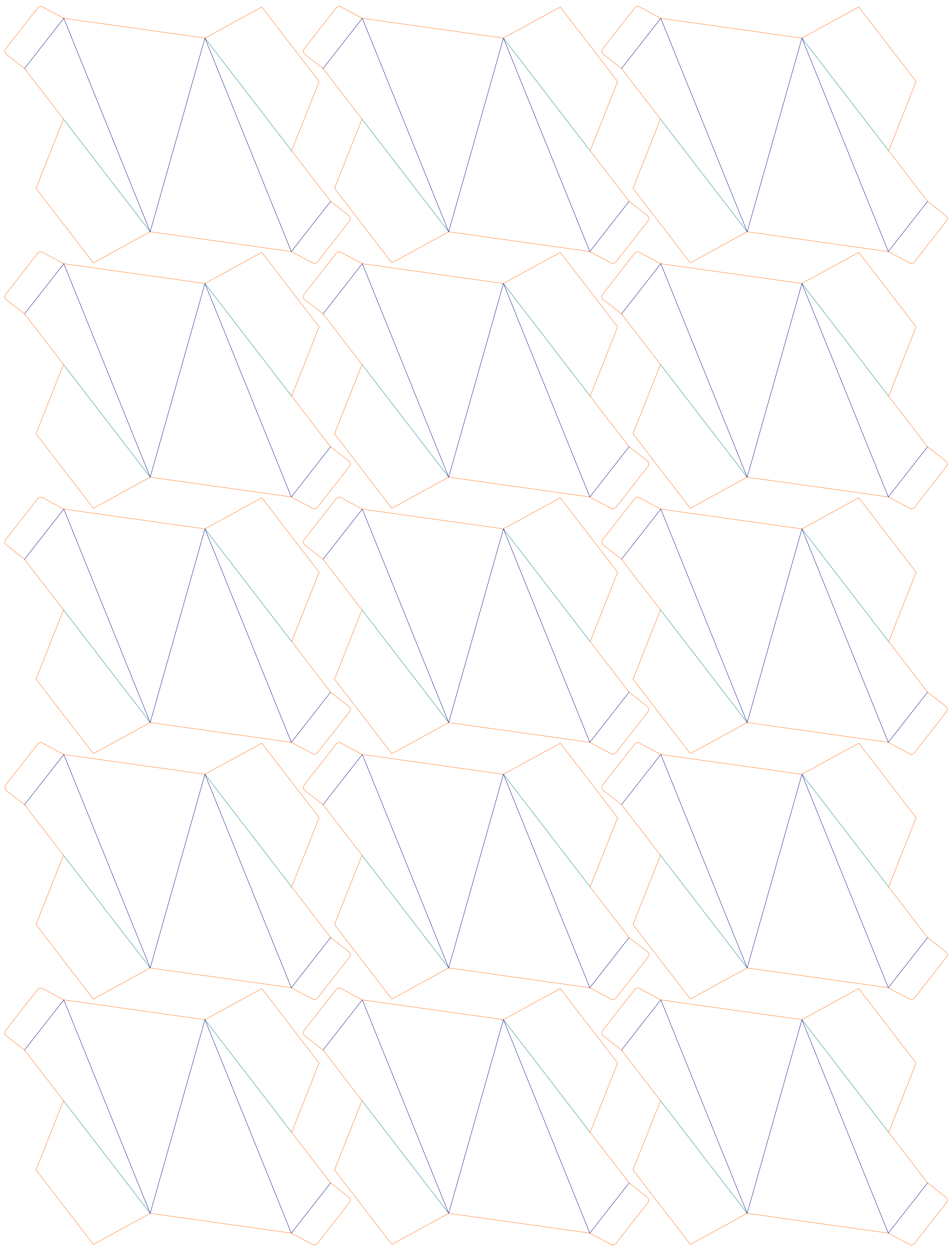
1. Making a band trefoil knot without gluing.
2. Gluing a stella octangula. This pattern is interesting because the net itself is topologically nontrivial.
3. Gluing an excavated dodecahedron.
4. Gluing the 14th stellation of icosahedron. Two copies (30 units total) of this sheet are needed.











Truncated flexagons: The hat flexagon and the heart flexagon

G4G15 Gift Exchange

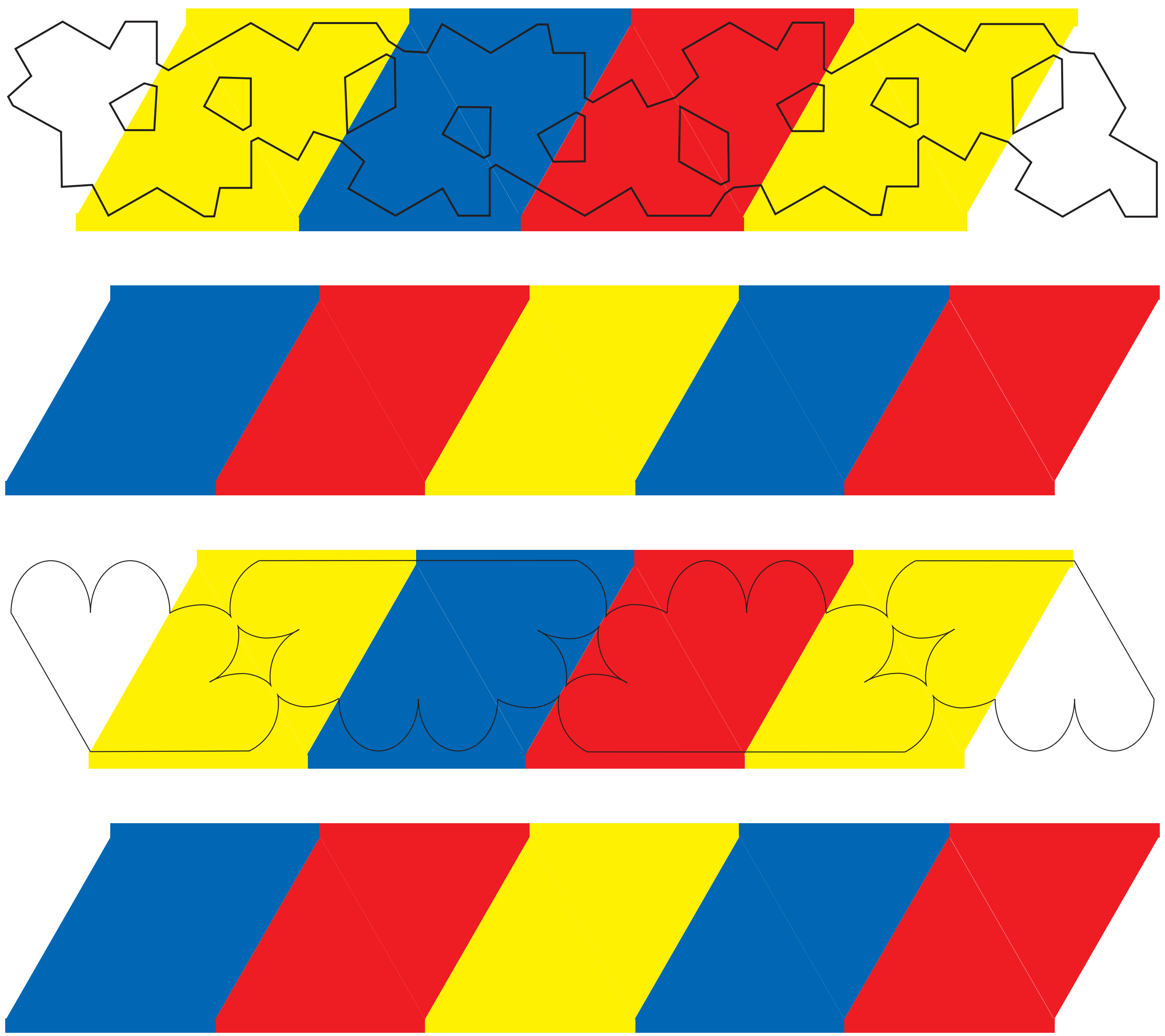
Yossi Elran

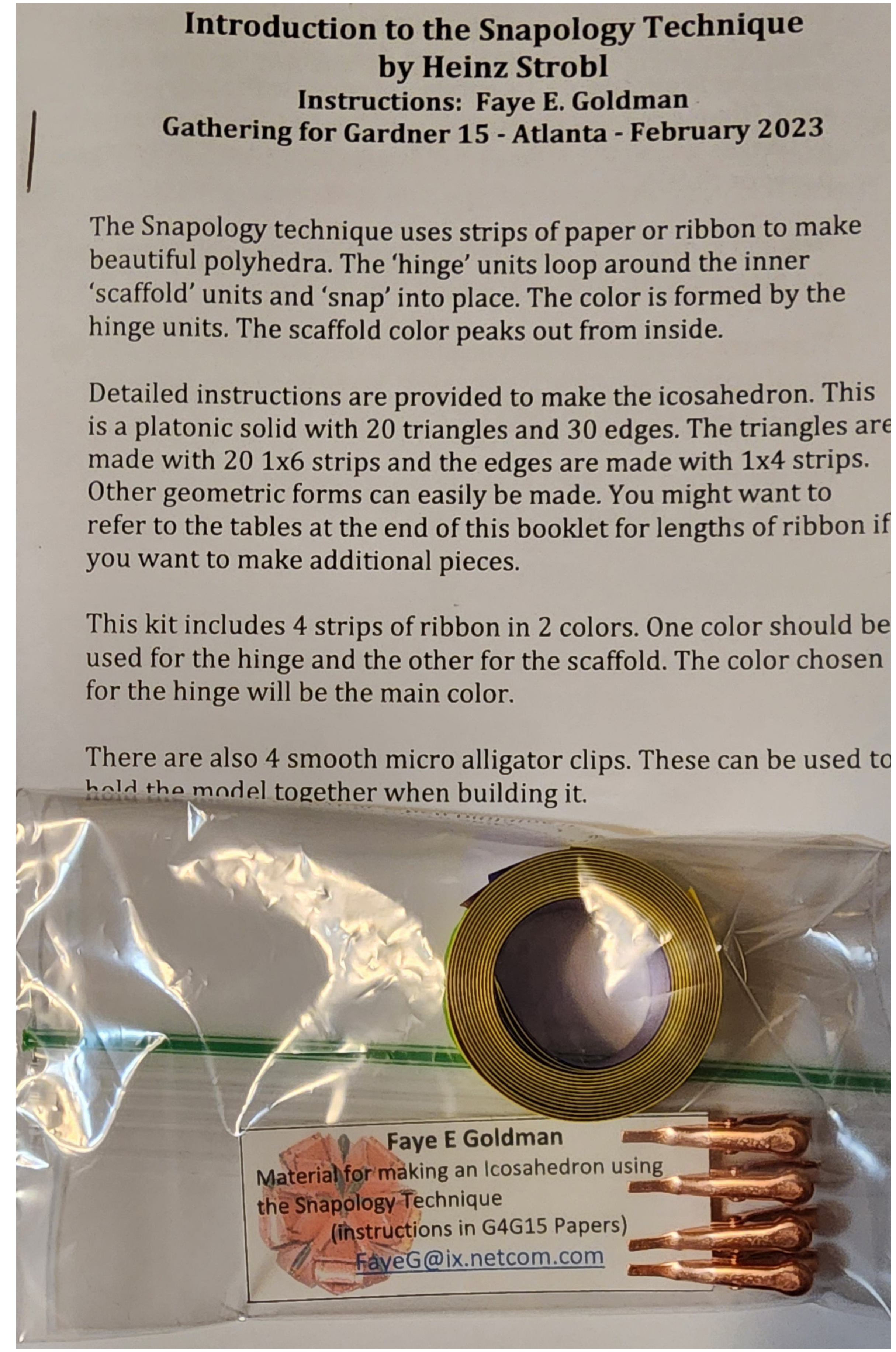
Truncated flexagons are flexagons where the initial polygon (triangle, square, etc) is truncated into another shape. Inspired by the recent discovery of the "hat" aperiodic monotile by David Smith, Joseph Samuel Myers, Craig S. Kaplan, and Chaim Goodman-Strauss, I truncated the triangle of a tri-hexa-flexagon to a shape resembling the hat. Due to the different rotations required so that the hats form neatly on the faces of the flexagon, the hat needed to be slightly adjusted; otherwise, some of the hinges would not be joined. I also made a "heart" tri-hexa-flexagon. When flexing the flexagons, beautiful patterns arise. Included are the templates for both flexagons. To create the flexagons, cut out the templates(preferably with a precision craft knife) along the bold lines. Make sure to cut around the whole boundary, and do not forget to cut out the small shapes within the strip itself. Crease the hinges between every two adjacent shapes (hats, hearts) back and forth. Do this gently so as not to tear the delicate hinges. Fold the yellow hats/hearts face-to-face and tape the white hats/hearts faces together. Your flexagon is ready!

Happy flexing!

Yossi

The Davidson Institute of Science Education, at the Weizmann Institute of Science (R.A)
P.O.B. 26, Rehovot 7610001, Israel | Tel. +972-8-937-8300 Fax. +972-8-937-8419





Introduction to the Snapology Technique
by Heinz Strobl
Instructions: Faye E. Goldman
Gathering for Gardner 15 - Atlanta - February 2023

The Snapology technique uses strips of paper or ribbon to make beautiful polyhedra. The 'hinge' units loop around the inner 'scaffold' units and 'snap' into place. The color is formed by the hinge units. The scaffold color peaks out from inside.

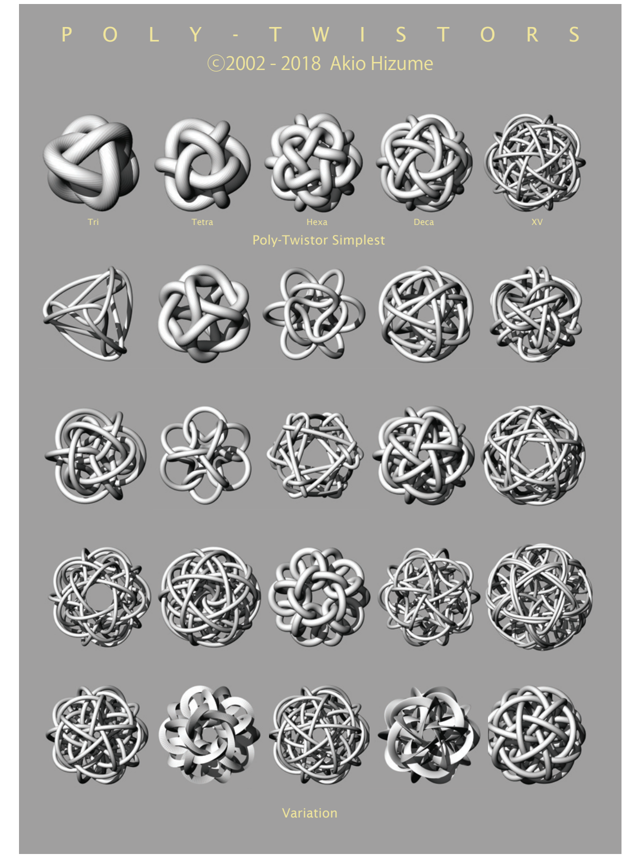
Detailed instructions are provided to make the icosahedron. This is a platonic solid with 20 triangles and 30 edges. The triangles are made with 20 1x6 strips and the edges are made with 1x4 strips. Other geometric forms can easily be made. You might want to refer to the tables at the end of this booklet for lengths of ribbon if you want to make additional pieces.

This kit includes 4 strips of ribbon in 2 colors. One color should be used for the hinge and the other for the scaffold. The color chosen for the hinge will be the main color.

There are also 4 smooth micro alligator clips. These can be used to hold the model together when building it.

Creative Constructionist Collage Making | Laura Hart

Recent Geometric Sculpture by Akio Hizume



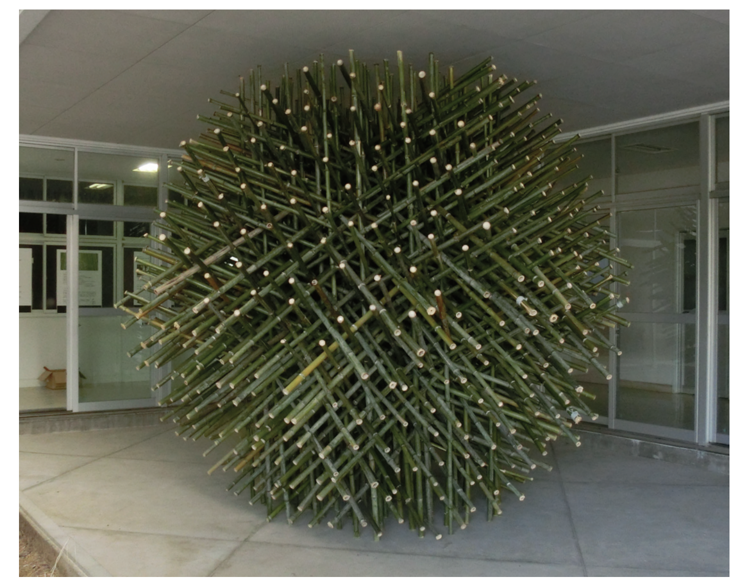
The "Poly Twistor" series of works is a three-dimensional integration of helical tori without intersections, according to the symmetry of regular polyhedra. The whole catalog was my exchange gift in G4G13 (updated in Jan. 2018).
I have already made over 50 different models on 3D printer. It is a formative study of the "three-dimensional torus" in topology.
I have concluded that 3D tori can be classified into five main types, eight more strictly, and 16 if all mirror images are included and counted.
It may be a form of explanation of the global structure of the universe as hinted at by Poincaré, and the precession of elementary particles.
The Poly Twistor model has the potential to request a new paradigm of physics.

Whole Catalog

http://starcage.org/poly-twistors_s.pdf

Youtube Movie

https://youtu.be/Ll1CZEwHPho



I have created giant quasi-crystal sculptures "MU-MAGARI" in Chichibu, Innsbruck, Zurich, Atlanta, Tokyo and Sao Paulo. They were always temporary exhibitions.
Finally, in 2022, I created my latest work as a permanent public sculpture at TAKE Labo, Ube, Yamaguchi, Japan.
Natural bamboo is usually individual and uneven. Taking advantage of this unevenness, I decided to try to construct a hyperbolic spatial arrangement of MU-MAGARI.
By placing the bamboos thinner in the centre and thicker on the periphery, the bamboos tend to curve slightly and the parallel lines become separated from each other. This is reminiscent of the nature of 'hyperbolic space'. The dodecahedral symmetry and curvature together create a stronger cohesion. It has 520 poles and is self-supporting.

Following on from giant quasi-crystal sculpture, I have created another giant bamboo structure at the TAKE Labo in 2023.
The Fibonacci Tunnel has been created many times before in Kyoto, Tokyo, Osaka and Los Angeles, but this time it is not a temporary installation but a permanent exhibition.
A double-helix tunnel based on a phyllotaxis. It consists of 200 bamboo poles. The diameter of tunnel is 3m and total length is about 8 m.

Youtube Movie

https://youtu.be/KOQVLMlAktw

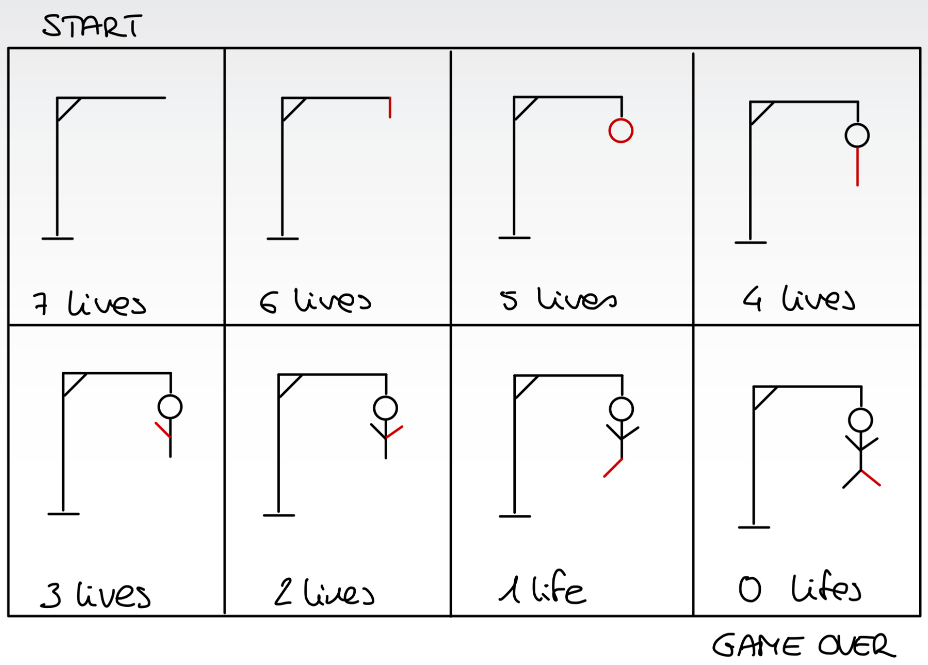
Apollonian dream

Bjoern Muetzel and Julian Muetzel
Eckerd College, St. Petersburg, Florida

Description:
Water drops fall into a lake and create ripples. For a fleeting moment these rings show an Apollonian gasket. The corresponding computer-generated video can be found in the link below.

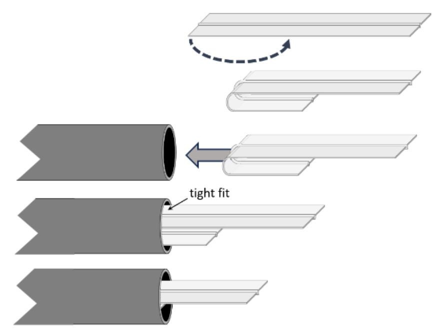
More info: http://natsci.eckerd.edu/~muetzeb@campus/gallery.php

GAMES



A Mathematical Hangman Game | L. Malato, F. Albuquerque | Page 40

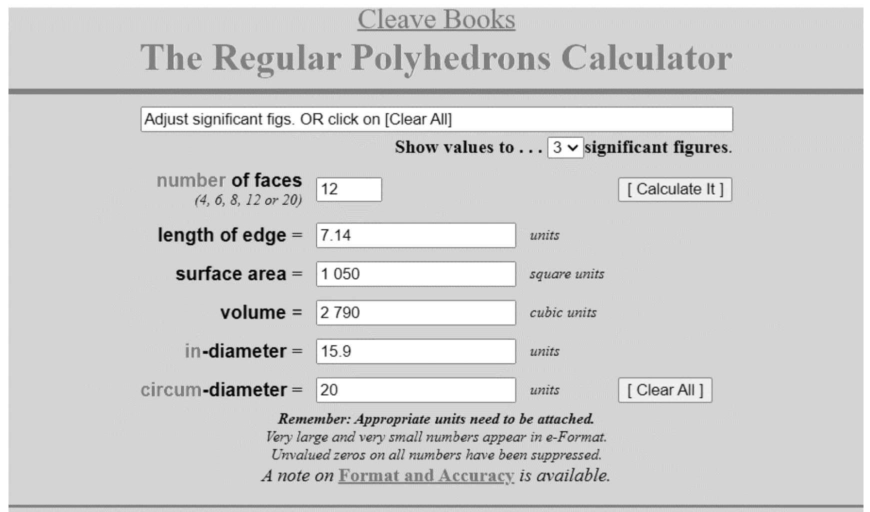
Coffee Stirrer Structures, Soap Film Configurations and the Birth of HyperTiles Robert Becker, HyperTiles LLC



A wide range of polyhedral frames can be built with coffee-stirrer straws and plastic-coated twist-ties.

It is important to find a coffee stirrer straw with an internal diameter that can just barely accommodate a folded-over twist-tie, as shown at right. It should take some effort to insert the folded-over twist-tie so that the friction with the inside walls of the straw will keep the assembly together.

These are intended to be temporary structures easily taken apart and reassembled into different structures. If desired, however, they can be made more permanent with the addition of small amounts of glue placed on the twist-tie bend before it is inserted into the coffee stirrer hole. [The author has found that E6000 Premium Contact Adhesive works quite well.]


Cleave Books

The Regular Polyhedrons Calculator

Adjust significant figs. OR click on [Clear All]

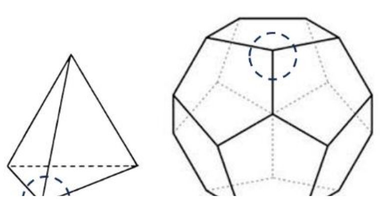
Show values to . . . 3 significant figures.

number of faces (4, 6, 8, 12 or 20)	12		[Calculate It]
length of edge =	7.14	units	
surface area =	1 050	square units	
volume =	2 790	cubic units	
in-diameter =	15.9	units	
circum-diameter =	20	units	[Clear All]

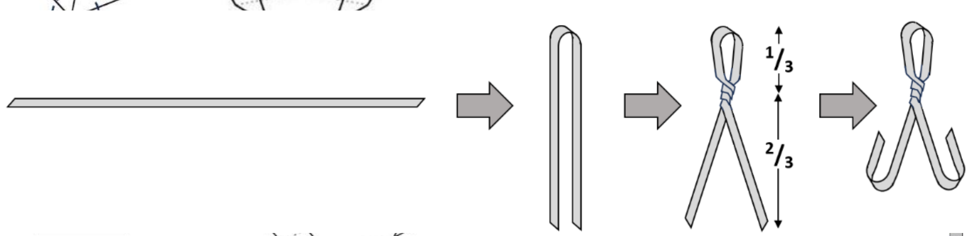
Remember: Appropriate units need to be attached.
Very large and very small numbers appear in e-Format.
Unvalued zeros on all numbers have been suppressed.
A note on Format and Accuracy is available.

The technique is simple:

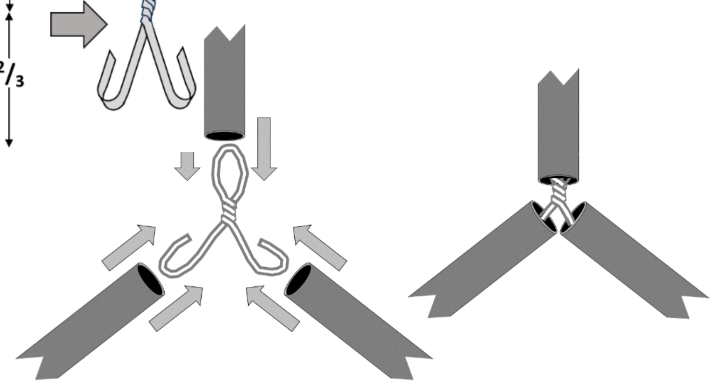
1. The coffee stirrers can be cut down to any desired size. Charts and "calculators" such as this one – that list the dimensions and edge lengths of various geometric solids – are quite common on the internet. So if you want to construct a dodecahedron (12-sided) that is 20 cm in circum-diameter, you will want to cut your each of your coffee stirrers down to 7.14 cm.



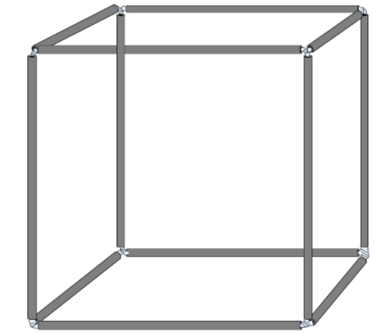
2. To make the twist-tie into a three-way connector (as might be used for the vertex of a tetrahedron, a cube, a truncated octahedron, a dodecahedron…), fold the twist-tie in half. Twist it with 3-4 *tight* twists about one-third of the way from the bend to the loose ends. Then take each of the two loose ends and bend them over.



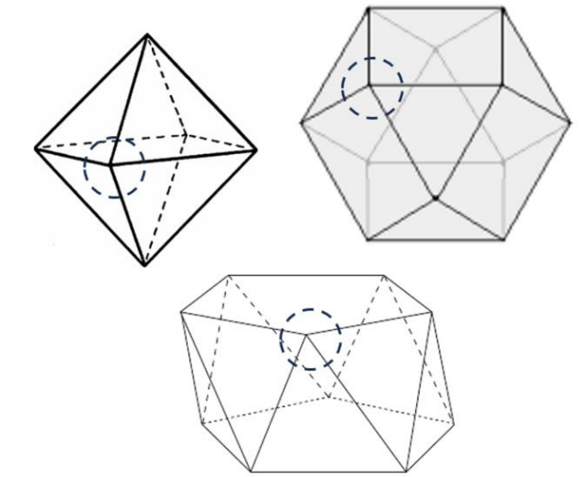
3. These three bent-over extensions can all be pressed flat and then pushed into the open ends of three separate coffee stirrer sections, as shown at far right. The fit should be snug enough to hold the vertex together.



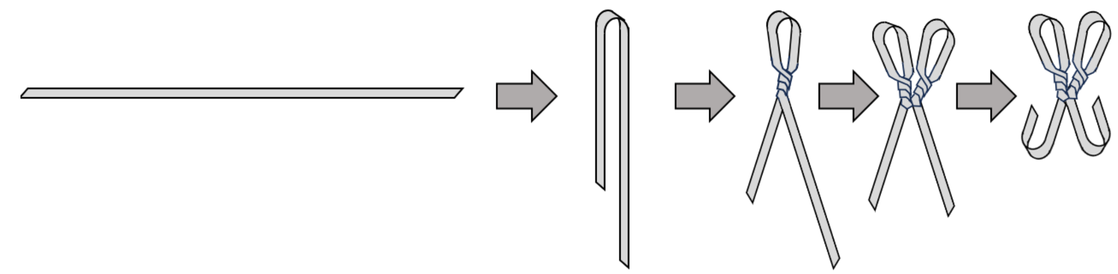
***Those participating in the G4G15 Gift Exchange have received a bag containing 15-20 coffee stirrers (13 cm long), 20-25 twist-ties (10 cm long), one pre-made three-way juncture – like the one shown at right, *and* two sample HyperTiles and two connectors – as described on page 3.**



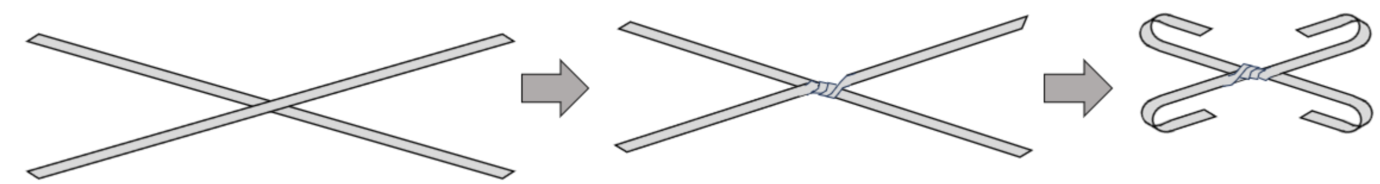
4. This process can then be repeated at each end of the coffee stirrer sections to create the desired shape, like the cube shown at left:



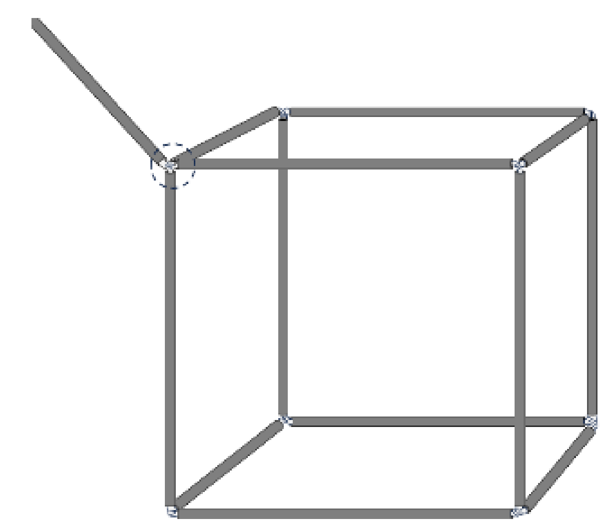
5. To make the twist-tie into a *four*-way connector (as might be used for the vertex of an octahedron, a cuboctahedron, a pentagonal antiprism…), fold the twist-tie in not-quite-half (think 40-60). Twist it with 3-4 *tight* twists about one-third of the way from the bend to the shorter loose end. Then make a second twisted loop using the longer loose end. Then take each of the two loose ends and bend them over.



Or… should one twist-tie prove too short to make this four-way configuration possible, use *two* twist ties. Cross them at their midpoints and then make 4-5 *tight* twists at their intersection. Then take each of the four loose ends and bend them over.



Also, a four-way connector can be used in place of one of the three-way connectors so that a handle can be added to a cube or tetrahedron. This is often quite handy for observing the patterns that form when these polyhedral frames are dipped in soapy water solution.



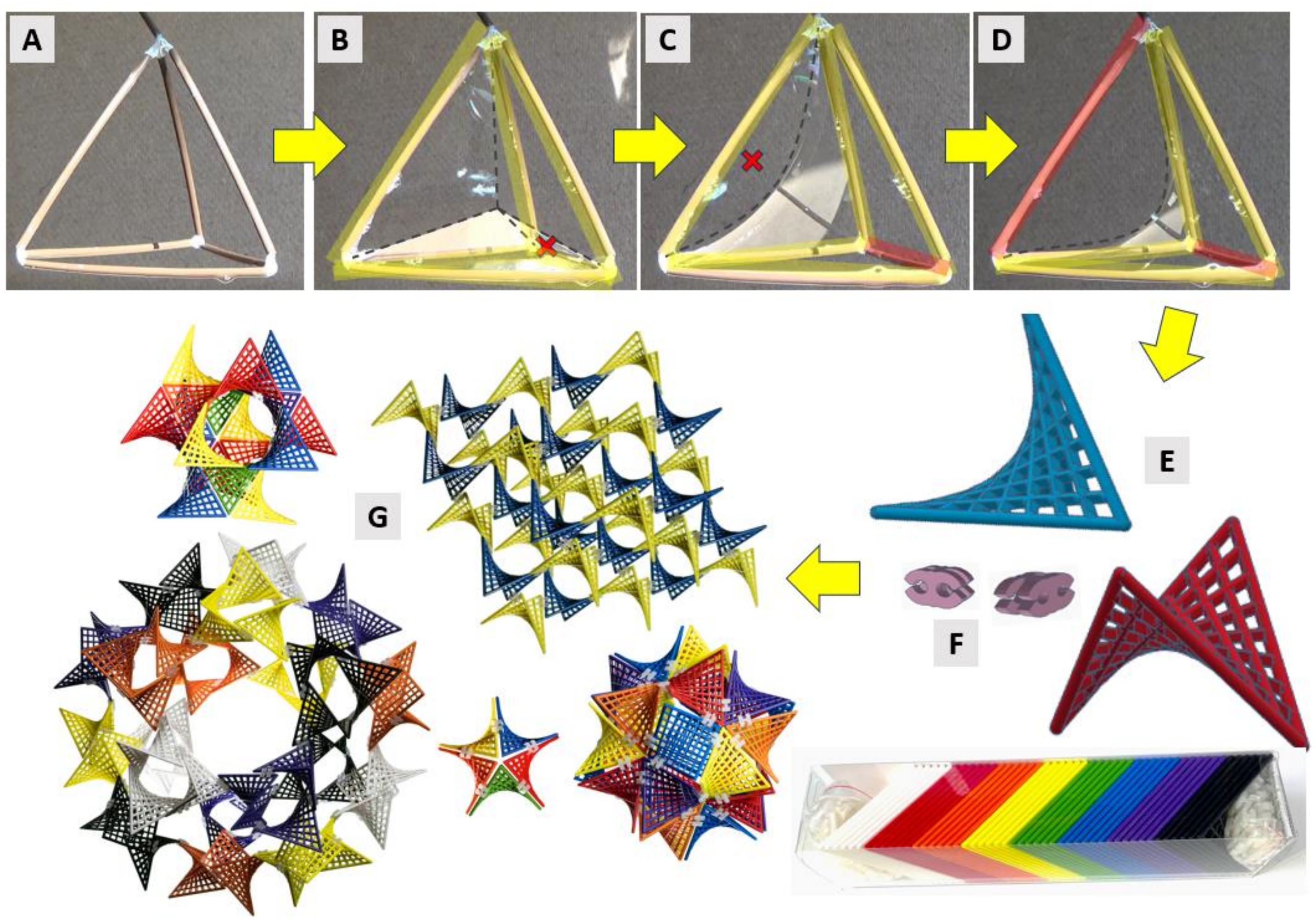
Speaking of which, dipping such frames in soapy water reveals the minimum surface structures that form on the frames. The soap films solve the problem of how to cover the edges of the frame with the least total surface area. These film configurations are sometimes made up of all flat surfaces, but often they will include hyperbolic surfaces – saddle shapes with negative curvature. The films will never have positive curvature unless the task also includes encapsulating a certain volume of air.

Many amazing explorations can be enjoyed by dipping these frames in soapy water, pulling them out and then popping various portions of the configuration and seeing what new configurations form. (5% Dawn dish detergent in tap water works just fine.) The cube shown above has well over twenty distinct and very beautiful soap film configuration's that can be produced upon it.

When a tetrahedral frame (A, below) is dipped in soapy water, what forms is a collection of six flat triangular films – all meeting at the central point (B). This is the best way to cover all six edges of the tetrahedron (highlighted in yellow) with the least surface area. When the lower righthand film (marked by first X) is popped, the arrangement instantaneously changes to one that involves just three films - one flat film (approximating a minor segment of a circle) and two somewhat hyperbolic films that all meet along a curved (parabolic?) line (C). This is the best way to cover the five tetrahedral edges highlighted in yellow with the least surface area. When the the upper left film (marked by the second X) is then popped, the arrangement immediately changes to a single film that approximates a hyperbolic paraboloid (D). This is the best way to cover the four tetrahedral edges highlighted in yellow with the least surface area. [This video shows this transformation.] This particular hyperbolic shape on a tetrahedral frame is what inspired the author to create a construction toy, HyperTiles (E).

Using just this one simple hyperbolic shape as building block, and a small flexible, hinge-able connector (F), a huge variety of structures can be created with all sorts of surprising mathematical features (G). Many of these are featured at the Gallery of Structures page on the HyperTile website.

Anyone interested in learning more about HyperTiles or possibly purchasing sets should visit the website at https://www.hypertiletoy.com/.



The Game Turing Machine

Lyman Hurd
iManage

Glenn Hurd
GQH Tutoring

January 20, 2024

1 The Game

Turing Machine[1] is a guessing game produced by Scorpion Masqué[2]. Players compete to deduce a three digit code where the numbers range from 1-5. The winner being the person who can deduce the number with the fewest clues. The game is completely self-contained and everything needed to play is provided. The answers are provided by a series of cards with obfuscated values and the game does not require any form of app, although one can expand the list of possible games from the given twenty printed in the rulebook to millions by means of the companion website[3]. The way verification is accomplished is ingenious in its own right and I encourage people to watch the numerous explanatory videos on BoardGameGeek[4]. Clues are provided by means of Criteria cards of which there is a deck of forty-eight cards and each game specifies between 3 and 5 cards for that particular game. In addition, the game specifies a verification card which enables players to query whether a number they present matches one of the criteria on the card. The rulebook uses "verification card" and "verifier" more or less interchangeably. See Figure 1.

Note that the digits in the secret code are always expressed in the order: (blue, yellow, purple), therefore on the criteria cards a reference to "blue" is interchangeable with a statement about the "first digit", yellow, "second" and purple, "third".

Here is an example of one such criteria card:

$$B > P$$
$$B = P$$
$$B < P$$

At least one of these statements is true of the secret code. In this case it is also true that at most one of these criteria can be true as the three possibilities are mutually exclusive. However, this is not the case with all such cards. For example another criteria card states:

$$B = 1$$
$$Y = 1$$
$$P = 1$$

In this case, a given keyword can match more than one of the criteria. The code (1, 1, 1), for example, would match all three criteria. Each of the criteria cards lists a minimum of two and a maximum of nine criteria. One point that escapes many players on first examination, is that the verifier card performs validation for a single criterion on the criteria card. It does not say anything about the truth or falsity of any other criteria on that card.

Figure 1: The Turing Machine Game

2 Analysis

There are some assumptions one can make simply by looking at the choice of criterion cards independent of verifiers, which is public knowledge at the start of the game. The rules narrow down possibilities by the following implicit facts. In the discussion that follows all references to criteria cards and verification cards mean the cards spelled out as associated with a specific game.

1. The secret number must give the answer yes to at least one criterion per criteria card.

2. Only one number will give the answer "yes" to all of the verification cards, or stated a different way, there is a unique solution that gives the answer yes to the criteria tested by each verifier.

3. If you remove any of the criteria cards (and its verification card), the solution will no longer be unique.

The second property means that while the rulebook includes a table of correct answers for each of the twenty sample games, showing that a number gives a "true" answer to each card suffices to prove it is correct.

It follows from the above assertions that if you know which criterion on each criteria card is being tested by its associated verification card, you would be able to deduce the solution. You could, in that case, compute for yourself the yes/no answers given by each of these criteria to all one hundred twenty-five possible codes and the only number that passed every test would be the answer.

3 The Program

The authors were curious as to how much information was available to the players before the game began based on some of the implicit statements described above. First, we represented every one of the forty-eight criteria cards as a list of assertions expressed as a Python lambda function which was simply a function from the variables (b, y, p) to true/false. For example, the first criteria card corresponds to the expressions:

lambda b, y, p: $b > p$
lambda b, y, p: $b == p$
lambda b, y, p: $b < p$

We chose for our example the most complex puzzle that only used four criterion cards. The four criteria cards provided were:

Card A	$B + Y + P < 6$
	$B + Y + P = 6$
	$B + Y + P > 6$
Card B	$B = 1$
	$Y = 1$
	$P = 1$
Card C	$B < 4$ or $B = 4$ or $B > 4$
	$Y < 4$ or $Y = 4$ or $Y > 4$
	$P < 4$ or $P = 4$ or $P > 4$
Card D	$B < Y$ or $B = Y$ or $B > Y$
	$B < P$ or $B = P$ or $B > P$
	$Y < P$ or $Y = P$ or $Y > P$

An interesting property of this particular set of criteria cards is that the conditions are symmetrical with respect to the letter b, y, p. There are $729 = 3 \times 3 \times 9 \times 9$ possible combinations of criteria that could have been expressed by means of the provided verifiers. We then set out to further narrow the possibilities. For each of the 729 possible combinations, we tested all 125 possible codes to determine how many codes would satisfy them. Of the 729 combinations we immediately ruled out 489 combinations which have no solution whatsoever. Also 162 possible combinations were eliminated as they did not lead to a unique solution as these combinations yielded between 2 and 14 possibilities). This leaves 78 combinations which would result in a unique secret code. However, the combination of criteria uniquely defines the codeword, the opposite is not true. There can be different combinations yielding the same underlying code. Thus, 78 combinations of criteria to only 27 possible codes At this point, however, we have not used the property that every one of the criterion cards is necessary, meaning that if you consider possible solutions for any subset the solution is no longer unique. We computed the list again but in this next run we only counted those combinations or criteria such that they led to a unique code but if you were to eliminate one of the four criteria cards altogether (this eliminates either 3 or 9 criteria depending on which card is removed) we further required that there would be more than one solution. Adding this constraint narrowed the list of 27 to the following 9 possible solutions:

$$(1,2,3),\ (1,3,2)\ (2,1,3)\ (2,3,1)\ (3,1,2)\ (3,2,1)$$

$$(3,1,3)\ (1,3,3)\ (3,3,1)$$

Note that the possibilities comprise all the permutations of $(1,2,3)$ and $(1,3,3)$. This property will not be true in general and is a consequence of the fact that for this game the criteria cards operated symmetrically on the three digits.

4 Implications for Game Play

The above analysis was conducted with no knowledge of the assigned verification cards. The deductions were made solely on the basis of the information that is available to all players before the game begins. This fact simplifies the goal of each player, assuming they are capable of this much mental calculation, to deducing which code among the list of possibilities is expressed by this particular choice of verifiers. Each time one queries one of the verifiers, one receives a single bit of information which means that the smallest number of inquiries needed to deduce the code should be: $\lceil \log 2N \rceil$ where N is the length of this list, and so, for example, with bad luck one would need four guesses for this game.

In the actual game, things are slightly more complicated as the players have to divide their guesses among rounds and in a given round one can choose a single number to test one to three verification cards. An obvious next step for the program is to determine the most efficient strategy for narrowing down the possibilities. Recalling that one of the rules implicit in our analysis was that removing any of the criteria cards would remove the uniqueness property. However, looking at the nine possible solutions one notes that they all give the same answer for the third criteria card and there is no reason to test using this card.

5 Conclusions

The authors performed the above analysis on all twenty of the initial games presented in the rulebook and determined that for four of twenty games, applying the logic above, there was a unique solution, in other words a perfectly logical player could state the solution having made zero tests, In an additional five cases, there are only two possible solutions meaning that a solution can be determined in a single guess. Although it was not the intention from the start, it turns out that the game we chose as our first focus has the maximum number of possible solutions of the first twenty.

We are also quick to point out that the calculations performed are not feasible for a player to actually carry out and that this paper should not be construed as a criticism. On the contrary, it is a fascinating game with, as we have noted, layers upon layers.

References

[1] Turing Machine `https://www.scorpionmasque.com/en/turingmachine`. Acccessed on 2024-01-20.

[2] Scorpion Masqué `https://www.scorpionmasque.com/en`. Acccessed on 2024-01-20.

[3] Turing Machine.info `https://turingmachine.info/`. Acccessed on 2024-01-20.

[4] BoardgameGeek `https://boardgamegeek.com/boardgame/356123/turing-machine`. Acccessed on 2024-01-20.

A mathematical hangman game

L. Malato, F. Albuquerque

lia.malato@hotmail.com, fmapic@gmail.com

February 1, 2024

Abstract

Everyone knows the hangman game! Well here is its mathematical counterpart. The goal of this game is to guess the right number. However, there is a catch... You are not allowed to guess the digits. How can you win then? You have to ask your opponent specific questions about the mathematical structure of the number you're trying to guess. Are you going to beat the game?

1 Introduction

The original hangman game is a two-player game, where player A secretly selects a random word and only displays how many letters that word has. Player B then has to guess the word letter by letter. But what if we didn't stop at letters?

It all started during a Sunday afternoon. The authors were having a hot drink at a local coffee shop when suddenly the first author had an idea to entertain their mathematical minds.

"What if we played a game of hangman, but with a little mathematical twist?"

The basic idea of the game is to guess a number instead of a word. In the original hangman game, the players guess letters in a way as to find a meaningful word. In a number variant of hangman, one might be drawn to guess the number digit-by-digit... which is something we will not do. Instead we will be playing with the fundamental properties of each number and how the secretly chosen number relates to other numbers. The goal is to guess the right number through well chosen questions.

2 Basic rules

In this game we have two players: player A and player B. Neither player needs to have any specific mathematical background. In what follows we will assume both players have high school mathematics knowledge.

To start the game, player A thinks of a number that player B will need to guess through a series of mathematical questions based on the structure of numbers. The questions must be made in a way that the answer can only be 'yes', 'no' or a number. Player A starts by drawing short lines (one line per digit) on a piece of paper. For example, if the number to be guessed is 490, then player A would write down

___ ___ ___

After this, the game repeats the following pattern:

1. Player B asks a question.
2. Player A answers with a 'yes', a 'no' or a number, depending on the question asked.
3. Player B gives a guess.
4. In case player B is right, the game ends. In case player B is wrong, the game continues and player B loses a life.

Player B has 7 lives. Just like with the common hangman game, with each lost life player A draws a line as follows:

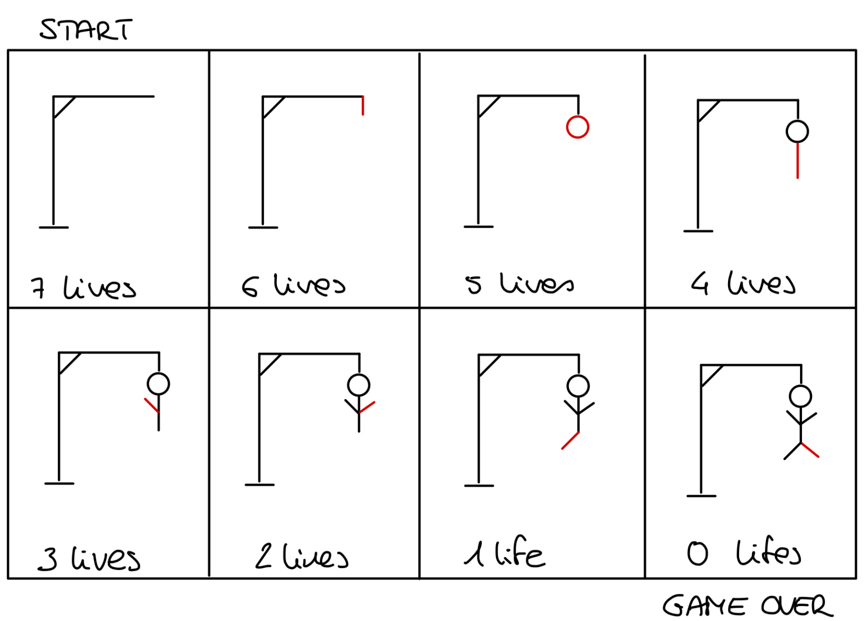


Figure 1: All possible drawings for the hangman game.

Bonus rule: If player B, at some point, gives a guess that is inconsistent with the previously gathered information, player B loses an extra life. This implies that it is possible to lose two lives in one guess!

3 An example

Consider two players: Saunders and Carl[1]. Saunders thinks of a number that Carl needs to guess. Saunders thinks of the number 476 and Carl will try to guess that number. Saunders starts off by writing 3 lines on a sheet of paper:

___ ___ ___

And so the game begins:

Carl: What is the lowest natural number above the square root of the number?
Saunders: 22 *(since* $\sqrt{476} \simeq 21.82$*).*
Carl: Is your number 400? *(Here Carl chooses any number below* $22^2 = 484$*)*
Saunders: No *(Carl loses their first life).*

Carl: How many distinct prime factors does the number have?
Saunders: 3 *(since* $476 = 2^2 \cdot 7 \cdot 17$*).*
Carl: Is your number 105? *(Carl tried* $3 \cdot 5 \cdot 7$*)*
Saunders: No *(Carl loses their second life).*

Carl: What is the greatest common divisor between your number and 30?
Saunders: 2 *(since* $30 = 2 \cdot 3 \cdot 5$ *and therefore* $\gcd(30, 476) = 2$*).*
Carl: Is your number 646? *(Carl tried* $2 \cdot 17 \cdot 19$*)*
Saunders: No and you just contradicted the answer to your first question.

Here Carl now loses two lives: one because his guess is incorrect and another one because his guess is greater than $22^2 = 484$*, an information obtained at the beginning of the game.*

Carl: What is the greatest common divisor between your number and 1001?
Saunders: 7 *(since* $1001 = 7 \cdot 11 \cdot 13$ *and therefore* $\gcd(1001, 476) = 7$*).*

From these four questions, Carl is able to conclude that the prime factorization of the secret number must be of the form $2 \cdot 7 \cdot p$*, where* p *is some prime greater than* 13.

Carl: Is the number 238? *(Carl tried* $2 \cdot 7 \cdot 17$*)*
Saunders: No *(Carl loses their fifth life).*

Carl now understands that one of the primes is repeated in the prime factorization.

Carl: Is any of the prime numbers in the prime factorization to the power of 2?
Saunders: Yes!
Carl: Is the number 476? *(Carl tried* $2^2 \cdot 7 \cdot 17$*)*
Saunders: Yes!

[1] The authors chose these names with great care as an homage to their favourite mathematicians: Saunders MacLane and Carl Friedrich Gauss.

In this game, Carl would have lost 5 lives and the hangman drawing would look as follows:

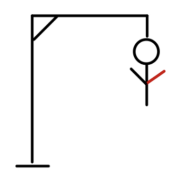


Figure 2: Final hangman drawing for Saunders' and Carl's game.

4 To ask or what not to ask: that is the question?

After this very exciting example, let's take a look at what the second player can and cannot ask.

4.1 Questions that are allowed

In short, any question with mathematical interest can be asked. In what follows, let $n = 490$ be the number to be guessed. Here are some examples.

- *Is the number a prime number?*
  Since 490 is not a prime number, the answer would be no.
- *How many (distinct) prime factors does the number have?*
  Since $490 = 2 \cdot 5 \cdot 7^2$, the answer would be 3.
- *What is the greatest natural number below the square root of the number?*
  Since $\sqrt{490} = 22.14$, the answer would be 22.
- *What is the lowest natural number above the square root of the number?*
  Since $\sqrt{490} = 22.14$, the answer would be 23.
- *What is the greatest common divisor between your number and the number* 2*?*
  Since $\gcd(490, 2) = 2$, the answer would be 2. This allows the player to know if the number is even or odd.
- *Is the number divisible by* 9*?*
  Since $\frac{490}{9} = 54.44$, the answer would be no.
- *What's the remainder of your number and* 11*?*
  Since $490 = 44 \cdot 11 + 6$, this means $490 \equiv 6$ $(mod\ 11)$, the answer would be 6.

Important rule: Questions about the greatest common divisor must follow a limitation in order for the game to not end after a small number of rounds. If the secret number has k digits, then we must use a number in the greatest common divisor that has at most $k+1$ distinct primes! For example, if we want to guess the number $n = 490$ (three digits), we could ask for $\gcd(n, 210)$ since $210 = 2 \cdot 3 \cdot 5 \cdot 7$, but not for $\gcd(n, 2310)$ since $210 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$.

4.2 Questions that are not allowed

In short, any question related to the digits of the number is not allowed. Moreover, any question that makes it too easy to guess the number is also not allowed. For example:

- *Are all the digits the same?*
- *Are all digits distinct?*
- *Does your number end with a* 0*?*

Questions like these three are discouraged. However, if it they can be formulated in terms of more interesting mathematical concepts, such as congruences, then it may be asked. For example, the third question *'Does your number end with a* 0*?'* could be reframed as *'Is the remainder of your number modulo* 10 *equal to* 0*?'*.

Although this basic set of rules has shown to be most productive in order to obtain an interesting game of mathematical hangman, this set is not closed. It is possible to adapt the rules based on the mathematical knowledge of each player.

5 What are the most interesting questions to ask?

As could be seen from the above, some questions are more useful than others. Some even allow to obtain multiple clues at once. Let's discuss this.

5.1 Strategy 1: Greatest common divisor

Thanks to the Fundamental Theorem of Arithmetic, we know that every natural number can be written as a product of prime numbers, and that this product is unique, up to the order of the factors. So, if one knows the right prime factors, one can guess the secret number easily.

Bearing this in mind, naively, we could focus on questions like:

- *"Is the number divisible by* 2*?"*
- *"Is the number divisible by* 11*?"*

This type of question allows us to check if a given prime (such as 2 or 11) belongs to the prime factorization of the number to be guessed. However, since player B only has 7 lives, this might not be optimal.

A powerful tool to help us in this task is the greatest common divisor between the number to be guessed and a number player B carefully chooses. For example, let $n = 490$ be the number to be guessed. One could ask the $\gcd(n, 2)$ which is 2. Although we now know the secret number is even, it is possible to formulate a better question. If we were to ask the $\gcd(n, 6)$, we would now know that 490 is divisible by 2 but not by 3, since $\gcd(n, 6) = 2$.

From this remark, we can deduce one of the most useful questions:

"What is the greatest common divisor between n and M?",
where M is a carefully selected number.

If player B chooses $M = 210$, since $210 = 2 \cdot 3 \cdot 5 \cdot 7$, this question will tell the player whether or not the secret number is divisible by 2, 3, 5 or 7. This works very well, because we not only sieve out primes from the prime factorization, but we can also get confirmation that certain primes belong to the prime factorization.

After some calculations, player A can tell player B that the answer is 70. Now, player B knows that 3 is not a divisor of the number to be guessed but 2, 5 and 7 are (since $70 = 2 \cdot 5 \cdot 7$).

When carefully choosing the number M, player B should choose M with a useful prime factorization. If one picks random numbers, we might exclude a prime factor twice, which is penalized by the game's bonus rule. For instance, in the previous question, we had already sieved out 3 as a prime factor. In the next question, if player B asks "What's the greatest common divisor between your number and 33?", we are not using the previous information that 3 is not a factor. So, our guesses will be inconsistent and we might lose two instead of one life.

5.2 Strategy 2: Root estimate

Another good strategy is to limit our search for numbers in terms of upper and lower bounds. A well-known result can be paraphrased as:

For every natural number m, there is always a prime factor that's less than or equal to $\sqrt{m}$. Any prime factor greater than $\sqrt{m}$ is unique.

Asking for an estimate of the square root of the secret number n allows us to:

- Predict which primes will be good candidates to be divisors of n. For instance, for $n = 490$, we saw $\sqrt{490} \simeq 22.14$. This means that the *prime suspects* would be $2, 3, 5, 7, 11, 13, 17, 19$.
- Get an estimate for the number n. For instance, if we know that $\sqrt{490} > 22$, this means that $490 > 22^2 = 484$. So our guesses must only be numbers greater than 484.

Asking for an estimate of the square root, also allows us to get both a lower bound and an upper bound:

For every natural number m, if we know the greatest natural number less than $\sqrt{m}$, we can add one to that estimate and we get the lowest natural number greater than $\sqrt{m}$.

In our example, since we know $\sqrt{490} \simeq 22.14 > 22$, we can also deduce $23 > \sqrt{490} > 22$. Which ultimately leads to $529 > 490 > 484$. So our guesses need to be only numbers greater than 484 but less than 529. With one question, we get two bounds.

This could be generalized to any root (cube root, fourth root...) but since the powers grow rapidly, this tends to be not very helpful sometimes.

An untested scenario worth checking would be if rational indices would be even more helpful. For instance, asking for an estimate of $n^{2/3}$.

6 What are the less interesting questions to ask?

In this section, we look at questions which do not really help us in our quest to find the secret number. These are questions that are allowed and interesting from a mathematical standpoint, but which are rarely useful. Here we follow the motto:

"Just because you can, does not mean you should."

For example, to ask whether or not a number is prime is not an ideal question. By the Prime Number Theorem, we know that primes get rarer and rarer as we go along the number line. Let $x \mapsto \pi(x)$ be the well-documented function which counts the number of primes less than x. Let's calculate the probability of randomly selecting a prime number:

- $\pi(10) = 4$ tells us that in the first ten numbers, there are four primes $(2, 3, 5$ and $7)$. Hence, we have a probability of $4/10 = 40\%$ of selecting a prime number less than 10.
- Since $\pi(100) = 25$, we have a probability of $25/100 = 25\%$ of selecting a prime number less than 100.
- Since $\pi(1000) = 168$, we have a probability of $168/1000 = 16.8\%$ of selecting a prime number less than 1000.
- Since $\pi(10000) = 1229$, we have a probability of $1229/10000 = 12.29\%$ of selecting a prime number less than 10000.

When playing the game with secret numbers with 3 or 4 digits, these odds are not enticing enough, and they only get worse as the number of digits grows. Also, since the number is chosen by the other player, this player could precisely avoid prime numbers altogether.

We could generalize this to other special number sets. One could ask questions like:

- *"Is it a triangular number?"*
- *"Is it a Catalan number?"*
- *"Is it a Fibonacci number?"*

All of these questions have a low chance of actually working, due to a similar phenomenon: These special numbers only get rarer and rarer as the number of digits of our secret number grow. So, one should stick to more "down-to-earth" properties of numbers (e.g. prime factorization).

7 Some considerations

The game's difficulty increases with the number of digits the player needs to guess. For this reason, we suggest to allow certain questions to be added to or removed from the base rules when guessing numbers with more or less digits. For example, when guessing a number with two digits, it should neither be allowed to use the greatest common divisor nor to ask for a lower or upper bound of the square root of the secret number. For example, if the number to guess is 72, then asking these three questions almost always allows the player to find the correct number too quickly.

Carl: What is the greatest natural number below the square root of your number?
Saunders: 8 *(since* $\sqrt{72} \simeq 8.49$*)*.

From this question, Carl already knows that the number lies between 64 and 81 since Carl deduces $9 > \sqrt{n} > 8$ (refer to section 5.2).

Carl: Is your number 68?
Saunders: No.

If Carl now asks a well-formulated question related to the greatest common divisor, the game is nearly over.

Carl: What is the greatest common divisor between your number and 210?
Saunders: 6.

Now Carl knows that 2 and 3 must be in the prime factorization of n. Knowing that only three numbers between 64 and 81 are divisible by 6 (that is, $66, 72$ and 78) which makes it now easy to guess that the number is 72.

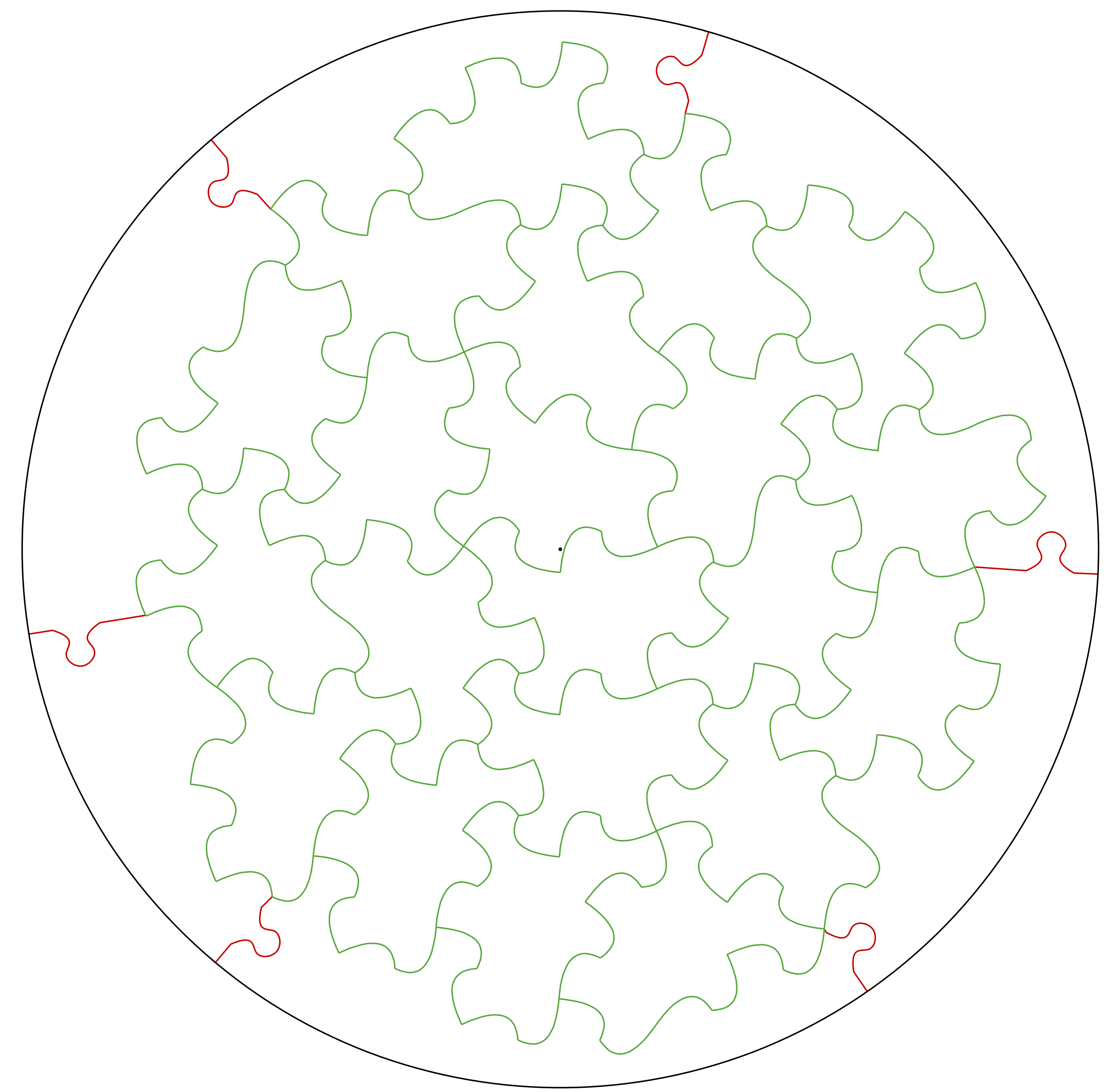
8 What's next?

The mathematical hangman game can be used as a tool to bring number theory closer to students that are dealing with structural properties of numbers for the first time. It can also be used as a tool to consolidate students understanding of basic number theory while doing so in a fun and competitive way and can be adapted to all ages.

This article outlines the first version of the game. However, both authors agree that there is much more to explore. If this game interests you and you would like to collaborate with its development or simply share some ideas, feel free to contact the authors!

Spectre Circle

Marti Reis



Tacorama

A game of recipe making and matching

Overview

Tacorama is a card game for one or more players to find matching families of recipes at a Tex-Mex restaurant. Players are employees angling to impress their boss and get a promotion to night manager. There are several variants of game play, but they all have the basic Tacorama matching rule in common: three recipes go together if the ingredients for two of them can combine to make the third. As these are recipes, it is the ratios of ingredients that matters, not the absolute amounts.

Story

In Tacorama, you and your fellow players are employees of the local *Tacorama* restaurant, a budget-conscious choice for Tex-Mex food of questionable authenticity and quality.
Headquarters has shipped your location a brand new *Taco-Maker 5000*, an automated taco maker that threatens to put you out of a job! However, your boss has made you and your coworkers an offer: whoever figures out how to prep the most recipes with the Taco-Maker 5000 will receive a promotion to night manager, while everyone else will have to look elsewhere for employment.
The Taco-Maker 5000 is a sophisticated machine that takes between one and four prepackaged kits of two different recipes and prepares a variety of tasty options with different combinations of ingredients. Your task is to take two such recipes and figure out all of the other meals it can prepare. Like all Tacorama meals, Taco-Maker 5000 recipes are made from various ratios of just three ingredients: tortillas, cheese, and ground beef. Each recipe may contain 0 or 1 tortilla, and 0 to 4 helpings of cheese and beef. Recipes must match exactly, without excess ingredients; however, if any ingredient has in excess of five units, the extra ingredients are packaged up in batches of five and output for shipment back to HQ (thus leaving the remainder when divided by five).

Matching Rule

Each Tacorama card lists a meal that can be made from a certain ratio of ingredients, shown in the top left corner of each. The ingredients are tortilla, cheese, and beef, going from top to bottom. On each card, the first ***present*** ingredient is the base and must be singular, and other ingredients are expressed in whole number increments relative to the first. Every card is unique and there are a total of 31 cards.
Every pair of two cards defines a recipe family of six cards. All of the recipes in a family can be made by different combinations of any two recipes from that family.
The Taco-Maker 5000 is limited to four helpings of each ingredient in a recipe; any combination of recipes that contains five or more helpings of any single ingredient will wrap back around to zero for that ingredient and then continue counting up. Note however that the ratios of ingredients must be simplified first, before determining any wrap-around to zero. Also note that the first ingredient present (in the order of tortilla, cheese, beef) must be expressed as one unit, with subsequent ingredients in whole-number increments thereof. If there are multiple of the base ingredient, you must scale the recipe until the base ingredient wraps back around to one.

Ways to Play

Solitaire

Waste Not, Want Not

Shuffle all 31 cards. There are three areas: the remaining deck, the cards-in-play, and the discard pile. Nine cards are dealt from the remaining deck, face up, in the cards-in-play area. The player attempts to find full or partial families (3 to 6 cards), moving them to the discard pile and filling in the open spaces from the unused deck, until all of the cards are used. Once a partial family is moved to the discard and new cards are dealt, it cannot be extended. The goal is to use all of the cards. Reduce the number of cards in play to increase the difficulty.

Key Ingredient

Shuffle all 31 cards. There are three areas: the *key ingredient*, the stacks, and the families. Deal a single card face up as the key ingredient. Deal the remaining cards face down into 6 stacks of 5, then turn the top card of each stack face up. The families will initially be empty, but will eventually take all 30 cards from the stacks in groups of 5. A face up card may be moved onto another if and only if the two form a family with the key ingredient card. Once all five non-key-ingredient cards in a family are together, they can be moved off to the family area. Face down cards can be flipped when open.

Simultaneous Common Play with Any Number of Players

Night Manager

Shuffle all 31 cards. Deal 7 cards face up in the center play area so that all players can see. Players look for three or more cards belonging to the same family according to the matching rule, yelling “Tacorama” when finding a family. The player points out the cards and gets confirmation that it is indeed a family from the other players, then collects the cards and moves them to his own recipe collection. New cards are dealt face up until 7 cards are again in play. Play continues until all cards are used up, or players agree that the remaining cards (less than 7) cannot form a family. The player with the most recipes wins.

Day Manager

Similar to Night Manager except that players may extend existing families of their own, or form families from a mix of cards in play and those in other players’ recipe collection.

Hand-based Play with 2 - 5 Players

Cero

The goal of *Cero* is to discard your entire hand. Players are dealt an equal number of cards, with any extras as a draw pile (there will always be at least one card starting in the draw pile). Flip the top draw card over to serve as the discard pile. Starting with the first player, then going clockwise, players take turns discarding partial families that match the top one or more cards on the discard pile. A player can play a single card if that card makes a family with the top two cards on the discard pile; a player can play two or more cards if those cards form part of a family with the top one card on the discard pile. Play continues around until a player runs out of cards, at which point he yells “Cero!”. If a player is unable to play, he passes to the next player in sequence. If play advances to a player who passed his last turn without any play taking place (all players passed their turns), the discard pile is shuffled and all players are dealt an equal number of additional cards, leaving at least one card over to restart the discard pile.

Taco Poker

A variety of poker-like games are possible. Players are dealt individual hands of cards, up to 6, and look for recipe families among their hand (and possibly any cards in common). The player with the highest ranking hand wins the round.
Named hands are as follows, see below for rankings:
2-cap: 6 cards with no families possible
Ring: 6 cards forming 3 overlapping families
Double Ring: 6 cards forming 4 overlapping families
Family of 6: 6 cards forming a single family
Neighbors: two sets of 3 cards forming separate families
Cross of 6: a set of 3 and 2 cards that each form a family with the 6th card
Family of 5: 5 cards forming a single family
Cross of 5: 2 pairs of cards that make different families with a fifth card
Family of 4: 4 cards forming a single family
Family of 3: 3 cards forming a single family

Mathematics Behind the Game

Underlying Geometry

Tacorama is based on the unique order 5 finite projective plane, referred to as PG(2, 5). The cards consist of points in PG(2, 5), while groups of 6 matching cards correspond to the lines. The proportions of ingredients on each card correspond to the homogeneous coordinates of the point (reversed from convention). There are 1 + 5 + 25 = 31 cards in total. Each pair of cards belongs to a set of 6 cards, and there are 31 such groups.

Lines in PG(2, 5) can be expressed as linear combinations of any two of their points, and this serves as the matching rule for Tacorama. As is typical with homogeneous coordinates, a special dimension (tortillas in this case) must be either 0 or 1, and furthermore the coordinates must be taken modulo 5.

The points can be thought of as as belonging to three disjoint groups. First, twenty-five normal points have a single tortilla, with each possible combination of zero through four for cheese and beef. Next, five points with no tortilla, a single helping of cheese, and zero through four beef are considered the line at infinity. Finally, the single beef point corresponds to the double-infinity point and is the sixth member of the line at infinity. Besides the one line at infinity with no tortillas, every other line will contain 5 normal points with tortillas and one infinity point without a tortilla. When placed in a grid of 5 by 5, each line goes through points with a certain ratio of ingredients, wrapping around at the edges until 5 points are crossed through. The slope of the line determines which infinity point it is paired with. When the beef does not change, the line is paired with the cheese only point. When the cheese does not change, it is paired with the beef only point.

Extension

An extension to Tacorama, called Tacorama Picante, can be constructed through the addition of another dimension, called "hot sauce". This corresponds to PG(3, 5) and has 1 + 5 + 25 + 125 = 156 points. Where in normal Tacorama there is correspondence between points and lines, Tacorama Picante has a correspondence between points and planes, where planes are formed by any three non-colinear points. There are $_{156}C_2$ / $_6C_2$ = 156 * 155 / (6 * 5) = 806 distinct lines in Tacorama Picante.

Unlike Tacorama, it is possible to partition Tacorama Picante into a "spread" of 26 lines, in which all of the lines are completely disjoint.

To make a copy of Tacorama Picante, simply print out five copies of the included cards, marking the addition of hot sauces zero through four packets of hot sauce, plus one "hot sauce" card all by itself.

Cap Sets

An interesting question is how many Tacorama cards one can have without having a set of three or more, known as the maximal cap set. In an order n finite projective plane, PG(2, n), an **oval** consists of n+1 points, no three colinear. For odd n, this is the maximal cap set; for even n, there is a **hyperoval** with n+2 points, no three colinear. Thus, for Tacorama the maximum number of cards with no three belonging to a common recipe family is 6. An example of such a set can be found with the homogeneous equation for the conic $X^2 = YZ$, where X is tortillas, Y is cheese, and Z is beef. Plugging in 0 for X, we can see that either Y or Z must be 0, giving the two points (0, 0, 1) and (0, 1, 0). Plugging in 1 for X, we find the four points (1, 1, 1), (1, 4, 4), (1, 2, 3), and (1, 3, 2).

Stepping up to Tacorama Picante, the maximal cap set is considerably larger. For an order n finite projective space, PG(3, n), an **ovoid** consists of n^2+1 points with no three colinear. As before, it can be expressed as a conic equation $Y^2 + Z^2 + W^2 = 3X^2$, where W is hot sauce. With X = 0, there are 2 points (**0**, **0**, **1**, {2, 3}), 2 points (**0**, **1**, **0**, {2, 3}), and 2 points (**0**, **1**, {2, 3}, **0**). With X = 1, there are the 8 points (**1**, {1, 4}, {1, 4}, {1, 4}), 3 permutations of YZW for the point of (**1**, 0, 2, 2}, the 3 permutations of YZW for the point (**1**, 0, 3, 3), and the 6 permutations of YZW for the point (**1**, 0, 2, 3).

Hand Outcome Probabilities

Monte Carlo simulation was used to estimate the probability of various outcomes from hands of 6 cards and 5 cards, using 10,000 samples.

Outcome *: exact value	6 Card Hands		5 Card Hands	
	Tacorama	Tacorama Picante	Tacorama	Tacorama Picante
Cap Set	0.42%	56.14%	13.13%* (80 of 609)	76.01%
Family of 3	8.32%	37.05%	54.23%	22.59%
Family of 4	6.37%	0.81%	7.33%	0.23%
Family of 5	0.59%	0.01%	0.11%	0.01%
Family of 6	0.0421%* (1 of 23,751)	0.00000444%* (1 of 22,533,126)	n/a	n/a
Cross of 5	37.27%	5.16%	27.12%	1.07%
Cross of 6	12.95%	0.11%	n/a	n/a
Neighbors	6.26%	0.58%	n/a	n/a
Ring	25.57%	0.13%	n/a	n/a
Double Ring	2.24%	0.01%	n/a	n/a

Authors

Tacorama was designed by Slowpoke Games, consisting of the father and son pair of Tim and Christian Snyder. It is their fourth card game designed together. Two of their earlier games, *Everybuddy* and *ZONI*, are based on the order 2 and order 3 projective geometries of 5 and 3 dimensions, respectively.

Tim Snyder is a professional software engineer and math hobbyist. He has been exploring recreational mathematics since elementary school. Mathematics first led Tim to programming, and is often part of his home programming projects today.

Christian Snyder is currently a STEM Academy high school student with lifelong exposure to and interest in recreational mathematics and a nack for design.

Tim and Christian formed the Slowpoke Games brand to tie their game and puzzle design efforts together.

Slowpoke Games can be found online at http://www.slowpoke.games.

Professionally printed cards can be purchased at:

- ***Tacorama*** - https://www.thegamecrafter.com/games/tacorama
- ***ZONI*** - https://www.thegamecrafter.com/games/zoni
- ***Everybuddy*** - https://www.thegamecrafter.com/games/everybuddy

MAGIC

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Fried and Seek

Matt Baker

Effect and Presentation: On the table is a deck of cards which David is invited to shuffle.

"David, in a moment I'm going to have you make a series of decisions. We're going to eliminate cards until we're down to just one. I'll then try to divine which card you're left with."

*"Now, despite what some people think, I'm not an **actual** wizard: behind everything I do there's a method. In this case, my prediction will be based on observing how you make decisions. So, David, I'm going to take out a few cards and then ask you to make some choices."*

The magician spreads the shuffled deck face-up and removes three cards, which he places in front of David. *"We have two red cards, both low values, and a black face card. On the count of three, you're going to place your hands down on **two** of these three cards. One… two… three."* David places his outstretched hands on the two red cards.

"Interesting… I'm not sure if you chose those because they're both red or because they're both number cards. Let's try again… this time all three cards will be red." The magicians removes three more cards. *"Again, place your hands down on any **two** of the cards, on the count of three. One… two… three."* David places his outstretched hands on two of the cards.

"I wonder, did you go for those cards because they were the closest to you, or because they're both diamonds? Let's try one more test to see." David and the magician repeat this 'observation test' one last time. *"Exactly what I expected you to do - I'm getting a good read on your decision-making process now."* The magician returns all the cards which have been removed to the deck.

"I'm also going to have you do some cutting, because I haven't observed you cutting the cards yet. Let's cut the deck into three piles: here's a small pile, a large pile, and a medium one." The magician cuts the deck into three different-sized piles.

"We'll start with the small pile." The magician pushes the small packet toward David and instructs him to give it a cut and complete, which David does. The magician watches the cutting process intently. *"OK, a little more than half… that's useful for me to know. And now let's do the large one. Once again, please cut and complete."* David does this. *"I thought you were going to cut about half again, but you cut deep that time. Alright, let's do the medium pile."*

David cuts the final packet. *"Just as I thought, you cut off only a few cards. I think I've got you figured out now! That means we're ready for the marquee event."*

The magician pushes all three piles toward David. *"We're going to eliminate two of these piles, because we only need one of them. So, with your left hand, pick up one of the piles."* David does this. *"And with your right hand, pick up a second pile."* David again does as he's been instructed. *"As I said, we're going to **eliminate** those piles, so please place them over here on the card box. That will be the discard pile from now on."* Two of the packets are placed onto the card box.

"Now, to make things even more fair, I'll turn my head." The magician turns his head away. *"David, if you would, please cut and complete the remaining packet. Have you done that? Good. And now deal into three piles, like you're playing a game of cards."* The magician, with his head still turned, pantomimes dealing back and forth as a dealer would do in a three-handed poker game. *"If you can't make three even piles and there are cards left over, just discard those. Have you done that?"* David says yes.

"Now put your hands out, just like we did before, and on the count of three you'll place your hands on two of the piles. One… two… three. And discard those two piles." David does this.

"Please cut and complete the packet remaining on the table." David says OK. *"And again, deal into three piles. As before, if there are any extra cards left just discard them."* When David is finished, the magician continues, *"Once again, place two hands out, and you know what to do: one… two… three. And discard."*

"Alright, you've got a pretty small packet left, probably just two or three cards. I'm guessing two?" David confirms. *"Place those two cards next to each other, and this time just put one hand out. On the count of three, place your hand on **one** of those two cards. One… two… three. And discard."*

*"We're finally down to just one card, as promised. Without even looking at the face, I'd like you to just **sit** on that card, so there's no way I or anyone else could see it. When you've done that, I'll turn around."* David says OK, and the magician turns his head back toward David again.

*"David, you just made a whole series of choices while my back was turned. You cut the cards several times, you dealt them back and forth, and you chose which piles to eliminate. Now, before we began, I carefully observed how you make such decisions. Actually, the only thing I **didn't** observe is whether you deal cards from left to right or right to left. Don't tell me what you actually did, but – before I commit to a prediction – please deal just a few cards into my hand."* The magician briefly pantomimes what David is to do. David picks up the discard pile and deals a few cards into the magician's outstretched hand.

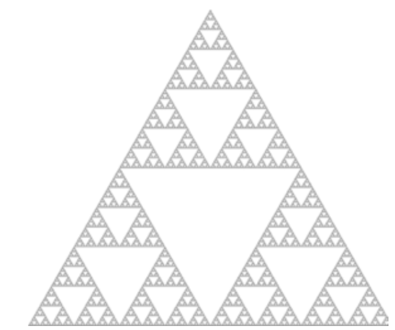
"OK, you can stop. It's all clear to me now." The magician hands the cards back to David. *"So, you cut the cards **this** way, dealt them **this** way, and put your hands down like **this**…"* The magician pantomimes various actions as he says all of this. *"…which means that you **must** have ended up with the Jack of Clubs! Go ahead, take a look."*

David takes out the card he's been sitting on and turns it over. It's the Jack of Clubs.

Preparation: None. A regular shuffled deck is used.

Overview of the Method: The initial phases, where the magician is apparently observing how the spectator makes certain decisions, are surreptitiously used to prepare an 18-card stack on top of the deck. Only nine cards have to be arranged, however. (This will make sense in a moment).

Equivoque is then used to force one of the three piles (the 18-card stack). Thanks to what I call the "Sierpiński principle", all of the cutting, dealing, and discarding that the spectator does with the magician's head turned ends with a selection that will always be the **mate** of the top card on the discard pile. It remains to glimpse the latter card; this is done in the process of 'observing how the spectator deals cards.'



Let's break all of this down a bit further now.

Setting the secret stack

Spread the shuffled deck face-up from left to right in such a way that you're able to see the indices of all the cards. The first three cards you pull out of the spread will be the **mates** of the top three cards in the deck (i.e., the leftmost cards in the spread). The next three cards you pull out are the mates of the next three cards, and similarly for the final three.

When you pull the cards out of the spread, place them in order from left to right (the same order in which their mates appear in the leftmost portion of the spread). Have the spectator place his hands on two of the cards and comment on his decisions, as in the sample script above. You should always be able to say **something** interesting about why you think he went for those specific cards, as opposed to a different pair. This part of the trick can drag if you let it, but it can also be quite engaging if you really play up the 'observation' thing.

After the spectator has made his decision, and you've commented wittily upon his choice, pick up the three cards from **right to left** with your right hand and place them face-up to the side. Do the same with each of the next two sets of three. Situation check: at the conclusion of the opening phase, you should have a stack of 9 face-up cards which are the **mates** of the top 9 cards of the deck, and in the same order. Close the spread, turn the deck face-down, and place the 9-card stack you just created face-down on top. As you do this, remember the **face card** of the 9-card stack – this will serve as a key card in just a moment.

You can false shuffle here if you wish, but I usually don't bother.

Situation check: the top 18 cards of the deck should now be in the pattern

A B C D E F G H I A B C D E F G H I

where identical letters correspond to mates; e.g., the top card is the mate of the 10th card, and so on. The leftmost 'I' card, which is the 9th card from the top, is your key card. Its mate is the bottom card of your 18-card stack.

Splitting the deck into piles

Spread through the deck face-up, and after about 10 cards, split the deck, turn these 10 or so cards over, and place them into a face-down pile on the table. This is your 'small' pile. Now spread until you reach the mate of your key card, and do the same thing, creating a 'big' pile of about 24 cards. The final 18 cards, which make up your secret stack, get placed down as the 'medium' pile.

Have the spectator cut and complete the small pile, then the large one, and finally the medium pile. Push them toward the spectator in such a way that the medium pile is to his left.

You now force the medium pile using a 'magician's choice', or equivoque, procedure.

Equivoque

Although most magicians think they understand equivoque, most magicians also do it rather poorly. This is because they either don't have a script or haven't rehearsed it enough. It's important to have a script! The following is based on the script which Eugene Burger used in this kind of situation.

"We're going to eliminate two of these piles, because we only need one of them. So, with your left hand, pick up one of the piles.

If the spectator picks up the 'medium' pile that you're trying to force, say, *"You went for the medium pile! I admire that 'Goldilocks' approach. OK, we'll discard the other two – place them on top of the card box."*

Otherwise, without missing a beat, say, *"And with your right hand pick up a second pile."*

Now, one of two things can happen.

1. If the spectator picks up another non-force pile, say, *"As I said, we're going to eliminate those piles, so place them over here on the card box."*

2. If the spectator picks up the medium pile, say, *"OK, you've narrowed it down to two piles. When I snap my fingers, hand one of them to me."*

Again, one of two things can happen.

2a. The spectator hands you the non-force pile. In this case, say, *"Alright, so you want to keep the medium one. Hold onto that pile, and we'll discard the other two."*

2b. The spectator hands you the medium (force) pile. You say, *"Well, I said I only needed one pile and you've given me the medium one. Please discard the other two."*

Regardless of which choices the spectator makes, you're now left with the medium pile.

Cutting, Dealing, and Discarding

This phase of the routine is self-working, so I don't think much else needs to be said about the handling. After you turn your head, just follow the script above. (There will never be extra cards left over in the dealing, by the way, since the spectator begins with exactly 18 cards.)

Once the spectator is down to a single card, turn your head back around.

Glimpsing the mate of the selection

As I've already mentioned, the spectator's selection will always be the **mate** of the top card on the discard pile. Why? Well, that's the beauty of the Sierpiński Principle. I won't spoil the fun: try it a couple of times with the cards face-up and you'll see exactly what's happening.

It remains to glimpse the top card of the discard pile. One way to do this would be to use marked cards. A sneakier way, which doesn't require a marked deck, is to have the spectator deal a few cards into your hand. You justify this by saying, *"Actually, the only thing I **didn't** observe is whether you deal cards from left to right or right to left. Don't tell me what you actually did, but – before I commit to a prediction – please deal just a few cards into my hand."* When he's dealt a few cards, one by one, onto your outstretched palm, tell him to stop. Hand the cards back to him, catching a glimpse of the bottom card in the process.

You now name the **mate** of the glimpsed card as your 'prediction'. Et voilà.

Final thoughts

(1) The first phase can be significantly sped up by beginning with the 18-card stack already on top of the deck and performing a false shuffle. The tradeoff is that the trick is then no longer 'from a shuffled deck in use'. Up to you to weigh the pros and cons, depending on your performing situation.
(2) If you are a memorized deck user, you can perform this routine with the top 18 cards of your memorized stack instead of with mates. The stack number of the selected card will differ by 9 from the stack number of the card you glimpse (and will belong to the top 18 cards in your stack).
(3) Instead of using mates, you can take advantage of "invisible deck pairing". (I assume most readers can figure out what I'm getting at here.) This has the advantage of camouflaging the setup a little better, and it also leaves no visible trace of a setup at the conclusion of the trick

(whereas in the standard handling there are some pairs of mates on top of the discard pile at the end).

Credits

I also teach this trick in my Vanishing Inc. Masterclass, under the name “Single-Fried”. The routine is a new presentation and handling for the trick “Triple-Fried”, from my book *The Buena Vista Shuffle Club.* That trick was originally inspired by Alex Elmsley’s “Animal, Vegetable, and Mineral”, published in *The Collected Works of Alex Elmsley, Volume 2.* Max Lukian provided the inspiration for switching from a 27-card stack to an 18-card stack, and the idea for setting up the stack on the fly comes from Juan Tamariz’s impromptu version of “Triple Coincidence”, explained in Disc One of his DVD set *Magic from my Heart.*

Math is Magic?

Martin Gardner liked to write about card tricks done with math ‘magic’. His first book “Mathematics, Magic and Mystery” was published by Dover in 1956. He also got the contract for his “Mathematical Games” column in Scientific American that year, and his first column was about flexagons. This greatly influenced me to experiment with twisted flexible circuits. Every time I wrote a letter to Gardner he would answer and often suggest a source for a question I asked him. I also wrote letters to Isaac Asimov and Buckminster Fuller and others and got answers. But Gardner’s work was what I liked most, and his influence was felt by an untold number of people, young and old, interested in anything logical, mathematical, magical, or scientific. One observation about what mathematicians do is to reveal how a magic math trick works by proving it with known math. Thus, revealing math tricks using other revealed tricks or logic. Not sleight of hand but exact reasoning by proof of a theorem with exact reasoning, using postulates and axioms and previous proofs. Many proofs require new supporting theorems that must be proven before the proof is complete. A proposition by Pierre de Fermat around 1637 is known as Fermat’s last theorem. It took over 300 years to prove and a very long proof by Andrew Wiles, involving math that did not exist in Fermat’s time. In a case like this and much new math the magic cannot be fully revealed but an astute author like Gardner can reveal in a simple way how the basic techniques worked without causing the reader angst.

Mirrors and Observers

Are mathematicians magicians? One answer is yes and one is partly, and one is sometimes, and one is no, and all four are sometimes, are the answers. Here is a riddle: What is not necessarily created by an artist, needs no power source, and has many different pictures and movies to show? Well it might be kind of an obscure riddle and maybe not well posed but the answer is a mirror. When you look in it you see a picture of yourself. Move and sway, blink a little and you are watching a video of yourself watching you. Whenever there is light the mirror is active, showing a picture. It could be a still life or any motion observed by the mirror. In the distant past some royal rulers would not allow lay people to own mirrors. Often lay people were very suspicious of mirrors believing that the mirror could steal away some of their soul when they were not looking in it so they kept it blindfolded when not in use. Even today some people avoid looking in a mirror except when necessary. The person in the mirror is the person outside the mirror, (but not quite!) and there might be a bit of suspicion that the mirror person is critical of the real person and vice versa. Who is really looking at who? Does it mean we are born with a quantum theory instinct about entanglement of soul and light? Louis Carrol used mirrors with his masterworks “Alice in Wonderland” and “Through the Looking Glass”. When I was in Africa, I started to take a picture of a lady making peanut butter. She immediately waved me away. My African friend said she may think the camera can steal a soul. In fact she probably just thought her appearance was not good that day.

Quantum Theory and Elementary Geometry

Mathematical magic long ago started with the need to find or stake out property lines and building perimeters and so forth. From these beginnings the Pythagorean triplet was born, and was later turned into the Pythagorean theorem by the Greeks. Thus the logical postulate of a point was born. Simple and utterly devoid of anything like magic. Yet modern Quantum theory says a point with no size cannot exist. It must at least have Planck size to exist. Also, it cannot be precisely located. In fact, it can and must be several places at once, existing as a standing wave or moving rapidly as a traveling wave, not a one-dimension mathematical line. Thus, points and lines have varying sizes, and probability attributes, and cannot be precisely located. Nature simply refuses to be roped down! But mathematicians love the lines and points. They do magical wonders in math. They cannot ever be tossed away in favor of natures system. Thus, there you have it, mathematical geometry was magical from its very beginnings. The same extends to infinitely thin planes, perfect polygons and polyhedrons, and all polymorphs. These things give us immense power because nature does not 'know' of them and instead tries in vain to find them with its main law that all must always move. That is not to say we cannot have Planck size moving points in math, but instead that perfect points are the anchors for Planck size things. How could we have a concept of size and location if we have no way to refer it to zero size and exact location? The same reasoning extends to Einstein's curved space. Without being able to refer it to flat space the curvature would not be so easily conceived. In fact physicists often use flat space to make curved space calculations simpler, similar to how you use a topo map to calculate how much dirt must be excavated to flatten a plot of land.

Euler and the Real World

Leonhard Euler was a bold math innovator and creator. He was not afraid to wrestle with and reveal new mathematics using untested techniques, often with no proof other than his method produced numbers that worked and were gems of mathematical substitution and manipulation. He wrote an important equation which was much later the starting inspiration for String theory. His polyhedral formula opened up, actually more correctly, created the field of topology. It is an amazingly simple formula relating the faces, edges and vertices of a polyhedron, E = F + V – 2. He proved this formula, and it is now recognized as a gem of mathematics because it created the science of topology. It is easy to count E, F and V on the 5 Platonic solids and see that this holds true. Is it true for all polyhedra? Yes, if they are convex, meaning that all dihedral edge angles are greater than 180 degrees, and all vertices point outward, and all faces are flat. For instance, a hexagon shaped antiprism could have the top and bottom 2 vertices pointing inward and touching each other making a single vertex. Thus, the formula is off by 1.

Euler took all this into account in proving his formula. In the real world you can have a box with one face missing to allow use as a container. The missing face can be called a null face, Nf. To make the formula work for Nf's we can include Nf = 1 in the formula as E = F + NF +V – 2. This works if two Nf share an edge by retaining the edge thus a skeletal polyhedron has zero F! Similarly, you could have a box with a lid and the formula still works if Nf is zero with the lid shut, and we make the lid a face when it is open. When the lid is shut Euler is polyhedral. When it is open, we have a lid face, Lf so we have 15E, 5F, 10V, 1Nf, 1Lf, and 15=5+10+1+1-2. It is not 'true Euler' but works anyway. Eulers proof took some effort because how do you know that some very odd polyhedron with a multitude of elements might not be Euler? Another question is, can two different polyhedrons have the same numbers of E, F, and V? That question is what creates an essential feature of topology. You can, indeed have two polyhedrons with the same E, F and V as a cube! This is not possible with a tetrahedron.

A magnetic and gravitic world

When I was about 6 or 7 years old my older brother showed me an old rusted V magnet he found in the yard. It was from the magneto of an old car. I was amazed, dumfounded how it could attract nails and bolts. From then on my interests were a battle between art and science. Gravity was also a battle for me. I thought it was on my side but found out otherwise after breaking my arm 4 times climbing trees. My dad got tired of the doctor bills. One time the doctor was drunk and my arm healed crooked. Doc

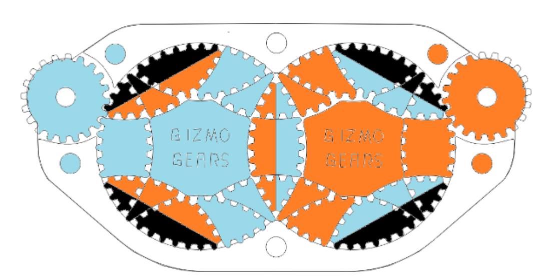
put it against a bar and snapped it, then reset it correctly. The pain from the snap was sharp but quickly subsided. I had dreams about gravity. Floating high in the sky, gravity would suddenly return, then I would fall but always woke just at contact, a real nightmare. It began to seem that gravity must be respected. Evil Knievel would agree.

Two disk twisty puzzles

After college Hofstadter's column in Scientific American was about Rubik's Cube. That was my inspiration to invent Engel's Enigma, a name invented by Alexander Keewatin Dewdney who described it in his Scientific American column 1986. Finally, with a patent it was produced by Go Images. Royalties were minimal. Rubik was, and still is the great rage. It is easy to see why. The cube is magic, combining both a hidden mechanism and the mathematics of group theory. You hold a 'crystal' with 27 'atoms' in your hand with . The 3x3x3 cube is still the best selling twisty puzzle and has inspired contests, yearly events, magician stunts, and solving records of a few seconds. It inspires a constant stream of new designs. It would be great if you could afford to purchase choice new twisties. The Twisty Puzzle forum has collectors' inventories, puzzles for sale, new introductions and much more. You can start your own discussion by getting an account. A self-published 'Circle Puzzler's Manual' was written. It surprised me that twisty enthusiasts liked it. Jaap Scherphuis put it on his web site and I put it on mine. Meetings would be held. I attended one in California headed by Bram Cohen and Oskar Van Deventer. Tom Rodgers was there. CPM was discussed, and the words bandaged, and unbandaged were discussed. Bram wanted to know if a puzzle can always be unbandaged meaning that if you find a position where a cut is required to continue movement you make the cut to remove the 'bandage'. If not does some or all the puzzle pieces continue to get cut until they 'turn to dust', I believe, in Bram's words.

That meeting was partly my inspiration to design Battle Gears and Gizmo Gears twisty puzzles. Also it was my intention to add an extra challenge. This was financially a bad idea. Jaap produced a solution for Battle Gears on his website but had a difficult time with it. Jaap did not like it, since no simple algorithm worked, like with a cube. BG required a long careful analysis for several different situations, not a simple symmetrical set of a few repeated moves. It looks symmetrical but it is not. Very few were sold. Gizmo Gears was much worse, less than 5 or ten sold. My conjecture presented here for the first time is that it cannot be solved without reversing many or all the moves made to mix it up. It has dead ends where you **must reverse** and take a different path to make new moves. Here solved implies that every piece is returned to its starting position and orientation. What is the simplest coloring required to force a full solution?

Fractal Unbandaging Gizmo Gears



Bob Hearn, a master coder got wind of the unbandaging problem. He attempted to unbandage Battle Gears and later, someone found a solution. Oskar Van Deventer alerted me and so I sent the Gizmo Gears puzzle layout to **Andreas Nortmann** of the Twisty Puzzle Forum to have the coders try Gizmo Gears. This started a 'Gizmo Gears' on the Twisty Puzzle Forum. A short while later Bob revealed Gizmo becomes fractal but just past the critical radius. It became the first twisty puzzle to reveal the crystallographic mix-master of 2 disk fractals. Bob now sells this beautiful artwork printed on T-shirts on his Etsy store. The 'Gizmo Gears' forum went on for several years and has hundreds of posts with cool art by Bob, Brandon Enright, Jason Smith, and several others. The coders put great effort rendering the fractals, some taking a week or more of PC time. I did not attempt to do my own coding. My Kindle book "Incendiary Circles" uses, with credits, some figures and partial posts from the Gizmo forum reviewing the incredible work of the collaborators. I see it as the radii of the two disks increasing, like two single crest expanding waves, and surrounding waves all around also expanding to create an infinity of pieces. The fractals on the two circles are what the coder's work displays. Amazingly, Bob found Penrose tiles with n=5 rendering. Gizmo Gears is 3*4=12, so not surprising it was fractal. What is cool is that GG used the same lozenge design as a puzzle patented in the late 1800's. I just divided the lozenges using two symmetries 4 and 3 thus 3x4 or 12 when you attempt to unbandage. Bob came up with the idea of compound groups to describe bandaged twisty puzzles. He wrote this up along with several co-authors and published it on ArXiv, a site for preprinting new papers that may not have yet been peer reviewed by a journal. I downloaded and printed it. It is very well written and clear, with some beautiful color fractal renders. Bob has done presentations about these fractals with video renderings, one was at a G4G I was unable to attend.

Gardener's pulls back the curtain

Martin Gardener had a very simple writing style making you feel like you had entered a hidden world of treasures. In that regard his math writing style revealed the magic illusions behind math magic. Instead of describing the fearsome mental strain required to produce a proof he peeled back the layers hiding the inner secrets of much mysterious math. Math can be a difficult subject but if you are willing to research and master the current knowledge of a problem then you can really begin to contribute to, and possibly find a proof, and if not a proof, possibly contribute to what is known about the problem. You too, must pull back the curtain. Gardener could also pose a riddle that was so simple yet cofounding that even his solution left many people confused. For instance, he asked "When you look in a mirror why is your left and right reversed but not your up and down?" It is a psychological as well as a math kind of question. Gardener explains that it is due to our bilateral symmetry. For some people that does not satisfy. My own idea is to lean over sideways. Now your left and right is up and down reversed. What if your up and down really were reversed. You would have a hard time combing your hair but could easily see where you need to clean your toenails. You could examine your feet and knees without lifting them down much. You bend your head down and see your self looking up at you near the bottom of the mirror. It is a good thing our top and bottom are **not reversed**.

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Jason Smith's site: https://www.deluxerender.com/2015/12/gizmo-fractals-in-2d/ (Renders 2 disk twisty puzzles, in the form of a captivating video, all fractal n's to about n=20.
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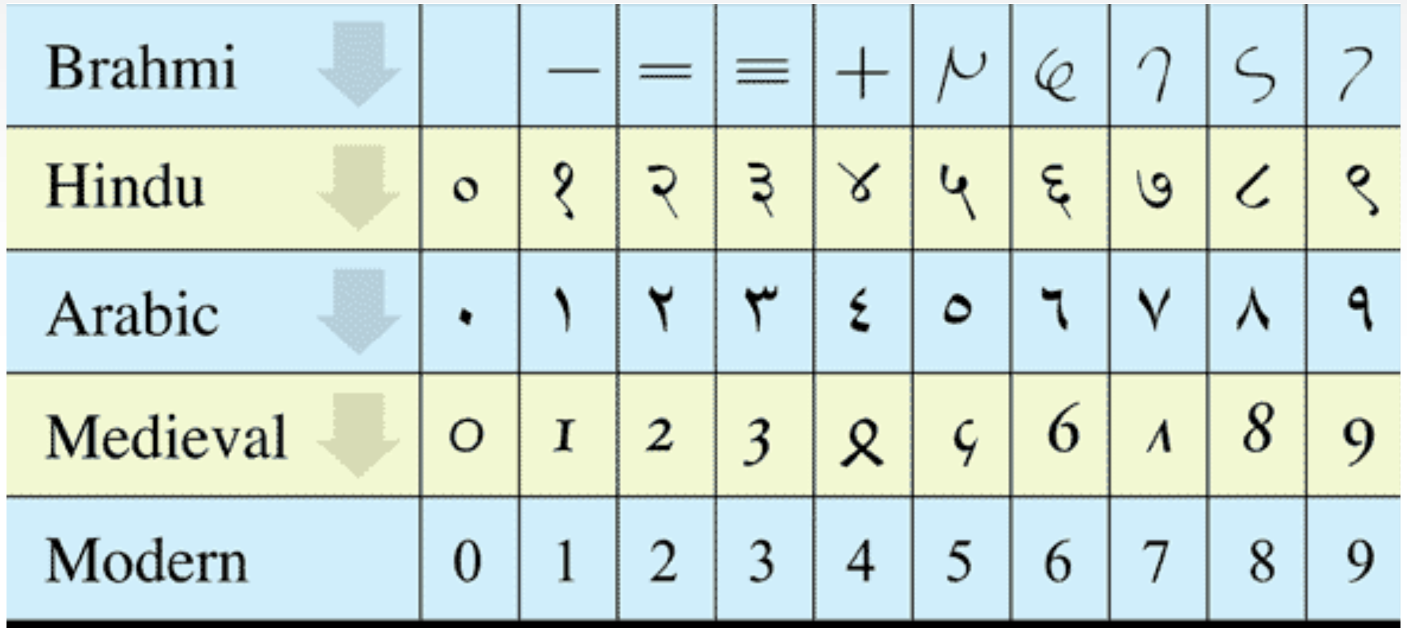
MATH



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Ten Great Revolutions in Mathematics

Suggestions for Engaging Math Students

David Albert

Introduction

How to engage middle-school and high-school students in mathematics has long been a topic of interest to math teachers. The traditional curriculum is often seen as dry, boring, and irrelevant by those students who have not already been exposed to mathematics recreationally and historically. An alternative approach may help to pique the interest of additional students. This paper suggests several topics in mathematical history, each of which can be the focus of many productive days of exploration at the middle-school and high-school level. For the sake of time I have had to make hard choices, so this paper focuses on a much smaller set of topics than I wish I could include: Counting; Place Value; the Pythagorean Theorem, Fermat's Last Theorem; the Goldbach Conjecture; Sizes of Infinity; Irrational Numbers; One-Way Cryptographic Functions, Pascal's Triangle; and Fractals.

Although my paper begins with some of the oldest topics in the history of mathematics, it does not proceed strictly chronologically. Many topics were developed over time, and I have chosen to lay out the topics in the order in which I might wish to teach them; for example, I follow the Pythagorean Theorem with a look at Fermat's Last Theorem because of the close relationship between the two.

Within each topic, I focus, after a history and explanation, on some of the methods one might use to bring the topic to life in a middle school (or high school) mathematics curriculum, and to interest and engage students in the 11-17 year old age range.

Sources

Sources used for a single topic will be referenced with footnotes within each topic article. One source, however, is used regularly throughout this paper and is not explicitly referenced separately each time general ideas are used from it. This is *The Britannica Guide to the History of Mathematics*, ed. By Erik Gregersen, 2011. Where a specific footnote is required, this will be referred to as [Britannica].

And finally, with apologies to Fermat: there are hundreds, if not thousands, of other important revolutions in mathematics whereof one could write. I have discovered truly remarkable facts about many of them, but this paper is too narrow to contain them all.

1. Counting

It is hard to imagine any more primitive mathematical operation than the simple act of counting. Research has shown that some non-human animals, including birds and monkeys, can count[1], from which one can assume that humans have been counting, in one form or another, since the dawn of humanity's existence.

Being able to record what you have counted, however, turns counting from what might be an instinctive behavior into a human one. The earliest-known artifact that displays a record of counted objects is the Ishango Bone, approximately 22,000-25,000 years old[2], a fossil with a series of marks that are clearly an attempt at recording numbers, even if their exact use is uncertain.

Figure 1 - Ishango Bones, ca. 20,000 BCE

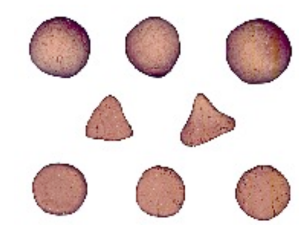


Figure 2 – Syrian Counting Stones, MS 5067, ca. 8000-3500 BCE

The recording of numbers in some form may have begun as the concept of one-to-one correspondence. It is theorized that a shepherd, keeping track of his sheep, might have used a "pouch full of stones. As the sheep left one-by-one in the morning, he could place one stone in the pouch for each sheep; in the evening, he could reverse the process"[3] and so determine if any sheep were missing (or, perhaps, if someone else's sheep had come to join his flock, though one wonders if he would have cared).

Many students are familiar with tally marks for counting beads or books, or may have seen them used (stereotypically, but accurately[4]) by prisoners in movies for counting days of incarceration.

Figure 3 - Graffito showing tally marks, Maltese prison

In the middle school classroom, activities centered around counting (an activity already hopefully ingrained in the vast

[1] "More Human Animals Seem to Have Some Ability to Count: Counting may be innate in many species", Michael Tennesen, Scientific American, Sept. 15, 2009. http://www.scientificamerican.com/article.cfm?id=how-animals-have-the-ability-to-count

[2] Various dates are proposed. See http://mathworld.wolfram.com/IshangoBone.html and http://humanorigins.si.edu/evidence/behavior/ishango-bone, for two references.

[3] Introduction to Abstract Mathematics, 2nd ed. 1990, John F. Lucas, Ardsley House, p. 119

[4] http://www.heritagemalta.org/sites/oldprison/oldprisoncoll.html – see "Graffito showing Tally Marks" towards the bottom of the page.

majority of the students at this age) would not of course involve teaching the students how to count. After looking together at the Ishango Bone, it could be fun to explore methods of providing a one-to-one correspondence and recording the counting.

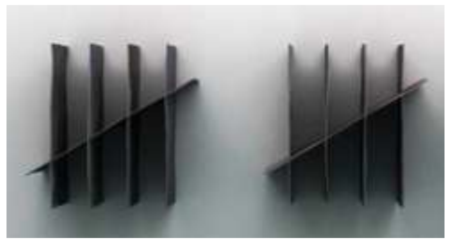


Figure 4 – Bookshelves shaped like tally marks

Students familiar with tally marks may know the familiar "counting by fives" system[5] in which a diagonal line is drawn as the fifth mark. Students may have other ideas for keeping track of numbers, and some of these ideas may lead into the study of place value (see next topic).

<u>Student Activities and Approaches:</u> The use of stones or other objects to keep track of one-to-one correspondence could be explored by using any small classroom object (pencils, paper clips, Unifix cubes, etc.) to keep track of students as they exit and enter the classroom. Discussions of whether these methods would require someone to have words for the numbers (they do not) and how such words for numbers might subsequently develop could also engage the more verbal (or argumentative) students. This discussion can also segue nicely into the topic of place value, and of the number zero.

Figure 5 - Unifix cubes

2. Place Value

Whereas counting itself is a prehistoric activity, and tracing its roots into antiquity is not fully possible, a lot more is known about the origins of number systems that involve place value. Without some system of place value, recording a very large number becomes essentially impossible -- recording the number 1,000,000 using tally marks or counting stones would require a very large bone, or an enormous pile of stones, not to mention an unusual amount of patience.

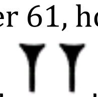
Perhaps the earliest known place value system is that of the Babylonians, who recorded numbers using a base-60 (sexagesimal) system[6], a number possibly chosen due to its divisibility by 2, 3, 4, 5, and 6. Using the Babylonian system, the symbols for 1 and for 10 were combined to form numbers up to 59. The number 61, however, would have been represented as a two-place number, , with the first symbol representing one 60, and

Figure 6 - The Babylonian number 150 (two in the 60s place, and 30 in the 1s place)

[5] (a bookshelf product inspired by prisoners' tally marks arranged in groups of five can be seen at http://www.trendhunter.com/trends/5-dias).
[6] [Britannica], pp. 24-25.

the second one 1, for a sum total of 61. It is not clear from early records exactly how the Babylonians could differentiate between the number 61 as represented above, and the number 2. Presumably, additional space would be left between the two symbols to make it clear that they represented different place value columns, but a degree of ambiguity may have remained as to whether that was the number 2 or 61.

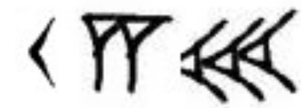
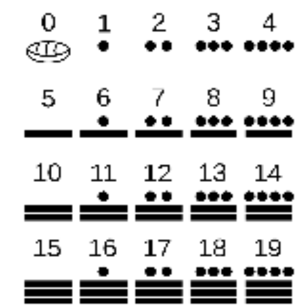


Figure 7 - The number 36,150

Higher places (3600, or 60 squared) could also be represented. The symbols in Figure 7 on the left could represent the number 36,150 (a 10 in the 3600's place, plus the same 150 shown in Figure 6). Even more ambiguity is introduced, as the Babylonians did not at first have a placeholder value for empty (zero) place values[7]. Thus, the number we called 150 in Figure 6 (2x60 + 30) might in fact have been intended as the number 7230 (2x3600 + 30), with an unwritten zero in the 60's place; [Britannica] asserts that "the context determined which power was intended" but it seems possible that the answer was not always clear.



numerals, including a placeholder 0 symbol.

The Mayans, too, had a place-value system[8] that operated on a base of 20. Their system did include a place-holder symbol to keep track of places with a valuation of zero. The symbols they used are shown at left.

Although the Ancient Egyptians are sometimes said to have used place value, their system of numeration actually involved different symbols for each power of 10. The order of the symbols was immaterial, as each symbol carried its power of 10 along with it [Britannica, p. 32]. Therefore, this is not a true place-value system, in which the exact position of the symbol within the number determines its value. The order of the symbols, though typically written from greatest to smallest values, is in fact immaterial in determining the total value of the number written.

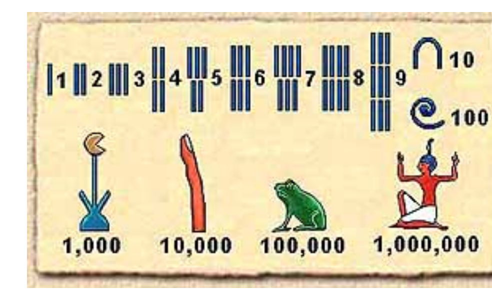


Figure 9 - Egyptian numerals (http://www.eyelid.co.uk/numbers.htm)

The same can be mostly be said of the system of Roman Numerals (M, D, C, L, V, I), although an interesting discussion could be had with students on this latter topic since there is a small amount of positional value determination (XI = 11, while IX = 9, for instance) and it could be worth bringing this issue up and comparing it to a true place-value system.

[7] [Britannica], p. 25: "The Babylonians appear to have developed a placeholder symbol that functioned as a zero by the 3rd century BCE".

[8] Knill, Oliver, *Worksheet to Arithmetic lesson*, retrieved 11/15/2012, http://math.harvard.edu/~knill/teaching/mathe320_2012/handouts/01-worksheet.pdf

In India, however, a decimal system of numeration, with place value, was developed at least as early as the second century CE [Britannica, p. 186]. The following chart shows some of the history of the evolution of numerals across the next thousand or so years:

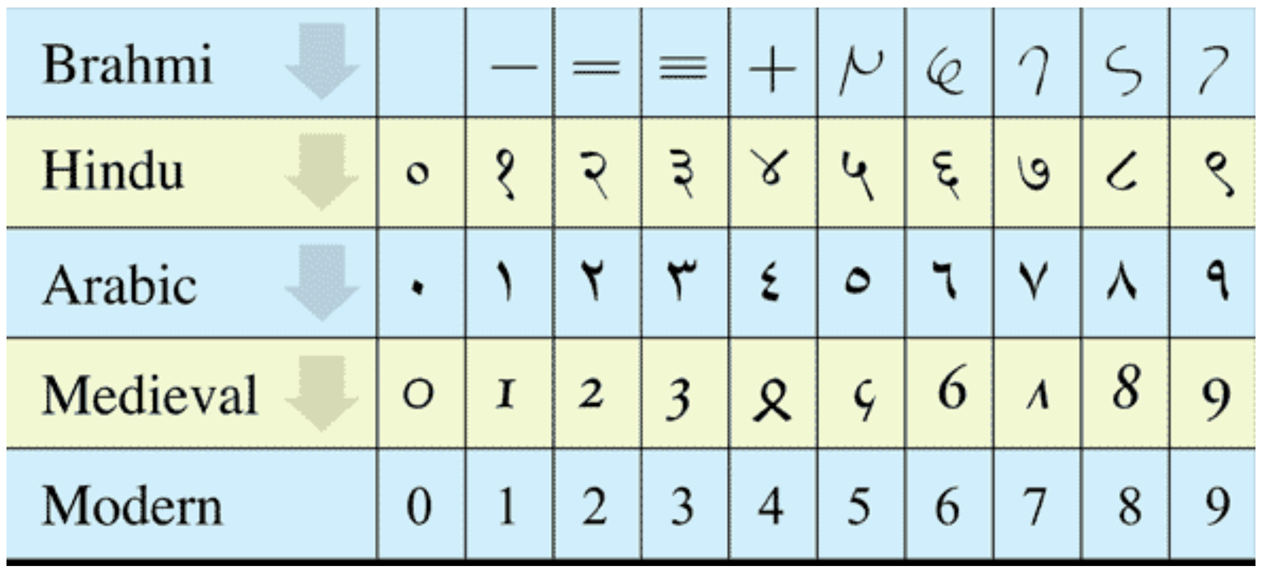


Figure 10 - Evolution of numerals from India to Europe (http://archimedes-lab.org/numeral.html)

Although the "zero" place-holder did not appear at first, this chart shows that it came into existence during the Hindu period, perhaps in the 9th century CE.[9]

Student Activities and Approaches: Once students understand that the Hindu/Arabic system of numeration we currently use is not fixed in stone, neither as to symbols used nor as to base, a fun activity to cement this understanding would be to have students each create their own system (with or without place value). Students can be encouraged to create their own symbols, using any base they choose. Some practice time using Babylonian, Egyptian, or Mayan numerals would also be useful.

Discussions of comparative merit of these systems can then ensue naturally out of the students' own systems. The teacher might present some very large numbers to see how each student's system can handle it. If students have created non-place-value systems, some of them may run across the problem that their system requires a very large number of symbols to represent a large number. Students who have created place-value systems rooted in an unfamiliar base may have difficulty determining the value of higher-valued columns. All of these issues can be discussed and illuminated.

An advanced activity could include decryption of each other's numeral systems. If the numbers have been written completely out of context, little to no decryption may be possible; for instance, when presented with the single fact that a farmer has cows, there is simply no way to determine what number these symbols are intended to represent.

[9] [Britannica], chart at top of page 186.

However, if you are given sufficient addition facts such as:

> Farmer A has cows. Farmer B has cows. Between them, they have cows altogether.

then it may be possible to begin decoding both the individual symbols, and the base of the place-value system. Certain ground rules (e.g. that the digits are written with the highest-order on the left, and the units place on the right) would be useful, as that is otherwise not an absolute requirement of mathematics. Students could create several such facts and then pass their papers to other students to decode. Although the above example is not sufficient in and of itself to decode an entire numeral system, there is already present in it some useful information. Notice that the symbol appears in the third (leftmost) column of the response. As the sum of two two-digit numbers, it is thus presumably numeral "1". From this we can also determine that the two symbols and , whatever their values, must sum to a number higher than or equal to the base of the system in question. The symbol in the rightmost column of the answer, as it is equal to the sum of two symbols, must equal 2, and so forth. The sum is thus the number 112, but in what base? Continued experimentation may allow students to arrive at the equation 21+31=112 in base 4.

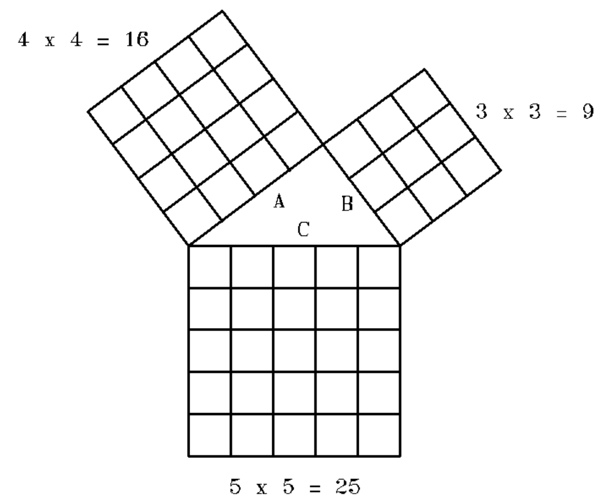
Cryptarithms can also be introduced. They may be somewhat easier (and for some students, more engaging) than the above activity. In a cryptarithm, typically grounded in base 10, each of the digits 0 through 9 is represented by a unique letter of the alphabet, and it is the task of the solver to determine which letter goes with which digit, when given an arithmetic problem. It is fun when these problems involve letters that make real words and when the problems make sense in English, such as:

```
 O N E
 O N E +
------
 T W O
```

in which one can begin by reasoning that "E" plus "E" must equal either "O" or ten plus "O", and thus "O" must be an even digit. Furthermore as "O" + "O" equals "T" and there is no fourth digit in the response, "O" must be the digit 2 or 4 (it cannot be 0 as we do not write leading zeros on numbers) and thus "E" must be 1, 2, 6, or 7 since those are the only numbers which, when doubled, end in a 2 or 4; and so forth. The teacher and students can invent cryptarithms, but as a starting point, good sources for them exist online[10].

[10] The website http://www.cryptarithms.com/ actually exists and is a good initial source, but a quick Google search will turn up thousands of additional sites and discussions. There are also online cryptarithm solvers, such as the one at http://bach.istc.kobe-u.ac.jp/llp/crypt.html, which will solve these puzzles using techniques from Artificial Intelligence.

3. The Pythagorean Theorem



The Pythagorean theorem states that the sides of a right triangle of length a, b, and hypotenuse c, hold the relationship $a^2 + b^2 = c^2$. This theorem is so famous that nearly every adult seems to have heard of it, and it has engendered hundreds upon hundreds of fundamentally different proofs. One book on the theorem[11] claims that "[b]y any measure, the Pythagorean theorem is the most famous statement in all of mathematics," and references an older book by Elisha Scott Loomis entitled "The Pythagorean Proposition" containing 371 separate proofs of the theorem, and a website called Cut The Knot[12] has collected 956 separate approaches to the proof with interactive illustrations.

Whether or not it is the *most* famous, it is certainly famous – nearly every adult appears to have heard of it, and even to understand the basic premise, whether they know how to prove it or not. Jokes and riddles have even been written about it.[13]

What appears to be uncertain is the degree to which Pythagoras should be credited with the discovery of the theorem. Certainly, the basic concept was well known long before his time; according to Cut The Knot, "the Theorem was discovered on a Babylonian tablet circa 1900-1600 B.C", and that "Pythagoras (c.560-c.480 B.C) or someone else from his School was the first to discover its proof can't be claimed with any degree of credibility."

Student Activities and Approaches: Students with prior exposure to algebra can handle nearly any of the proofs of the theorem. Such students can be shown them the Cut the Knot website referenced above, and each student in a class or seminar could be asked to choose a proof they enjoy and understand, and prepare the proof for demonstration to the class. An entire semester's mini-seminar could be profitably spent this way.

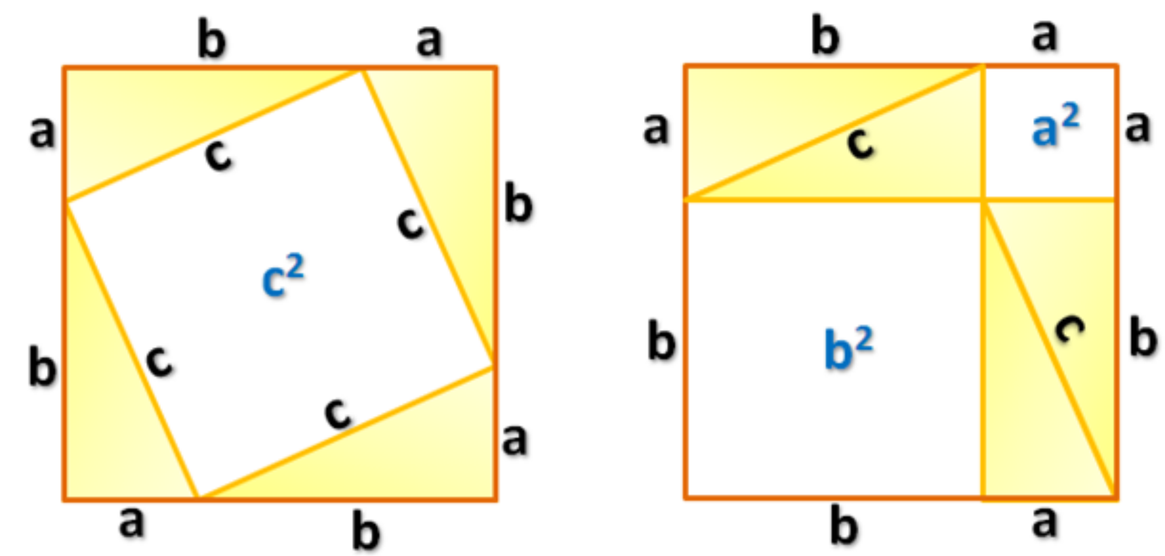
Students with less algebraic background (especially early in middle school) can still

[11] *The Pythagorean Theorem: A 400-Year History*, Eli Maor, Princeton University Press, 2007

[12] http://www.cut-the-knot.org/Pythagoras/index.shtml

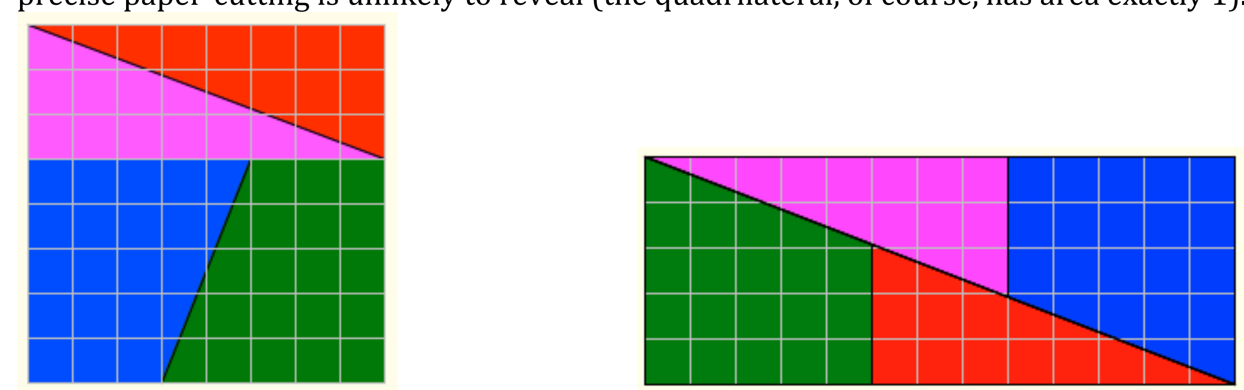
[13] For instance, the following riddle, written here in my own words but based on a joke I heard as a child: Three country gentlemen went for an outing in the country, bringing picnic blankets made of skins from animals they collected on an African safari. When they reached their destination, they stretched out on the grass, one upon the skin of an antelope, one on a lion skin, and one on a hippopotamus skin. If the first two gentlemen weighed 150 lbs. and 175 lbs. respectively, how much did the third gentleman weigh? Answer: 325 lbs., because the squire on the hippopotamus is the sum of the squires on the other two hides.

understand and enjoy many of the geometric proofs. For instance, the cut-and-rearrange proof such as the one illustrated here is relatively straightforward[14]:



In this proof, the large square is $(a+b)^2$, and the four triangles with bases a and b can be rearranged to leave either c^2 or a^2+b^2. One beauty of such a proof for use with children is that they can actually DO it – just give them scissors and card stock, and they can easily convince themselves that it works at least in the specific instance of the squares they happen to create.

One danger of such proofs is that very similar "proofs" can be given of clear fallacies. Actually cutting and pasting can lead to results like the famous 64=65 result, in which one can apparently cut up a square of size 8x8 and rearrange it into a rectangle of size 5x13. Cut the Knot has a good page on this fallacy[15] that students might enjoy examining. It is caused by a very thin quadrilateral along the main diagonal of the 5x13 rectangle, that even precise paper-cutting is unlikely to reveal (the quadrilateral, of course, has area exactly 1).



[14] http://mathandmultimedia.com/wp-content/uploads/2010/02/pythagrean41.png

[15] See http://www.cut-the-knot.org/Curriculum/Fallacies/FibonacciCheat.shtml for an interactive display of these figures.

4. Fermat's Last Theorem

If the Pythagorean Theorem is the most famous statement in all of mathematics, Fermat's Last Theorem, which on its surface looks closely related, must have been the most famous of the unsolved theorems for all the centuries between the time it was first proposed and the very recent time when it was eventually solved.

Figure 11 - Pierre de Fermat, 1601-1665

Fermat's conjecture, written by Fermat in 1637, "famously in the margin of a copy of *Arithmetica* where he claimed he had a proof that was too large to fit in the margin,"[16] states simply that there are no solutions in positive integers to the equation

$$a^n + b^n = c^n$$

when n is larger than 2. When n equals 2, of course we have the famous Pythagorean triples examined in the previous section of the paper, and it is not difficult, once you have found one solution (e.g. $3^2+4^2=5^2$), to show that there is an infinite number of such solutions. However, it is not at all obvious to the beginning mathematician that there should be no such solutions for n larger than 2.

According to [Britannica], searches for proofs for specific exponents, such as for cubes, predated Fermat by hundreds of years.[17] Euler is credited with the proof for n=3, and Fermat himself with the proof for n=4.[18] It is impossible to know for sure whether Fermat actually had a proof of the general case, and the use of the term "Theorem" rather than "Conjecture" for this equation, at least prior to its eventual proof, seems extremely generous. It was not until 1995 that the mathematician Andrew Wiles proved it for the general case,[19] finally laying to rest one of the most famous unsolved theorems of the 17th, 18th, 19th, and 20th centuries.

The theorem is so famous as a canonical "black problem,"[20] that it features in literature and in short stories not necessarily intended for a mathematical audience. In one such story, "The Devil and Simon Flagg" by Arthur Porges[21], the devil agrees to solve any solvable question within 24 hours. If the devil fails, then he must provide Simon Flagg with health, wealth, and happiness for the rest of his life. If he succeeds, then as per the usual arrangement, Flagg's soul is forfeit. Flagg's question is, "Is Fermat's Last Theorem correct?".

[16] Wikipedia, "Fermat's Last Theorem", http://en.wikipedia.org/wiki/Fermat's_Last_Theorem, 3/29/2012.

[17] [Britannica] p. 78: "Greeks and Hindus had studied indeterminate equations, and the translation of this material [led] writers like Abu Kamil, al-Karaji, and Abu Ja'far al-Khazin (first half of 10 century) … to attempts to prove a special case of what is now known as Fermat's last theorem – namely, that there are no rational solutions to $x^3 + y^3 = z^3$."

[18] Wolfram, http://mathworld.wolfram.com/FermatsLastTheorem.html, "Euler proved the general case of the theorem for n=3, Fermat for n=4, Dirichlet and Lagrange for n=5."

[19] [Britannica], p. 177.

[20] A "black problem," according to [Knill, E-320], is one that consumes all of a mathematician's energy, even as it remains unsolved, until eventually the solver goes crazy.

By the end of the story, the devil no longer cares about winning Flagg's soul:

> "It certainly gets you," he said [of the problem], avoiding Simon's gaze. "Impossible to stop just now. Why, if I could only prove one simple little lemma--."

But the black problem itself wins in the end, as Flagg joins the devil in the never-ending pursuit of the solution:

> "Mrs. Flagg sighed. Suddenly the devil seemed a familiar figure, little different from old Professor Atkins, her husband's colleague at the university. Any time two mathematicians got together on a tantalizing problem . . . Resignedly she left the room, coffeepot in hand. There was certainly a long session in sight. She knew. After all, she was a professor's wife."

Student Activities and Approaches: Since the proof itself is well beyond the capacity of middle-school students, the best approach here is to bring students to an understanding of the problem, and to an exploration of some simple examples. After studying the Pythagorean Theorem, one could ask students to try to find pairs of cubes that add to another cube.

Explorations of number theory in general may come out of this exploration. For example, students may find numbers such as 1729 that can be written as the sum of two cubes in more than one way, and can be introduced to G. H. Hardy's anecdote about the mathematician Ramanujan:

> I remember once going to see him when he was lying ill at Putney. I had ridden in taxi-cab No. 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways."[22]

[21] "The Devil and Simon Flagg," Arthur Porges, *Magazine of Fantasy and Science Fiction*, 1954; reprinted in *Fantasia Mathematica*, edited by Clifton Fadiman, 1958. The story is also available online at http://www.math.iupui.edu/~bramsey/SimonFlagg.html

[22] *Gödel, Escher Bach*, Douglas Hofstadter, p. 564, 1989. The book notes that the corresponding number for fourth powers is quite large: $635{,}318{,}657 = 134^4 + 133^4 = 158^4 + 59^4$.

5. The Goldbach Conjecture

If students' appetites are whetted by the prospect of unsolved problems, it doesn't seem fair to leave them with one that has in fact been solved (albeit hundreds of years after it was proposed). Why not leave them with a truly unsolved problem, and the prospect of their being the first ones to solve it at some point in their own mathematical career?

One of the best candidates for such a position in a young student's math education is the Goldbach Conjecture, which states simply that

Every even number greater than 2 can be expressed as the sum of two prime numbers.

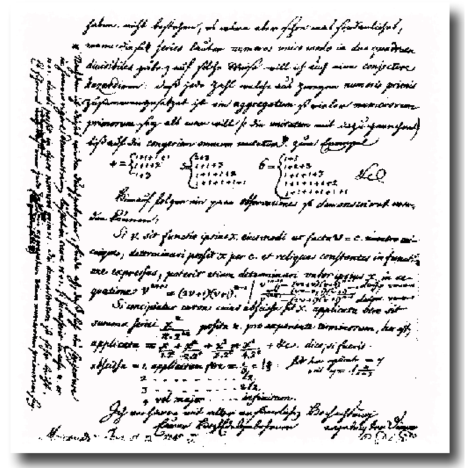


Figure 12 - Goldbach's Letter to Euler containing the Goldbach Conjecture, from http://www.mathsisgoodforyou.com/conjecturestheorems/goldbachs.htm

First proposed in 1742 by the mathematician Christian Goldbach in a letter to Euler[23], the conjecture is simple enough for any middle school student to understand, and easy to test for small numbers, but remains unproven as of the year 2012.

Student Activities and Approaches: Although most students have learned about prime numbers sometime before middle school, it is probably a good idea to begin this unit with a review of primality, and to explore what makes a number prime to begin with. Once those preliminaries are underway, students can begin by finding pairs of primes that sum to each even number beginning with the number 4. Students may notice (and be struck by the fact) that 4 = 2+2 involves even primes, while all the remaining examples involve, of necessity, two odd primes (since there are no other even primes, and since an even number must always be the sum of either two evens or two odds – a fact that might in and of itself be worth confirming with students and treating algebraically).

Students can then make tables of even numbers and of the primes that sum to them: 6 = 3+3, 8 = 3+5, 10=3+7 or 5+5, and so forth. When students reach 10, they may notice that there are two different pairs of primes that will solve the conjecture (if no individual student notices at first, the class may notice when students later compare results). Another question may arise, as to how many different pairs of primes can be found for any given even number. A function called Goldbach's Comet[24] shows, for any given even number X, the number of different pairs of primes that sum to X. A graph of Goldbach's Comet is shown on the next page as the function G(E) for even numbered input. The value of G(10), for instance, is 2, because there are exactly two different pairs of primes that sum to 10.

[23] http://mathworld.wolfram.com/GoldbachConjecture.html

[24] http://www.primepuzzles.net/puzzles/puzz_082.htm

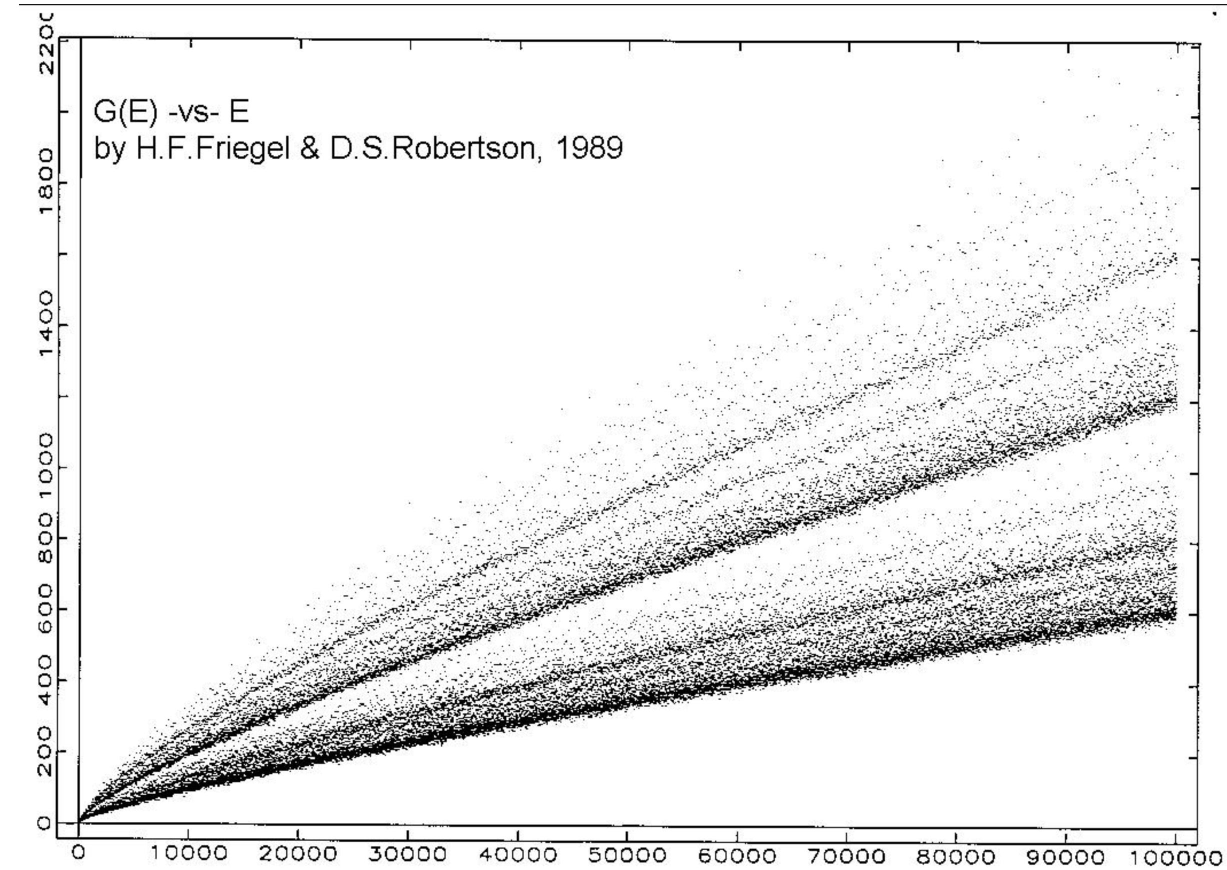


Figure 13 - Goldbach's Comet, from http://www.primepuzzles.net/puzzles/puzz_082.htm

Students may also ask why the conjecture does not apply to odd numbers. This is easily tested, of course – there is no pair of primes with a sum of 11, for example – but of interest would be to see if students can express reasoning along the lines of, "the number 2 is the only even prime, and an odd number must be the sum of one even and one odd number, so any number that is two more than a non-prime odd number (two more than 9, 15, 21, etc.) cannot be the sum of two primes."

A related theorem is that of Vinogradov, which states that every "sufficiently large" odd number can be written as the sum of three primes[25]. Students can explore this theorem with specific examples, and may also notice how Goldbach's conjecture, if true, would prove Vinogradov's theorem for odd numbers greater than or equal to 7. The theorem, however, has been proven only for primes > 10^{43000} (see footnote 25., p. 56).

Finally, students may wish to know about the current state of knowledge concerning Goldbach's Conjecture. According to a Wikipedia article[26] as of March 25, 2012, the conjecture has been verified (by finding pairs of primes) for all numbers up to 1.609 x 10^{18}, and a distributed computer search continues to run and find solutions higher still.

[25] http://math.stanford.edu/~ipetrow/V3.pdf

[26] http://en.wikipedia.org/wiki/Goldbach's_conjecture

6. Sizes of Infinity

When I told my 12-year-old that I was writing an article on the sizes of infinity, she replied, "Infinity means 'goes on forever.' How can forever have more than one size?" Such was the state of understanding in the mathematical world as well, until the question was posed and answered by Georg Cantor in 1874, in his article "On a Property of the Collection of All Real Algebraic Numbers."[27]

When working with infinite sets, the first thing one needs to consider is the definition of two sets having the same size. Two sets are the same size if you can put them into one-to-one correspondence. With finite sets, comparing their size is easy: "Even if you had no idea how many terms were in each of those two sets, it would still be easy enough to compare them. All you would need to do is look at Set A, match it to a term in Set B, and repeat the process until no terms are left in either Set A or Set B."[28]

With infinite sets, however, the comparison task can be trickier. There can be many different ways to compare two infinite sets by one-to-one correspondence. For instance, are there more natural numbers, or natural even numbers? A naïve answer would be that clearly there are more natural numbers, since if you match all the even numbers in the set E of all evens, to the same number in the set of natural numbers N, you achieve a one-to-one correspondence while still having all the odd numbers in N left over. However, you can also achieve a one-to-one correspondence by matching each element x in the set N with the element 2x in the set E:

```
N:  1    2    3    4    5    6    . . . . . .
    |    |    |    |    |    |    | | | | | |
E:  2    4    6    8    10   12   . . . . . .
```

This correspondence exactly uses up all of N and all of E, thereby proving that N and E are of the same size.

Cantor extended this concept further. His "first diagonal proof" showed that it was possible to put the entire set of rational numbers, Q, in one-to-one correspondence with the counting numbers. By placing all the fractions in a grid (see diagram) one sees that by the diagonal counting method shown in the diagram, one will eventually reach any given fraction with any numerator and denominator.

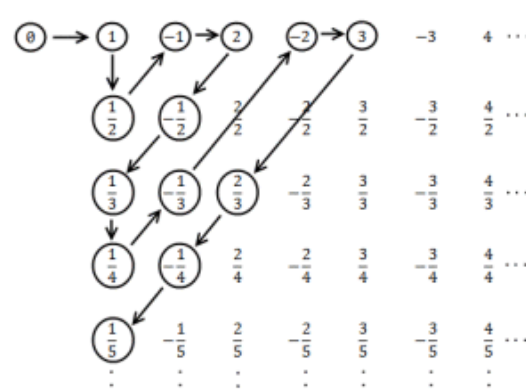


Figure 14 - Cantor's First Diagonal Proof (diagram from http://www.storyofmathematics.com/images2/cantor_bijection.gif)

[27] The reference is collected in the Wikipedia article, http://en.wikipedia.org/wiki/Georg_Cantor (titled "Georg Cantor") as of April 1, 2012. The original article is referenced as follows: Cantor, Georg (1874), "Über eine Eigenschaf des Ingebriffes aller reelen algebraischen Zahlen", *Journal für die Reine und Angewandte Mathematik* **77**: 258–262.

[28] "A brief introduction to infinity," http://io9.com/5809689/a-brief-introduction-to-infinity

The exact natural number (the "count") of any given rational is hard to determine a priori because of the need to skip fractions that are not in lowest terms, but one can rest assured that each fraction that is in lowest terms will eventually be counted.

While this result was impressive, it was not as earth shattering as that of his "second diagonal proof," with which he demonstrated that the set of real numbers R is *not* countable, and therefore, that there must be more than one size of infinity.

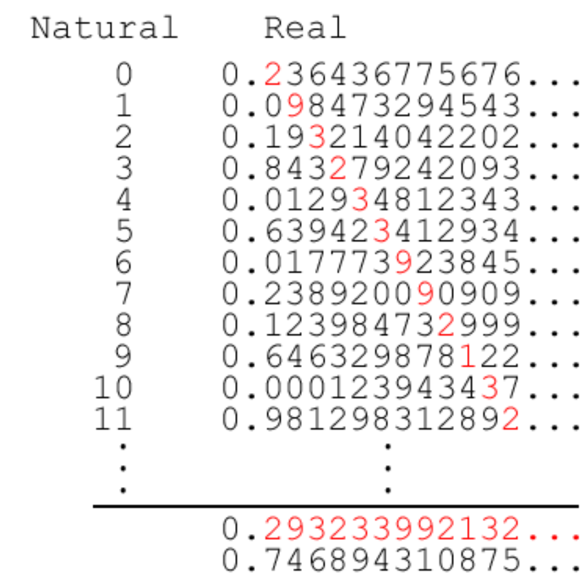


Figure 15 - Cantor's second diagonal proof (from http://www.irregularwebcomic.net/annotations/annot2292b.png)

The idea behind his second diagonal proof is to assume you can in fact count all the reals. If so, then there must be an ordering of real numbers, each associated with a natural number, and the numbers could be written in that order, as in the diagram to the left, where we restrict ourselves just to the reals between 0 and 1 for ease of discussion; also in the diagram, the count begins at 0 rather than 1 but that does not change the effect of the argument.

Whatever order you choose, next construct a new real number by modifying the first decimal digit of the first number, the second decimal digit of the second number, and so forth. In this example, the first two digits have been modified by adding 5 (modulo 10) to the digit, so that 2 becomes 7 and 9 becomes 4, but as we see with the other digits, any modification will do. The new real number we have constructed cannot be the first number (because it differs in the first digit); it cannot be the second number (because it differs in the second digit), and so forth – indeed, it cannot be *any* number in our list, thus contradicting the assumption that such an ordering is possible.

Cantor thus proved that there were at least two different sizes of infinity: that of the natural numbers (or rationals) and that of the reals. The first of these numbers is referred to today as $\aleph_0$ and the latter as $\aleph_1$. But there are additional sizes of infinity: by the same sort of diagonal proof, one can show that the number of *sets of numbers*

Figure 16 - The longest possible school bus ride song. Image from http://io9.com/5809689/a-brief-introduction-to-infinity

contained within $\aleph_1$ cannot have the same cardinality as $\aleph_1$ itself; that number is called $\aleph_2$, and so forth ad infinitum.

But how *infinitum*? According to some sources, "there are so many levels of infinity, that no level of infinity is enough to answer the question. No matter how hard anyone tries to come up with some incomprehendably large level of infinity, there are more levels of infinity than that."[29] Cantor himself is said to have had the "notion of an absolute infinite that transcended all attempts to express infinity within set theory. For his part, Cantor suspected that the absolute infinite was God."[30]

Student Activities and Approaches: For this topic, a good starting point after the basic principle of one-to-one correspondence with finite sets might be the Hilbert Hotel, described in numerous articles,[31] in which a hotel with an infinite number of rooms, all of which are full, needs to accommodate an additional guest, or finitely many new guests, or infinitely many new guests (the problem has also been presented as a work of fiction: "Welcome to the Hotel Infinity"[32]).

After the Hilbert Hotel has been explored, there is probably nothing better than diving right into Cantor's first and second diagonal proofs, both of which should be accessible to most lay audiences, and thus to the average middle school student as well.

On the topic of (countable) infinities in general, younger students can also be introduced to it through Chapter 15 ("This Way to Infinity") of *The Phantom Tollbooth* by Norton Juster[33]:

> "I think what you would like to see," said the dog [to Milo], "is the number of greatest possible magnitude."
>
> "Well, why didn't you say so," said the Mathemagician. "What's the greatest number *you* can think of?"
>
> "Nine trillion, nine hundred ninety-nine billion, nine hundred ninety-nine million, nine hundred ninety-nine thousand, nine hundred ninety nine," recited Milo breathlessly.

[29] http://www.xamuel.com/the-higher-infinite/

[30] op. cit., [30] "A brief introduction to infinity."

[31] http://en.wikipedia.org/wiki/Hilbert's_paradox_of_the_Grand_Hotel

[32] "Welcome to the Hotel Infinity" by Nancy Casey, 1991. http://www.ccs3.lanl.gov/mega-math/workbk/infinity/inhotel.html

[33] *The Phantom Tollbooth*, Norton Juster, 1961, Alfred E. Knopf, pp. 184-192

> "Very good," said the Mathemagician. "Now add one to it. Now add one again," he repeated when Milo had added the previous one. "Now add one again. Now add one again. Now add one again. Now add one again. Now add one again. Now add one again. Now add –"
>
> "But when can I stop?" asked Milo?
>
> "Never," said the Mathemagician with a little smile, "for the number you want is always at least one more than the number you've got, and it's so large that if you started saying it yesterday you wouldn't finish tomorrow."

7. The Discovery of Irrational Numbers

Irrational numbers are numbers that cannot be expressed as a fraction. The existence of irrational numbers was known to the ancient Greeks, who may however have thought of these quantities geometrically: "there is no length that could serve as a unit of measure of both the side and the diagonal"[34] of a square (see Figure 18 for an example – no matter what length "**a**" you choose, if that length can be laid down an exact number of times along the side of an isosceles right triangle, there is no exact number for the hypotenuse, and vice versa). Although the Babylonians could estimate the length of the diagonal of a square to great precision (see Figure 17), they did not have the concept of irrational numbers.

Figure 17 - Babylonian Tablet illustrating the square root of 2 (19th-17th century BCE), Yale Babylonian Collection YBC 7289

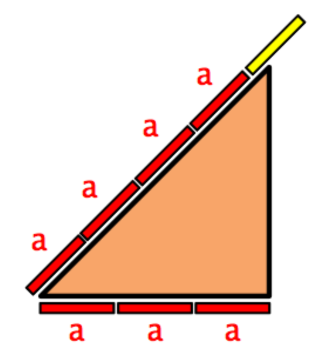


Figure 18 – example of non-commensurability. Drawing by David Albert.

The Greeks did, however. The irrationality of $\sqrt{2}$ "was already well known at the time of Plato, and may well have been discovered within the school of Pythagoras in the 5th century BCE"[35], and a geometric proof of the irrationality of square roots "is associated with Plato's friend Theaetetus"[36] because the proof is actually discussed in one of Plato's dialogs:

> ***Theaetetus****: Theodorus was proving to us a certain thing about square roots, I mean the*

[34] [Britannica], p. 43.

[35] Ibid., p. 43.

[36] Ibid., p. 44.

square roots of three square feet and five square feet, namely, that these roots are not commensurable in length with the foot-length, and he proceeded in this way, taking each case in turn up to the root of seventeen square feet; at this point for some reason he stopped. Now it occurred to us, since the number of square roots appeared to be unlimited, to try to gather them into one class, by which we could henceforth describe all the roots.[37]

One might, however, reasonably surmise from the above speech that the irrationality of the square root of *two*, specifically, was already known, as the above snippet mentions square roots beginning with three.[38]

Others credit the discovery, if not the proof, of the existence of irrational numbers to Hippasus of Metapontum: "The Pythagoreans were supposed to have been at sea at the time and to have thrown Hippasus overboard for having produced an element in the universe which denied the Pythagorean doctrine that all phenomena in the universe can be reduced to whole numbers or their ratios."[39]

Be that as it may, a well-known proof of the irrationality of $\sqrt{2}$ by Euclid survives, so the result is clearly at least as old as he is. The proof can be written in many different ways. In the box at the top of the next page is a straightforward explanation, written for use by homeschooling parents and students. It is short, sweet, and detailed enough for any early student of algebra. And yet it is rarely taught in middle school, and often not even in high school. These are the sorts of deep but simple proofs that can turn middle-school students into life-long mathematicians. In the next paragraphs, I will explore how we might approach this proof with an 8th grade student.

<u>Student Activities and Approaches:</u> If students can understand the proof of the irrationality of the square root of two, then they are ready for almost anything in higher mathematics. Thus, I think one of the best things we can do with this subject is to help them realize how easy it really is.

A few concepts should be taught before attacking the proof, so that when the proof itself is introduced, it will seem as easy and natural as breathing. These concepts include: (1) the general idea of proof by contradiction, which is probably the hardest part of the actual proof for a newcomer to grasp; (2) the lemma that the product of two odd numbers is always odd, while the product of two even numbers is always even; and (3) the definition of rational numbers as numbers that can be expressed as fractions in lowest terms.

[37] Plato's Theaetetus, as quoted on http://www.cut-the-knot.org/proofs/sq_root.shtml
[38] http://www.math.ufl.edu/~rcrew/texts/pythagoras.html: "Theodorus shows that the square roots of 3, 5, 6, 7, etc., up to 17 are irrational; but as he does not show that the square root of 2 is irrational, this must have been known already."

[39] This quotation from "Mathematical Thought from Ancient to Modern Times", by Morris Kline, was found at http://www.math.ufl.edu/~rcrew/texts/pythagoras.html on April 2, 2012.

The proof that square root of 2 is irrational:

Let's suppose $\sqrt{2}$ were a rational number. Then we can write it $\sqrt{2}$ = a/b where a, b are whole numbers, b not zero. We additionally make it so that this a/b is simplified to the lowest terms, since that can obviously be done with any fraction.

It follows that $2 = a^2/b^2$, or $a^2 = 2 * b^2$. So the square of *a* is an even number since it is two times something. From this we can know that *a* itself is also an even number. Why? Because it can't be odd; if a itself was odd, then *a* * *a* would be odd too. Odd number times odd number is always odd. Check if you don't believe that!

Okay, if *a* itself is an even number, then *a* is 2 times some other whole number, or *a* = 2k where k is this other number. We don't need to know exactly what k is; it won't matter. Soon is coming the contradiction:

If we substitute a = 2k into the original equation $2 = a^2/b^2$, this is what we get:

$$2 = (2k)^2/b^2$$
$$2 = 4k^2/b^2$$
$$2*b^2 = 4k^2$$
$$b^2 = 2k^2.$$

This means b^2 is even, from which follows again that b itself is an even number!!!

WHY is that a contradiction? Because we started the whole process saying that a/b is simplified to the lowest terms, and now it turns out that *a* and *b* would both be even. So $\sqrt{2}$ cannot be rational.

Figure 19 - A proof that the square root of two is irrational[40]

Probably the hardest concept in the proof is not the algebra, but the very idea of a proof by contradiction. Thus, it would be wise to precede the entire exercise with some other examples of proof by contradiction (for instance, the proof that there is no largest prime number[41], which I will not go into here).

Examining the product of odd numbers or even numbers can be done by example but is also worthy of a proof, looking at the definition of even and odd numbers (2k and 2k+1, respectively) and at what happens when you multiply 2k by 2m, 2k by (2m+1), etc.

The definition of rational numbers is the last major step; students may be more familiar with other definitions (e.g. "repeating decimals") if they've seen one at all.

With these preliminaries completed, the Figure 19 proof may be straightforward.

[40] This particular statement of the proof comes from the Home School Math website, http://www.homeschoolmath.net/teaching/proof_square_root_2_irrational.php

[41] There are many wonderful statements of this proof; one can be found here: http://delphiforfun.org/programs/Math_Topics/proof_by_contradiction.htm

8. One-way cryptographic functions (Public Key Cryptography)

History of encryption goes back at least as far as Julius Caesar, who is said to be the inventor of a simple encryption scheme called the "Caesar Cipher"[42], which encrypts each letter of the alphabet by replacing it with the letter 3 spaces to the left, treating the alphabet as cyclical (so that the letter E, for instance, would be encrypted as an H, while the letter Y would become B). Although Caesar's cipher rotated the alphabet by 3 spaces, one could clearly choose a different number and thus create 25 different ciphers (or 26 if you count the identity element in which each letter is represented by itself).

Figure 20 - A generic Caesar Cipher decoding device; image found at http://www.geocaching.com/track/details.aspx?id=3421742

Using Caesar's cipher, the phrase "I love math" would be encoded as "L oryh pdwk." If you know that someone has used Caesar's cipher, then cracking the code is easy – simply count backwards three letters from each of the letters in the encrypted version. If you know that a generic Caesar cipher was used but you don't know how far around the alphabet was rotated, you can simply try 25 different possibilities until you find one that makes sense (it is extraordinarily unlikely that two different decryptions will make sense in English for a cipher of any length greater than one or two words).

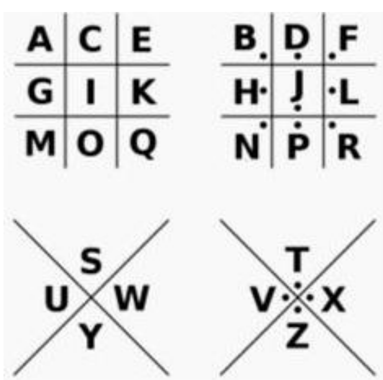


Figure 21 - A PIGPEN cipher

Caesar's cipher is a specific example of a substitution cipher, in which each letter is replaced with some other symbol, and the same symbol is used throughout for each letter. One popular substitution cipher used by children is the PIGPEN cipher,[43] so-called because the key to the cipher can be shown by symbols that resemble pigpens in shape. The key is shown at left in Figure 21; the encoding of "I love math" using this version of the PIGPEN cipher is shown in Figure 22.

Even without knowing the substitutions, however, it is often relatively simple to decrypt a simple substitution cipher, assuming you know the language and that there is sufficient encoded material to find patterns in the letter frequencies, word length, etc. Explanations of code breaking (cryptanalysis) go back as far as Al

Figure 22 - "I love math" using the PIGPEN encoding applet at http://www.purplehell.com/riddletools/pigpen.htm

[42] Codes and Ciphers", John Laffin, 1964, Signet Books, The New American Library, pp. 23-24.

[43] Ibid. "Codes and Ciphers", pp. 38-39; slight variants exist elsewhere and the one pictured in Figure 21 is version #3 from the site referenced in Figure 22.

Kindi (801-873)[44]. The most common letters in the English language, in order, are ETAOIN SHRDLU, and this knowledge can give you a leg up on finding a solution. It doesn't take more than a sentence or two of encrypted text to provide enough material to solve a simple-substitution Cryptogram.

Some ciphers, however, are completely untractable without access to the key. For example, imagine a Caesar cipher in which each letter is rotated to some random position in the alphabet (e.g. the first letter is rotated by 3, the next by 10, the next by 25, the next by 7, and so forth). Then, without knowing which rotation to use for each letter, any sequence of encrypted letters is potentially equivalent to any other sequence of the same length; "A aaaa aaaa" could be the encryption for "I love math" if we rotate the first letter (I) 18 places around to the "A", and then we rotate the "L" 15 places around to the "A", and so forth. But "A aaaa aaaa" could also mean "A blue coat" or "I want cake."

To encrypt and decrypt messages using this method, we need a one-time keypad – a codebook that tells the encrypter what rotation to use for each letter, and the decrypter what rotation was used. You can use as much of the pad as you need, and by never reusing the same part of the pad, all your messages are 100% undecodable by an outside eavesdropper. There is only one problem: both the encrypter and decrypter must have access to the same keypad. If you send the pad in the mail, it could be intercepted and all your codes broken. If you meet in advance and share codebooks, what happens if the book runs out of pages (which must be used only once) and you need to send another message?

Wouldn't it be nice if you could send out your encrypted message without having to send the decryption code, and rest assured that one and only one person in the world would be able to decode it? Enter the realm of *public-key private-key cryptosystems,*[45] developed by Whitfield Diffie and Martin Hellman in 1975.

A public-key private-key cryptosystem relies on a mathematical algorithm that generates a PAIR of complementary keys which produce mathematical functions) F(x) and G(x). For any data d, G(F(d))=d, and F(G(d))=d – the functions are inverses of each other. Both F(x) and G(x) can easily be computed if given the key, but each is a ONE-WAY function – that is, computing the inverse function without the key is believed to be impossible in a reasonable amount of time.

This property of the functions means that you can encrypt the data using either F or G as the key. Whichever one you use, the other function then becomes the decryption key. Applying the same function a second time, resulting, for instance, in F(F(d)), does not decrypt the data.

[44] [Knill], lecture of April 16, 2012.

[45] There are numerous references and entire books on this subject. One comparatively clear and readable source is at http://www.pgpi.org/doc/pgpintro/#p9. A more comprehensive look can be found in the book *Cryptography and Public Key Infrastructure on the Internet*, Klaus Schmeh, 2003, Wiley.

To make use of this system, an individual (let's call her Mary; see Figure 23) who wishes to receive some data D from another individual, John, must create a pair of keys, and designate one of those keys *public* and the other *private*. The key pair is wholly symmetrical, so it doesn't matter which key is which. Let's call them PUB and PRIV.

The private key must be kept secret by Mary, but the public key may be sent to anyone in the world, or even published on the Internet. Mary then sends John her public key. John uses that key to compute PUB(D), and sends it to Mary, or publishes it on the Internet – it does not matter in the least how many other people intercept this data, because the only known way to reconstruct it is to use the function PRIV(PUB(D)), which equals D and gives you back the original data. So as long as Mary has kept her private key truly private, this system is currently believed unbreakable in any reasonable amount of time.

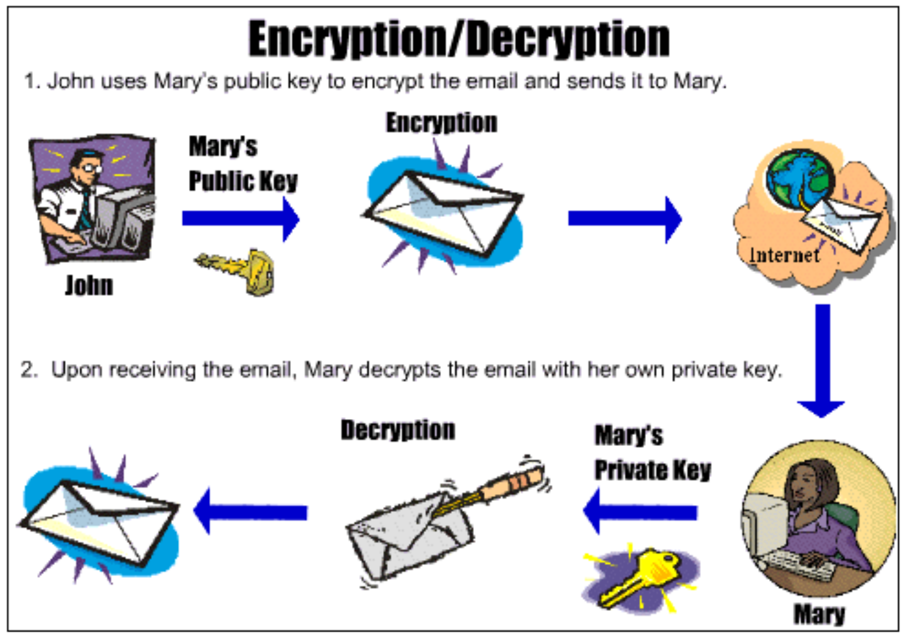


Figure 21 - Public Key / Private Key cryptosystem. Image from http://www.infosec.gov.hk/english/itpro/public_main.html

Without the private key, the only known way to decrypt the data is to try a brute-force method, painstakingly attempting every possible key. How long that will take depends on number of possible keys. In practical terms these keys are describable as binary numbers with some bit-length (e.g. 40-bits or 128-bits) with, respectively, 2^{40} or 2^{128} possible keys for those two lengths. The longer the key you use, the safer is your data. Longer keys take slightly longer to generate and to use, but only linearly, while the time to decrypt increases exponentially. In practical terms, a key of length 2048 will guarantee unbreakability by the brute force method.

<u>Student Activities and Approaches</u>: Before beginning a unit on cryptography, make sure students have a thorough grasp of functions in general. Then take a look together at inverse functions, such as doubling and halving, x^2 and sqrt(x), or the function f(x)=1/x which is its own inverse.

As a next step, students can experiment with simple substitution ciphers. With enough patience, students can build Caesar-cipher decoder circles (this could be a good opportunity to teach the use of protractors, but as the angle for each letter in a Caesar cipher circle needs to be 360/26 degrees, those can be tricky to build even with a protractor. Students can write and send each other messages to decode.

Students with access to computers can run algorithms that generate public-key private-key pairs. The exact mechanism for creating these would depend on what computing power is available. On PCs, as one example, the website *http://theillustratednetwork.mvps.org/Ssh/Private-publicKey.html* links to downloadable software for key pair creation and encryption/decryption.

Even without such software, however, it is possible to *simulate* the encoding and decoding process with a little imagination. Here is one suggestion for how to perform this task so that John can send Mary a message that only Mary can decrypt, even though everyone in the room sees them exchange the data. John and Mary should each be provided with a calculator for this exercise. Pretend for this exercise, with the class, that the concept of multiplication is well understood, that inverses (1/x) can be computed using a calculator, but that division is unheard of:

(1) Mary begins by generating two numbers X and Y such that XY=1. Let her pick a number at random between 0 and 2 , X = 1.127, say, and use the calculator to generate Y=1/x = 0.887311446. She designates one of these numbers as her public key, and one as her private key. It doesn't matter which is which.

(2) Mary sends John her public key, allowing the whole class to hear it. Let's assume she sends out the number X.

(3) John's "message", for this simulation, should be a number. (A separate class discussion can ensue at some point about how to turn words into numbers.)

(4) John multiplies his number by X, and sends the product to Mary, allowing the whole class to hear it.

(5) Mary multiplies John's product by Y, thereby retrieving his original number (subject to a small rounding error). A class discussion can then be held about whether (and how) anyone else would be able to recover the number, given only what they had heard and without the benefit of division.

9. Pascal's Triangle

Pascal's Triangle, according to Wolfram Science, "probably dates from antiquity; it was known in China in the 1200s, and was discussed … by Blaise Pascal in 1654 … in connection with probability theory."[46] The triangle is easy for any schoolchild to construct:

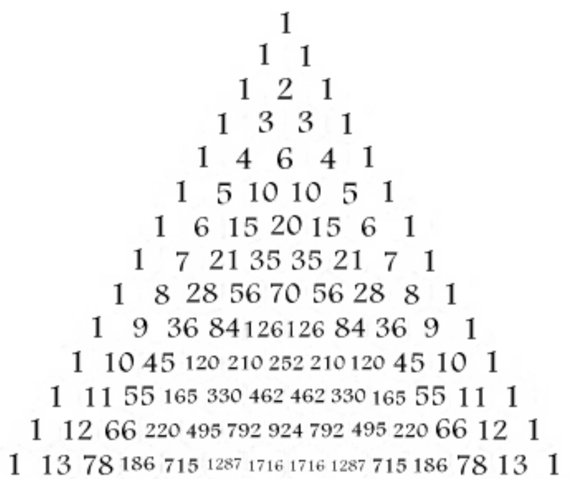


Figure 22 - Rows 0 through 13 of Pascal's Triangle. Image from http://www.mathsisgoodforyou.com/artefacts/pascalstriangle.htm

Starting with a solitary 1 in the top row (usually referred to as row *zero*), build a triangle of numbers by entering in the row *one*, between each two digits, the sum of the two numbers to the left and right in the row above. Assume 0s at the edges of the rows. Thus, row *one* contains two 1s, derived by adding 0+1 and 1+0. In row *two*, the numbers 1, 2, 1 are derived by adding 0+1, 1+1, 1+0, and so forth. Figure 24 shows the continuation of the triangle to the 13th row.

Building Pascal's Triangle is thus easy. Its interest, however, lies in the numerous patterns hidden within, and in the many uses to which it can be put. The triangle hides the integers, the triangular numbers, the tetrahedral numbers, the binomials, and combinatorics, to mention just a few.

Arguably the most important discoveries about the triangle, and the reason why it is named for Pascal, is his observation that it can be used to determine combinatorial probabilities; specifically, the number of ways of choosing k objects from a set of n objects. That number, written "n choose k" or or $_nC_k$, can be found by examining the k^{th} element in the n^{th} row of the triangle, where the top row consisting of just the number 1 is counted as row 0, and the leftmost 1 in each row is the 0^{th} element. For example, is 15; to see that, look in the 6^{th} row (1 6 15 20 15 6 1) for the 2^{nd} element (counting from 0).

The triangle can also be used in conjunction with binomial expansion: the coefficients of x in the formula $(x + y)^n$ are simply the elements of the n^{th} row of the triangle; $(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$, for instance. This theorem was known long before Pascal's time, but was published by Pascal in 1665.[47]

In about the same year, however, Sir Isaac Newton extended the results to include negative and rational numbers. "To deal with the case where *n* is a negative integer, Newton extended the table backwards by computing the difference between the j^{th} entry in each row and the $(j-1)^{th}$ entry in the row above it…. [W]here *n* is a fraction, Newton carefully studied the numerical pattern in Pascal's triangle until he was able to … interpolate the coefficients when n=1/2, 3/2, 5/2, and so on."[48] This extension permitted Newton to develop a theory of infinite series that, ultimately, led to an understanding of the number *e* and of irrationals and transcendental in general.

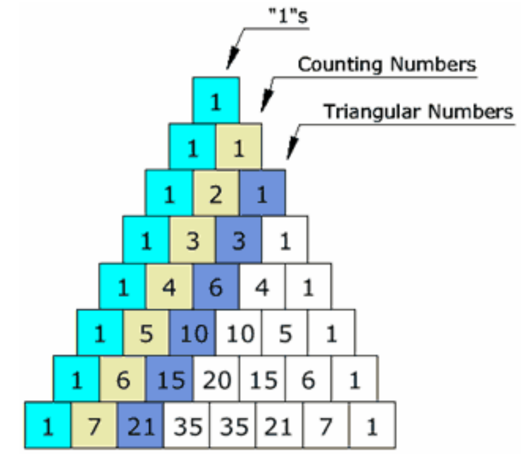


Figure 23 - Patterns in Pascal's Triangle. http://www.mathsisfun.com/pascals-triangle.html

Student Activities and Approaches: Students should first be given the general rule for building Pascal's Triangle, and asked to construct it down to the 6th or perhaps the 8th row; by that point there is plenty of data for analysis.

Several excellent descriptions of activities for children can be found on the website http://www.mathsisfun.com/pascals-triangle.html. The first step is to ask students to see what patterns they can discover in the triangle. Some ideas are shown in Figure 23, but there are many more. If students are familiar with triangular numbers they may discover the third (dark blue) diagonal, but are less likely on their own to discover the tetrahedral numbers in the fourth diagonal.

Some students might wonder about the numbers running along the vertical axis of symmetry (in even-numbered rows only): 1, 2, 6, 20, 70, 252 …. These grow very quickly, but are they in any sort of explainable pattern? I have yet to discover a simple explanation for this series, although it can be calculated directly[49] as $S_{n+1} = (2n)! / (n!)^2$ without need for recursion.

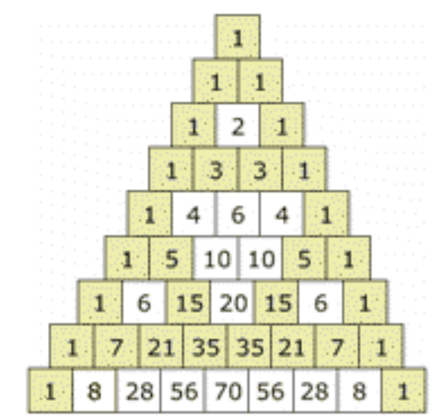


Figure 24 - Odd and Even number pattern in Pascal's Triangle (Ibid.)

But probably the most interesting activities to perform with students are combinatorial. Students can try, for example, to determine by exhaustive search the number of combinations of small groups of objects: , , and , for example, and compare them to the results from the triangle. Students can also play probability games with coin tosses, and learn to recognize that the probability, say, of tossing 4 heads out of 6 throws is the number divided by the sum of all the entries in the 6th row. There is not space in this paper, however, to go into more detail.

[46] http://www.wolframscience.com/nksonline/page-870e-text

[47] http://mathworld.wolfram.com/BinomialTheorem.html

[48] *e*: the Story of a Number, Eli Maor, 1994, Princeton University Press, pp. 71-72

[49] http://wiki.answers.com/Q/What_comes_next_2_6_20_70_252_924

10. Fractals[50]

How long is the coastline of Maine?[51] You can measure it with a ruler on a map, straight across the main diagonal, from the New Hampshire border near Kennebunkport to the Canadian border near Eastport. When I perform this exercise on a map, I come up with the approximate figure of 200 miles, or 300 kilometers. Or you can measure it with somewhat greater fineness, going in and out of the major bays, and reach double that amount. Or you can walk along the coast with a yardstick, measuring every little nook and cranny, and come up with a much higher number. You can go on *ad infinitum* with smaller and smaller rulers, and longer and longer measurements – though not quite *ad infinitum* in the real world, because at some point your ruler gets too small to hold, or even to see with an electron microscope.

Figure 25 - Maine (http://worldatlas.com)

Is there an official answer? According to the State of Maine, "Maine has approximately 3478 miles (5600 kilometers) of tidally-influenced shoreline."[52] Although that source does not state how it arrives at that particular figure (which, one might note, is in round numbers when expressed metrically, and hence presumably accurate only to the nearest hundred, making the rather more precise 3478 mile figure somewhat misleading), there may in fact be a specific guideline that determines the measurement.

The US Geological Survey, recognizing that the length of the ruler is intimately bound up in the length of what is being measured, has established specific criteria:

> GENERAL COASTLINE figures are lengths of general outline of the seacoast. Measurements were made with a unit measure of 30 minutes of latitude on charts as near the scale of 1:1,200,000 as possible. Coastline of sounds and bays is

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[50] The primary references for the material in this section not otherwise footnoted are the following:

[MM] "Mandlebrot Magic," "Visions of Julia," and "Mandlebrew and Mandlebus," from *The Magic Machine: A Handbook of Computer Sorcery*, A. K. Dewdney, 1990, W. H. Freemand and Co., pp. 3-37.

[WBFM] "White, Brown, and Fractal Music," from *Fractal Music, Hypercards and More,* Martin Gardner, 1992, W. H. Freeman and Co., pp. 1-23.

[51] Although this is a canonical problem for which I drew on generally-known material, I can also point to http://www.patheos.com/blogs/unreasonablefaith/2010/10/so-how-long-is-the-coastline-of-maine/ as an interesting resource for those new to the problem.

[52] http://www.maine.gov/doc/nrimc/mgs/explore/geography/index.htm#q7

> included to a point where they narrow to width of unit measure, and the distance across at such point is included.[53]

In 1977, with his book "The Fractal Geometry of Nature," Benoit Mandelbrot took the above idea and turned it into what has become not only a new field of mathematics, but a stunningly beautiful mechanism for generating abstract art, realistic-looking nature art, and even music. His invented term *Fractal* comes from assigning "each of the curves a fractional dimension greater than its topological dimension" (WBFM, p. 6); these fractional dimensions, inherent in the "self-similar" curves and spaces of many natural phenomena, that can be replicated mathematically by a variety of functions in which a pattern is repeated at smaller and smaller scale, literally *ad infinitum*, so that a simple line or closed figure can become, with successive applications of a modification rule, ever more and more complex. Some examples of applications of this rule are shown in the following figure:

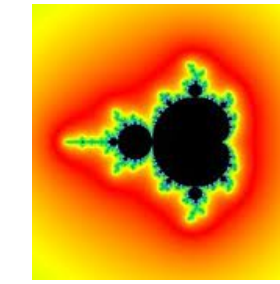

Figure 26 - The Mandelbrot Set

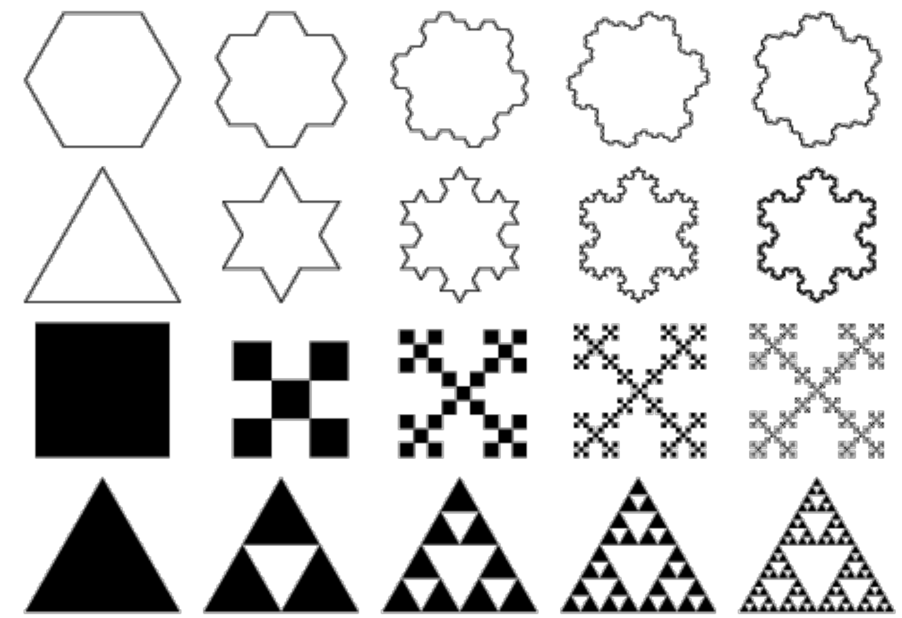

Figure 27 - Examples of Fractal designs, from http://mathworld.wolfram.com/Fractal.html

In the second line, for instance, a triangle is altered by building a similar triangle of 1/3 the size on each of the sides. That triangle is then further altered by building similar triangles, 1/3 the size of the new triangles, on each edge, and so on. It takes only a few applications of this recursive rule to create a stunningly realistic snowflake-like object.

Certain mathematical functions can generate pictures that, when colored to represent the function's values at certain two-dimensional points, look like abstract art or like futuristic landscapes. Two of these functions, the Mandelbrot set and the Julia set, are easy to code (MM, pp. 6-7) and have inspired numerous computer programs that allow

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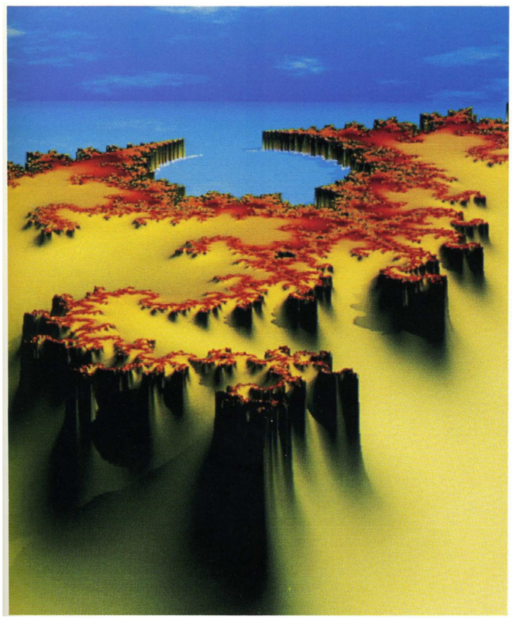
[53] http://gcmd.nasa.gov/records/GCMD_CGS_NCD_COASTLINE.html

users to explore the set interactively. As you dig deeper into each set, the same basic patterns recur at smaller and smaller sizes. Two of the images that can be generated from these sets, including a 3-dimensional realization of the Mandelbrot set as a landscape, are shown in the next figures:



Student Activities and Approaches: Although the mathematics underlying fractal sets is probably well beyond the average middle-school student, much of the beauty of the patterns, and some of the basic questions (such as the opening question of this section, "How long is the coastline of Maine?") are well within their power to comprehend and discuss. My own 12-year-old immediately grasped that one could come up with many different answers depending on the length of the ruler used to measure the coast.

There are also tie-ins to other subjects they may be studying. The small intestine, for instance, with its intricate network of coiled tract and protruding villi, is an excellent example of why the length of the "coastline" actually matters, and of just how much "coastline" one can squeeze, in real terms, into a very small area.

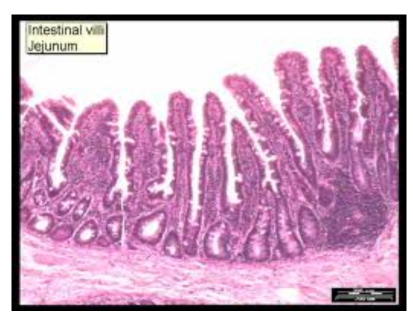


Figure 28 - Intestinal villi

What will probably grab students the most, however, is a chance to actually explore the Mandelbrot or Julia sets interactively. If software can be installed, http://xaos.sf.net/ is a great resource. If not, a web-based applet such as that found at http://aleph0.clarku.edu/~djoyce/julia/explorer.html, will suffice to get started. More advanced students might eventually be led to write their own programs in Logo or Starlogo, for instance using the http://education.mit.edu/starlogo/samples/sirpinski.htm project page.

Inverting binomial distributions

Ed Baranoski (ed.baranoski@gmail.com)

G4G15, February 2024

Abstract

The binomial distribution shows the likelihood of discrete success or failures for a given probability of success. This paper shows how to back out the probability density function of success probability from a binomial sample set. It compares this to real-world results, including Major League Baseball winning percentages.

1 Introduction

The binomial distribution is the well known way of showing the probability of a given outcome for things like number of heads when tossing a coin twenty times, or even the likelihood of a baseball team winning a best-of-seven series of games. It starts with an assumed success rate and then computes the likelihood of each possible outcome. The common test is to ask the likelihood of the next coin flip even after ten successive tosses yield heads. For something like a coin flip of a fair coin, it is reasonable to assume that the likelihood of a toss yielding either heads or tails is exactly 50% so it is still a 50/50 chance that tails will be the outcome of the next toss. However, what if you don't know whether the coin is fair or not? Perhaps it isn't a physical coin at all, just a poorly written simulation of one, or the process of flipping the coin isn't random. What is the likelihood of the actual success probability from a sample if you truly don't know the true success probability?

This paper shows how to use the Bayes Theorem to show the likely probability density function. The paper discusses two example use cases. The first is completely unknown success rates when you have no idea of what the likely success rate is. A coin toss obviously has physics behind the 50/50 heads-or-tails assumption, but other example like the likelihood of hitting a half court basketball shot are more instructive. For cases where the true success rate is close to 50%, the results aren't that surprising. For likelihoods closer to zero or 100%, the results are more interesting.

The second case uses historical records for baseball teams. Is the final record for a team really indicative of their *ideal* winning percentage, or were they lucky (or not) relative to their final record? This will also discuss the Pythagorean Expectation as a way of estimating a team's expected winning percentage based on runs scored and runs allowed.

2 Bayes Rule

The binomial distribution formula of $P(k|p) = \binom{n}{k} p^k (1-p)^{n-k}$ gives us the likelihood of getting k successes out of n trials when the probability of success is p for each individual trial. $P(k|p)$ is called a conditional probability of k given the parameter p. The term $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the how many combinations of k successes can be achieved out of n trials. For a given probability p, we can calculate the likelihood for all values of k. The question we address here is how do you solve the inverse problem; that is, for a given k successes in $n$trials, what is the likelihood of the probability being $p$or within any arbitrary range of values of p.

Bayes theorem is exactly what we need! Bayes rule states that you the conditional and relative probabilities are related–for a given n, $P(p|k) = P(k|p)P(p)/P(k)$. If we don't have any idea of the distribution of , we can assume that $P(p)$ is uniform between zero and one. Since we are starting with the value of k, that probability is one.

Ideally we want to compute the full distribution of $P(p|k)$ over the continuum of $p$from zero to one. The cumulative distribution function

$$C(p < P) = \int_0^P p(p|k)dp \tag{1}$$

would contain the likelihood of $p \leq P$, but a straight-forward implementation using the binomial distribution is very computationally expensive. Thankfully, you can use the ratio of binomial distributions so you only have to compute a single binomial probability. Appendix A shows a very computationally efficient way to compute this distribution. Appendix B includes code for computing the cumulative probability density function for the programming language R.

3 Use Case 1: Consecutive Successes/Failures

Suppose there is an series of event where you have no advance idea of what the success rate should be (unlike a coin flip, where the probability of heads or tails can be assumed to be 50%). If the first trial is a success, the likelihood that the actual probability of success is small (say under 10%) is much smaller than the likelihood the actual probability was high (over 90%, for example). The only information you have is that single success and are using the assumption that the actual distribution of the success probability is even from 0 to 1. Figure 1 shows this by looking at the cumulative density functions of the probability likelihoods for different numbers of consecutive successes (again assuming zero prior information on the actual probability density function). As the number of consecutive successes grows, the likelihood of the actual probability becomes closer to one.

You can then use these updated probability likelihoods to compute the likelihood that the next trial is a success. Table 1 shows the likelihood of the next success after a sequence of consecutive successes when you have no prior information on the actual probability function.

Table 1: Likelihood of the next success after a string of consecutive successes, assuming zero prior information on the actual probability density function

Consecutive Successes	1	2	3	4	5	6	7	8	9	10
Probability of Next Success	0.67	0.75	0.80	0.83	0.86	0.88	0.89	0.90	0.91	0.92

As we will see in the next section, large sample sizes (particularly with probabilities away from zero or one) do not need the inverse binomial distribution. You can approximate the continuous probability density function with the discrete points of the binomial distribution itself.

4 Use Case 2: Baseball Records and the Pythagorean Expectation

Baseball statistics are a great way to dive into large scale statistical analysis. Sources like www.retrosheet.com, baseballsavant.mlb.com, and www.baseball-reference.com offer treasure troves of quantifiable data for free. In this section, we will use look at probability distributions of winning percentages of teams over the years, the expected distribution around a "true" winning percentage, and comparison to a concept called Pythagorean Winning Percentage first introduced by Bill James [3].

A baseball season has been 162 games long since 1961 (excluding a few years due to strikes and the pandemic, which were shorter). A typical team wins between 60 and 100 of those games, with some exceptions on either end. Figure 2 shows the continuous inverse binomial distributions(continuous lines) and discrete binomial likelihoods for three different winning percentages, corresponding to an expected 60, 80, ands 100 wins. Note that the binomial distribution is a good discretized sampling of the actual continuous probability density function using the assumed winning percentages. Also note if a omniscient winning percentage was known in advance, the actual team's winning percentage can easily vary by a half dozen wins either way over the sample size of a 162 game season.[1]

[1]This is usually attributed to luck (either good or bad), but often correlates well with a team's performance in games decided by a single run as we'll see shortly.

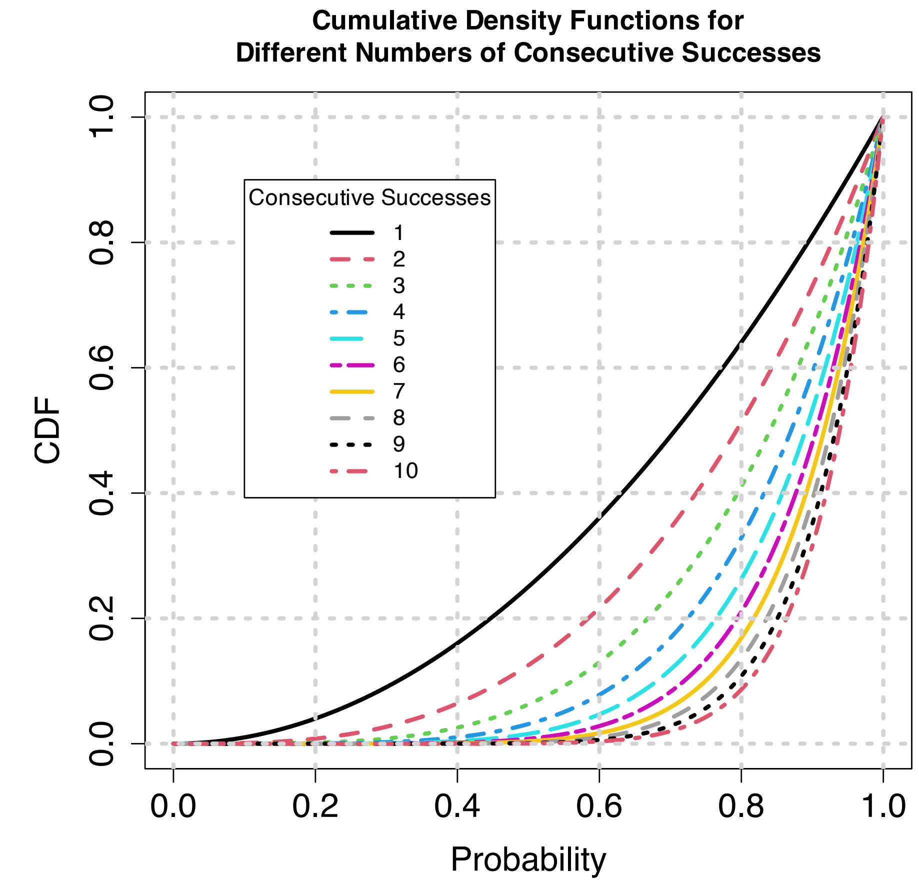


Figure 1: Cumulative density functions of the probability likelihoods for different numbers of consecutive successes, assuming zero prior information on the actual probability density function.

Bill James is largely credited with starting the SABRmetrics revolution in baseball analytics. (SABR is the Society for American Baseball Research.) His *Baseball Abstracts* introduced many analytical innovations, including what he called Pythagorean Expectation[3]. This was a heuristic approximation using number of runs a team scored (RS) and the number of runs they allowed (RA) to estimate a team's expected winning percentage using the formula

$$Pct = \frac{RS^2}{RS^2 + RA^2} \tag{2}$$

The name is due to the obvious similarity to the Pythagorean Theorem in geometry. Later analysis[4, 2] showed improvements,[2]including adjusting the exponent to 1.8 to better represent actual results.

This Pythagorean approximation holds amazingly well, and produces a tighter distribution of potential outcomes than the Bayesian distribution would indicate. To show this, we used Retrosheet[1] data for all 162 game seasons from 1961 through 2023. Over this time period, the number of teams varied from 24 to the current 30, giving us 1,329 end-of-year team records.[3] Figure 3 shows a histogram of the relative error between a team's actual record and what was predicted with the Pythagorean expectation. It also overlays the expected binomial distribution showing the expected variance assuming a team's actual record

[2]The exponent is generally low for low scoring games which have more variability in outcomes. For sports like basketball with much higher scores, there is less variability in outcomes and the exponent is closer to 15 instead of 2.

[3]A few seasons included one last tie-breaking game at the end of the season, or missed a game due to a rainout that was never rescheduled provided it had no impact on which teams advanced to the playoffs.

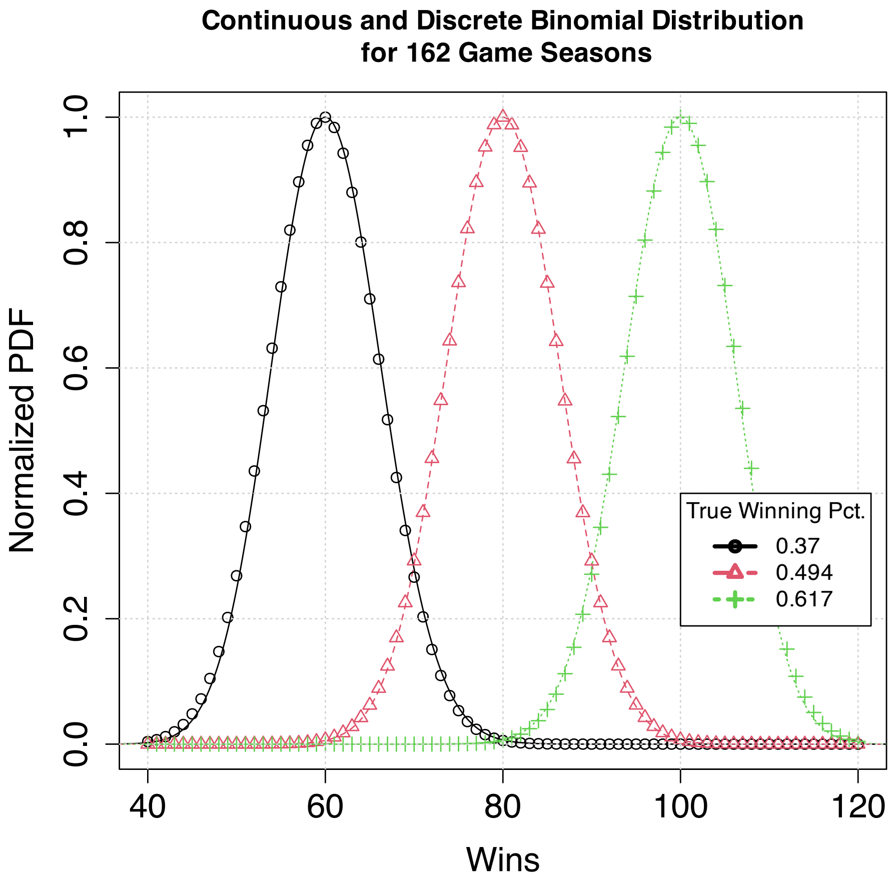


Figure 2: Continuous inverse binomial distributions(continuous lines) and discrete binomial likelihoods for three different winning percentages, corresponding to an expected 60, 80, ands 100 wins.

was indicative of the "true" winning percentage. This indicates that the variance around the predicted expectation is narrower for the Pythagorean Expectation than that predicted by the binomial distribution. This is due to the additional *a priori* information the Pythagorean Expectation has by using the actual runs scored and runs against. As expected, there is a correlation between a team's record in one-run games (games decided by a single run separating the winner and loser) and the offset between a team's predicted win using the Pythagorean Expectation and their actual record as shown in Figure 4.

5 Summary

This paper shows how to invert the binomial distribution to determine the density function for the true probability given a discrete outcome. This is most useful for small sample number, particularly for low probability (probability close to zero) or high probability (probability close to one) that are not represented well by a simple discrete binomial calculation around the success rate of the small sample size. We also evaluated the Pythagorean Expectation for baseball to show that it provides more information to predict a team's true winning record than a simple binomial sampling around a team's actual record would indicate.

References

[1] http://www.retrosheet.org.

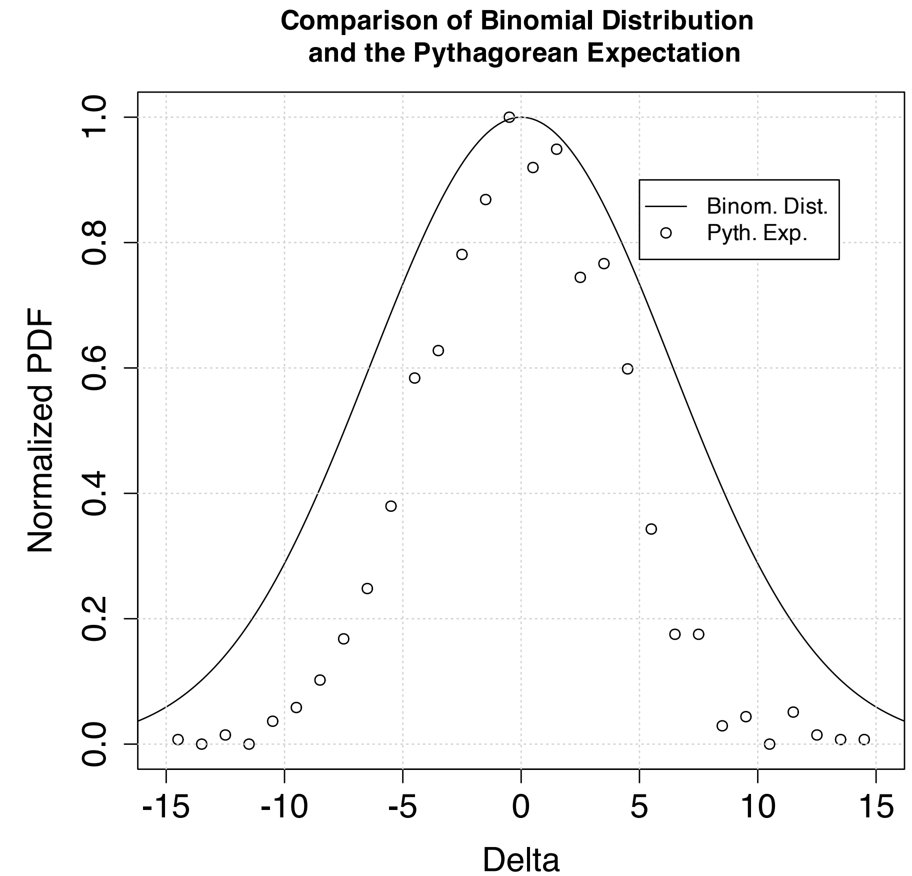


Figure 3: Comparison of the binomial distribution around a team's final record (solid line) versus a normalized histogram predicted by the Pythagorean Expectation (hollow circles) using Retrosheet data of all 1,329 of the 162-game seasons between 1961 and 2023. Note that Pythagorean Expectation has a narrowed variance around the true outcome.

[2] Campbell Gibson. Actual pennant winners versus pythagorean pennant winners, 1901-2020. *SABR Baseball Research Journal*, 50(1):69–74, Spring 2021.

[3] Bill James. *1980 Baseball Abstract*. self-published, Lawrence, KS, 1980.

[4] Stanley Rothman. A new formula to predict a team's winning percentage. *SABR Baseball Research Journal*, 43(2):97–105, Fall 2014.

A Ratios of Binomial Distributions

The direct calculation of a binomial probability can get to be numerically unstable when dealing with large values of n and k, so most programming languages have more efficient and stable ways of computing this. However, most of them are not vectorized to compute them for multiple values of the probability p. Thankfully, you can compute the ratios easily relative to a baseline. Starting with the basic binomial distribution

$$Q(p) = \binom{n}{k} p^k (1-p)^{n-k} \tag{3}$$

you can compute the desired result

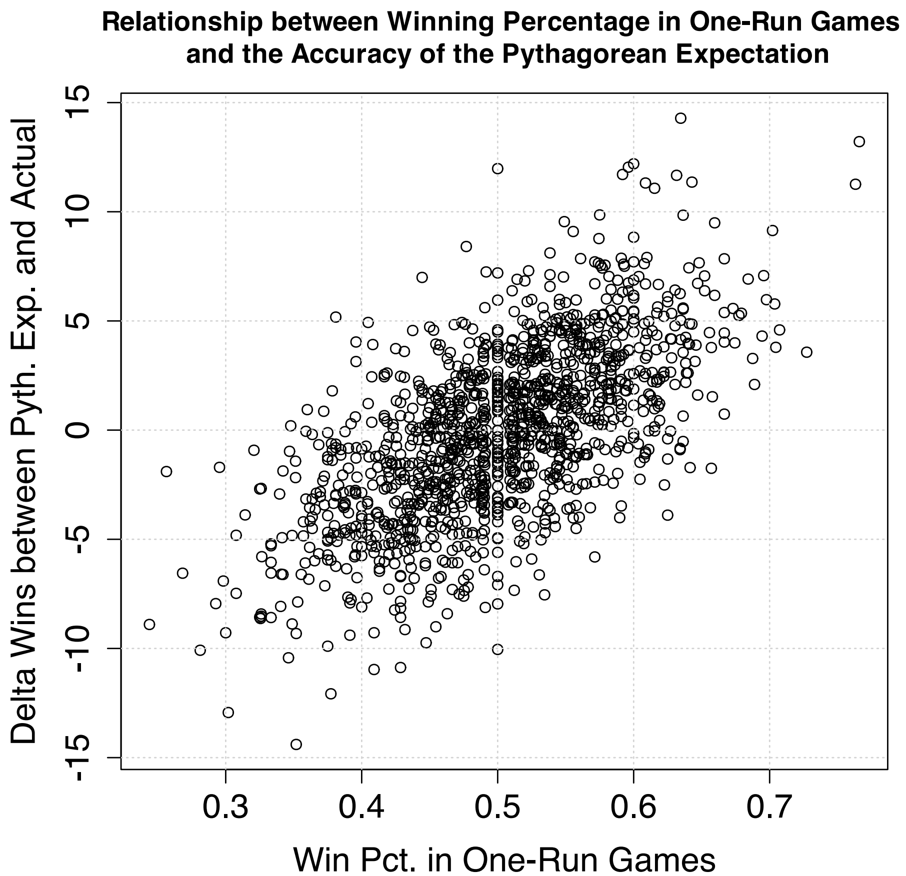


Figure 4: Relationship between winning percentage in one-run games and the difference between the Pythagorean Expectation and teams' actual records

$$Q(p+\delta) = \binom{n}{k}(p+\delta)^k\,[1-(p+\delta)]^{\,n-k} \tag{4}$$

using the ratio of $Q(p+\delta)$ to $Q(p)$

$$\frac{Q(p+\delta)}{Q(p)} = \left(\frac{p+\delta}{p}\right)^k \left(\frac{1-p-\delta}{1-p}\right)^{n-k} = \left(1+\frac{\delta}{p}\right)^k \left(1-\frac{\delta}{1-p}\right)^{n-k} \tag{5}$$

This formula gets interesting when $p = \frac{k}{n}$ (the best *a priori* guess for the center of the distribution), so we can use the basic substitutions $k = pn$ and $n-k = (1-p)n$ to yield

$$\frac{Q(p+\delta)}{Q(p)} = \left[\left(1+\frac{\delta}{p}\right)^p \left(1-\frac{\delta}{1-p}\right)^{1-p}\right]^n \tag{6}$$

The value within the square brackets is a curved function that peaks at 1 when $\delta = 0$. The overall function gets progressively narrower for larger values of $n$as expected. An example of this is shown in Figure XX for values $k = 70$ and $n = 162$ such that $p \approx 0.432$. The two curves show the value of the function within brackets as well as the overall value when $n = 162$.

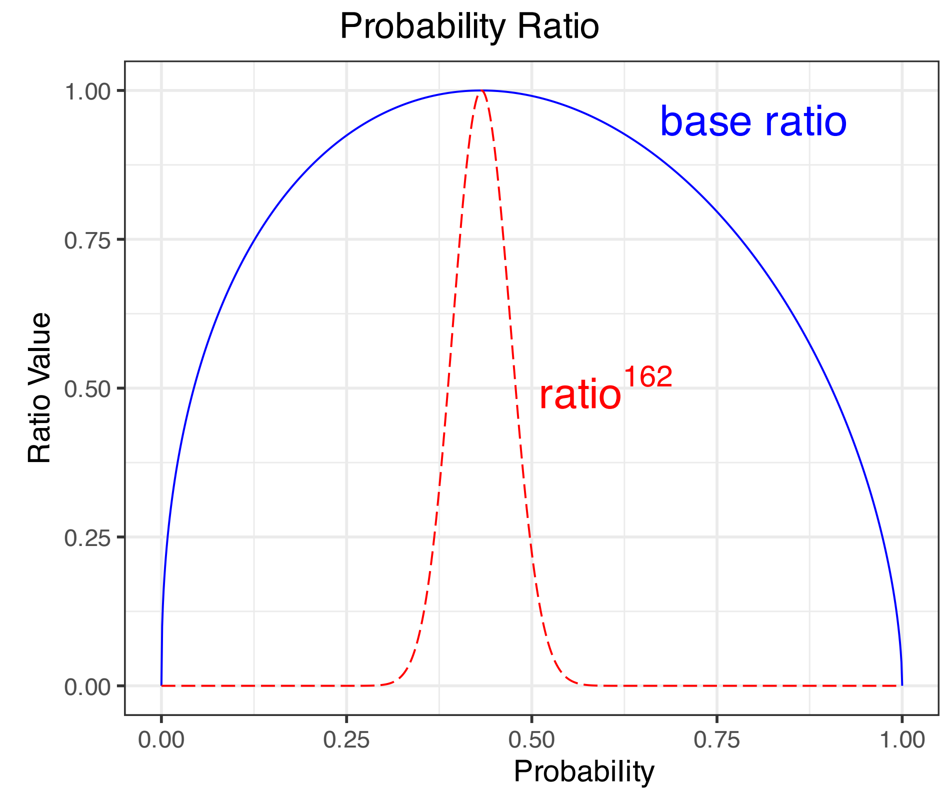


Figure 5: Sample case showing the base and full sample probability ratio when winning 70 games out of 160

B R Code for the Inverse Binomial

The following R language code returns the PDF (probability density function) and CDF (cumulative density function) for $n$successes out of N trials over the space specified by the variable *prob*. Note that the PDF values are correct for the given values of *prob*, but since they represent values on a curve they should be normalized such that the sum of all the PDF values is one when using them to compute averages.

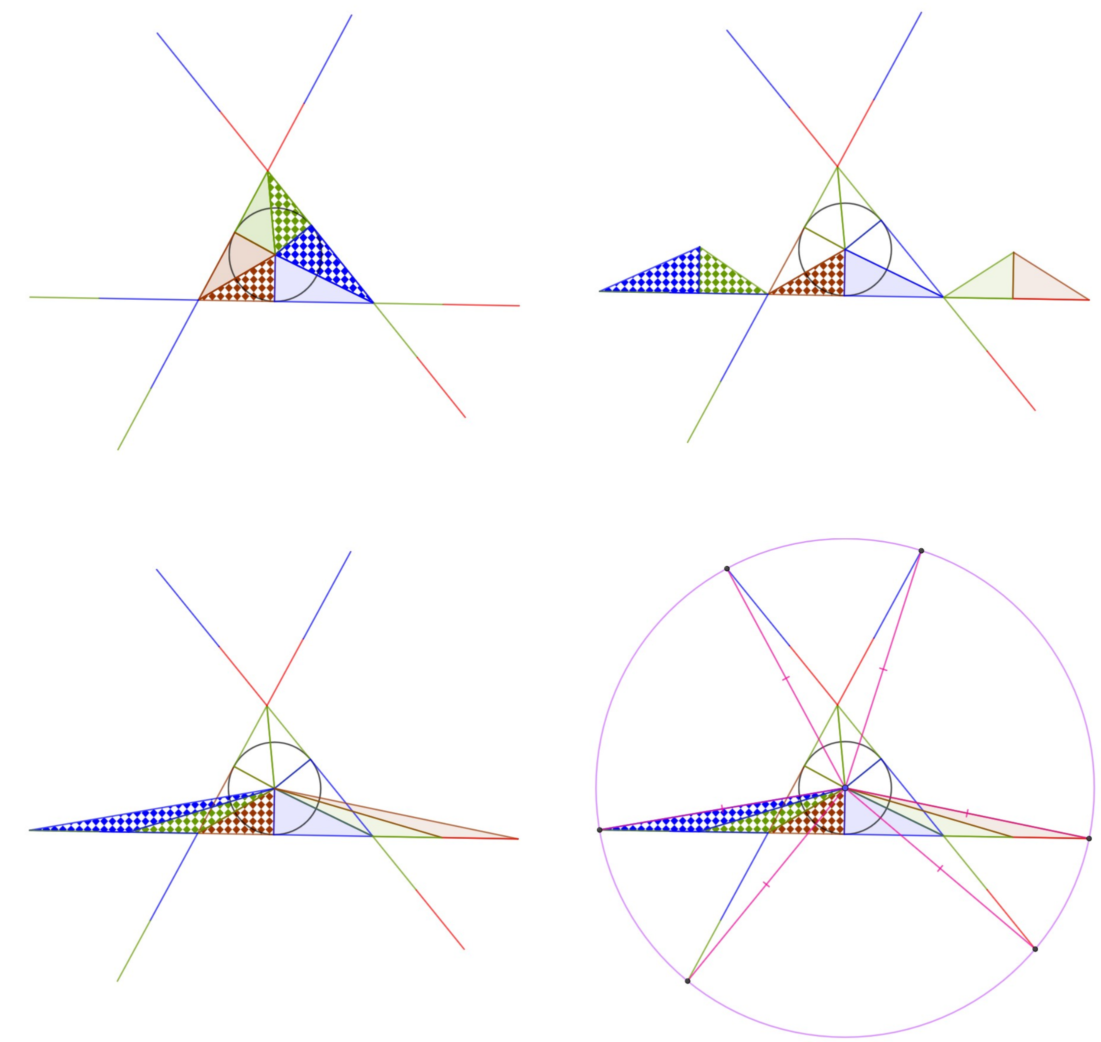
```
# compute the inverse binomial for N-choose-n centered around probabilities in
# the vector probs.  This returns a tibble whose columns are the values of prob,
# the pdf for each value in prob, and the cdf for each value in prob
require(tidyverse)
require(stats)
require(pracma)

invBinom <- function(N,n,prob=linspace(0,1,1001)) {
  # pick a suitable baseline probability to use for scaling
  # (it shouldn't be too close to the either zero or one)
  p0 <- min(c(max(c(0.1,n/N)),.9))
  # compute the baseline binomial probability at p0
  pdf0 <- dbinom(n,N,p0)
  # determine the offsets from p0
  eps <- prob-p0
  # implement (1+eps/p)^n * (1-eps/(1-p))^(N-n) in a more numerically stable way
  ratio <- (1+eps/p0)^(n/N)*(1-eps/(1-p0))^(1-n/N)
  pdf <- pdf0*ratio^N
  # approximate the cdf from the lower and higher edges of each interval
```

```
  cdf_low <- cumsum(pdf*diff(c(prob,1))*(N+1))
  cdf_hi <- cumsum(pdf*diff(c(0,prob))*(N+1))
  # return a tibble containing the pdf and cdf for each value in prob
  tibble(prob=prob,pdf=pdf,cdf=pmin(1,(cdf_hi+cdf_low)/2))
}
```

Conway's Circle Theorem: A Proof Without Words

Colin Beveridge

Conway's Circle Theorem:
When the sides meeting at each vertex of a triangle are extended by the length of the opposite side, the six endpoints of the three resulting line segments lie on a circle whose centre is the incentre of the triangle.



Animated version: https://bit.ly/g4g-cct

Sophie Germaine

and the

Mod n Cardioid

4 Zine printables

by Lhianna Bodiford

hand drawn art by Mahlon Bodiford

This PDF file is made up of "Parts 1-3" and "Extra Circles" zine-format printables. Plus "Parts 1-3 (low Ink) versions that do not have background color if you are printing in B&W or trying to use less ink.

These are intended to use with middle to high school aged students or adults as a fun exploration of Modular arithmetic. This concept is introduced through the story of Sophie Germain, and includes notes about mathematical thinking. They are intended to be used in series but each part can also stand alone.

You can print and cut-n-fold your own zine. It's easy! and I expect most people will enjoy the process. Be sure to check out online instructions for zines if you are new to zine folding. (No glue or staples required!)

Please share with friends, students, family..... and try it yourself!

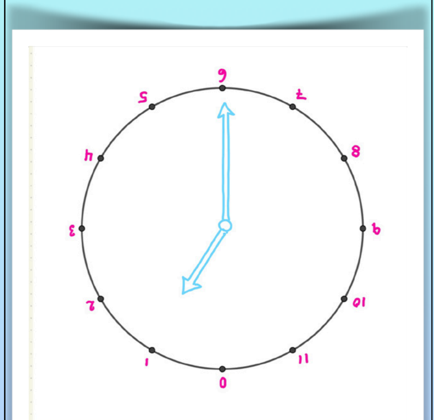
Thank you,

Lhianna Bodiford

TheLadyLhi@gmail.com

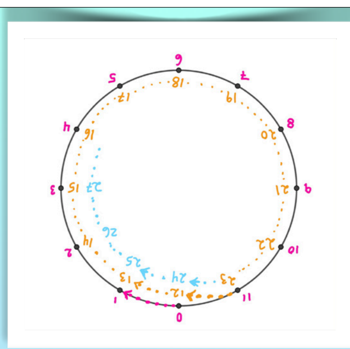
Sophie Germaine and the Mod n Cardioid

Part 2: ‘Exploring Her World’



Let’s take a break from the story to look at some of this colorful math...

- Have you ever seen numbers arranged in a circle like this before?....
- What does it remind you of?....

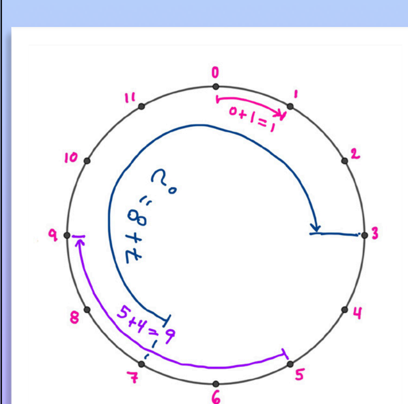


We call this “clock arithmetic” or “Modulo Arithmetic”

- A “clock” that goes from 0 -11 is called “Modulo 12”, or “**Mod-12**”.
- We could travel around the circle counting 1,2,3,4....etc until we get back to 0. But we don’t have to stop there!
- If we continue counting we would count 12 where the “0” is, 13 where the “1” is, 14 where the “2” is and so on.... We could keep going forever,... but maybe you get the idea.
- Count around the circle a few more times and label the numbers on the drawing. Do you notice any patterns?....

We can add the numbers around the circle just like we add regular counting numbers...

- 1+1 =2, 4+5=9,....etc. Try it!
- What about 7+8? What does 7+8 equal equal on the clock?...
- 3? 15?...27?
- Think about how you would add 16+24 using our Mod-12 ‘clock’.... what makes sense in this case?...

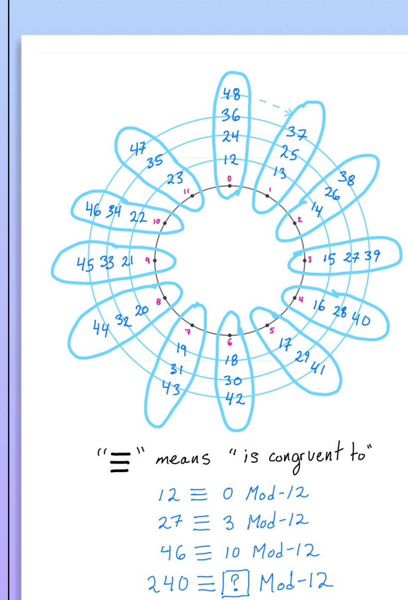


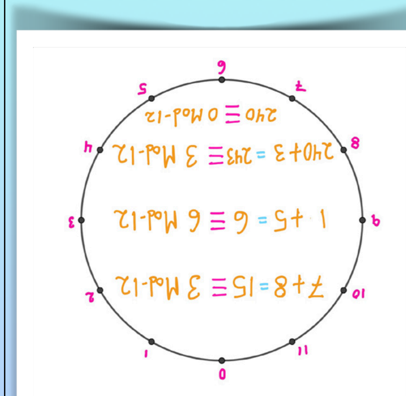
In Modular Arithmetic we focus on the inner circle, ‘clock numbers’, just like when we tell time...

- So 12 becomes 0 Mod-12,
- 13 becomes 1 Mod-12...etc.

- The **infinite family of numbers** that grows out from each inner circle number can all be thought of as equivalent to that single inner circle number.
- So 12, 24, 36... are all equivalent to 0 Mod-12.
- Likewise, 13, 25, 37... can all be thought of as equivalent to 1 Mod-12

- Mathematicians call this family of numbers a **congruence class**.
- We say: 37 “is congruent to” 1 Mod-12





Adding on a clock can be fun, but Modular Multiplying can be beautiful!....

(look for “Part 3: ‘Multiplying by Heart’”)

* * *for credits, links, references and questions, email: TheLadyLhi@gmail.com

Sophie Germaine and the Mod n Cardioid

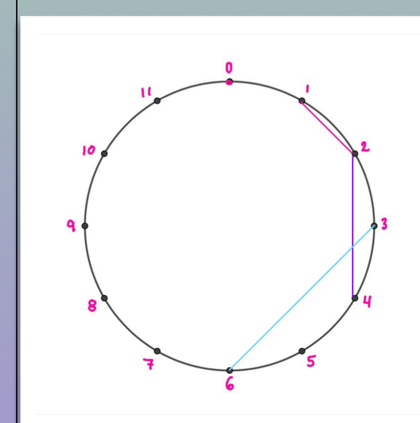
Part 3: ‘Multiplying by Heart’

Let’s connect the dots using a simple rule:

- Start at 0 and **multiply by 2**.
- 0 connects to itself because 0 * **2**=0 so we draw no line.
- Next multiply 1 * **2**=2
- **Connect 1—> 2** with a straight line.
- Next multiply 2 * **2**=4 and **connect 2—> 4** with a straight line.
- Now 3 * **2**=6 so **connect 3—> 6**.... And so on...

What happens at 6 * 2=12?...

- Remember 12 is in the same family as 0
- so 6 * 2=0 Mod-12
- We connect 6 —> 0 with a straight line.
- Can you see why 7—>2 ?...



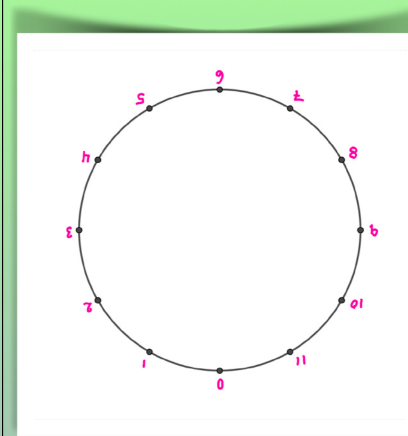
Keep going until you are repeating lines.

* *It may help to keep track of your regular and Mod-12 multiplication in a table like this one, on a separate paper.

#	*2	*2 Mod-12
1	2	2
2	4	4
3	6	6
...	...	...
7	14	2

Now step back and take a look at your lines...

- what do you see? ...
- Any patterns? shapes? symmetries?...
- Discuss your ideas with a friend...
- Write down what you observe....



What do you think will happen with a different number of dots?

- What about 32 dots?...
- Try multiplying each number by 2 again and connecting to the answer Mod-32
- What happens now?
- What is different?
- What is the same?
- Will this always happen?

- Try multiplying by 3 this time.... or 4... Experiment!

- Can you make any predictions?

Why is this happening?...

*You can use the Zine of extra circles to experiment.
or try one of these websites to play around:

https://mathsbot.com/tools/cardioids

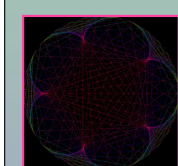
https://lengler.dev/TimesTableWebGL/

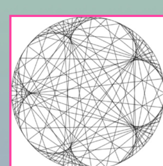
So... on the previous page we asked “why is this happening?”..... up until now we have presented some math of multiplication. It may have felt challenging, or easy, depending on the knowledge you had going into this. We were practicing multiplication tables...in an unconventional way... and finding beautiful shapes in the process. Nice.

...But something surprising happened! A beautiful shape appeared and we saw symmetry! And repeating patterns!... and maybe you were wondering “why did this happen?” ....
And THIS! This question: “WHY did this happen?”, is the START of your adventure!

You have seen something beautiful. You can let it alone and just admire it or forget about it. But if you ask “why” and try to answer... **NOW** you are doing mathematics!

Warning! There is no one here to tell you which way to go! You have friends to talk things out with. Perhaps a guide to give you hints and help sometimes when you are really lost.... The true idea here is that the adventure is all yours!





“How the heck...?”....
Likely you have no idea where you are going or how to start.... That’s ok... The first step in every math adventure is to write down what you know so far, and then to stumble about. Mess around. Experiment, until you find something that looks like something....

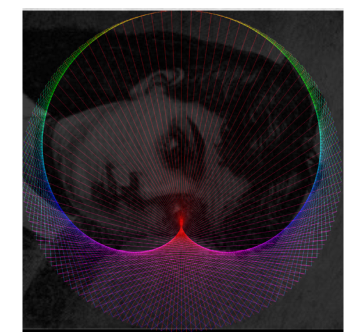
*Write down and organize your observations.
*Talk to others.
*Can you make any new conclusions with your observations?
*keep messing around....
* “wait, what was I looking for?”... go back and check in with your original question. Does it need revising? Are you getting any closer to an answer?.... Keep messing around!

*** * *Math is MESSY, and you are the one making the map! * * ***

* * *for credits, links, references and questions, email: TheLadyLhi@gmail.com

Sophie Germaine and the Mod n Cardioid

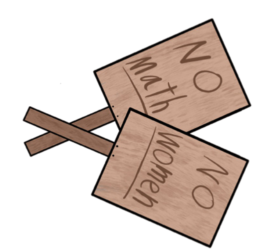
Part 1: 'The Story'

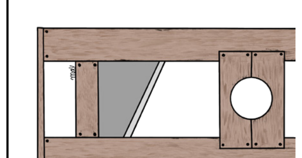


Our Story Begins...

"Once there was a young girl who wanted to study math... who was curious about the world... who found strange and unusual books on her father's bookshelf....

...but she was not allowed to study! ....you may get sick of your parents, or teachers telling you to study math... imagine if, instead, they were telling you to stay away from math because studying it would hurt your brain and make you sick....





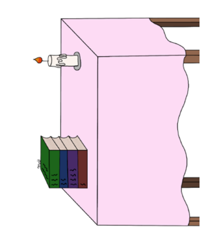
Sophie livid during the French Revolution and was made to stay indoors much of the time to protect her from the violence and chaos of the streets.

She was not given a formal education because she was a girl. Yet she had full access to her father's library and studied whatever she could find.

She was particularly drawn to his math books...

But when her parents found out she was studying math they ordered her to stop! They said that math was too dangerous for a female and would hurt her delicate brain and make her sick. She was so curious about what she had seen in the books on the shelf that she ignored her parents and tried to study anyway. She had to stay up late and sneak into the library in the middle of the night and study by candlelight. When her parents discovered her one night, they took away her candles and pajamas to discourage her from studying at night in the cold.....

Sophie just studied in the cold wrapped in blankets and using bits of candle she had hidden away.



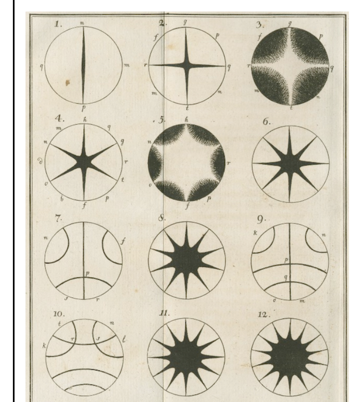
Math is rebellion!

Math is seeing the path laid out before you and choosing to take a different route.... An uncharted path.... Because you can see something that might be exciting off in the distance.

Do you ever just get curious about something and pick it up and start messing around to find out?...

Or have a thought that isn't quite clear so you get out pencil and paper to sketch your ideas?....

That is what math is....



Although Sophie had to push through these hardships to learn what she wanted, she also had an advantage.

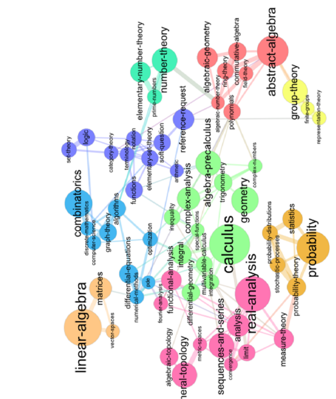
She did not have anyone telling her what to learn.

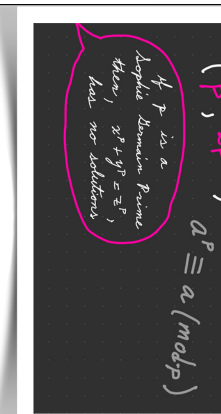
All she had was her own curiosity. Her study was guided by the questions she asked herself.

Instead of proceeding on a fixed path of knowledge, she was exploring the landscape of math on an adventure all her own!

Navigating the mathematical Landscape...

It is incredibly difficult to both ask and answer your own questions in mathematics. However many people find this kind of adventure so rewarding that they are willing to climb the mountains of difficulty.





Sophie's curiosity fueled her lifelong journey studying mathematics...

She spent many years attempting to solve Fermat's famous "Last Theorem".

She did not solve it, but along he way she discovered some beautiful patterns in primes and some powerful ideas in Number Theory using something called a 'PRIME MODULUS'....

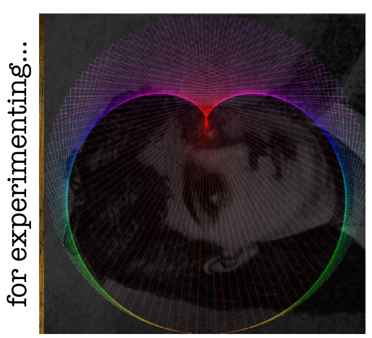
(the adventure continues in "Part 2: Exploring Her World")

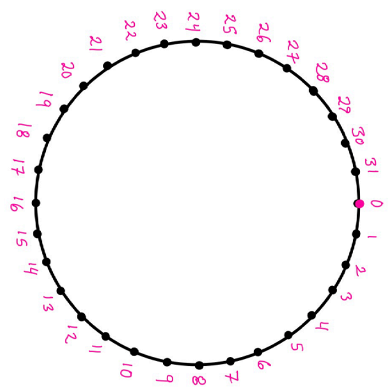
** hand drawn art credit: Mahlon Bodiford

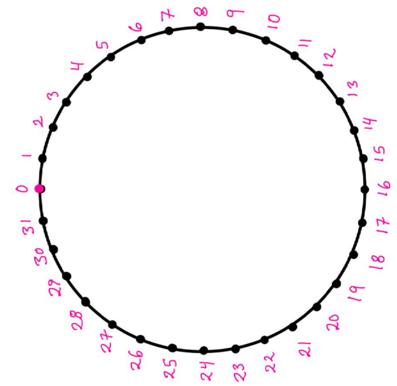
***for other credits, links, references and questions, email: TheLadyLhi@gmail.com

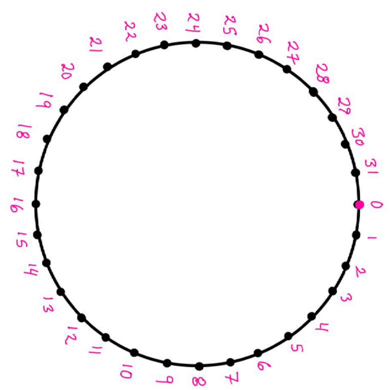
Sophie Germaine and the Mod n Cardioid

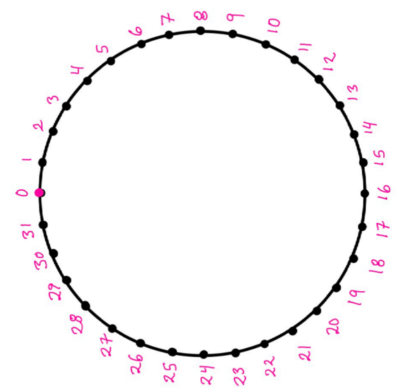
Extra Circles for experimenting...

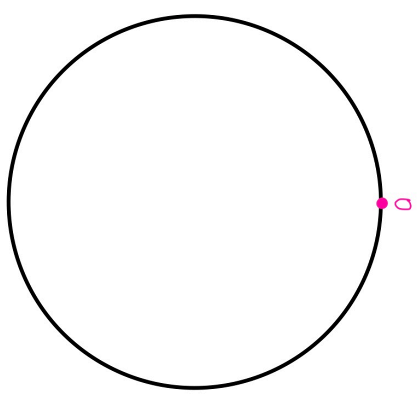


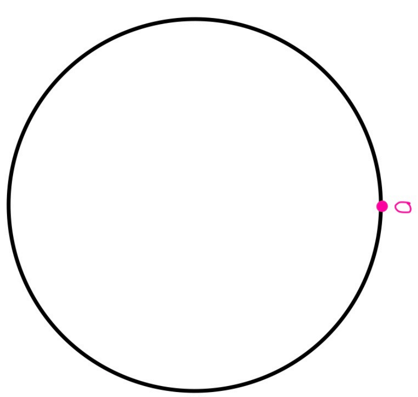


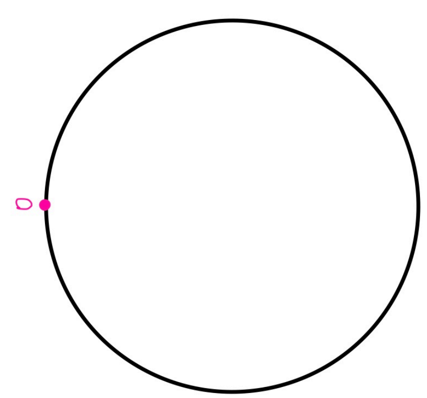










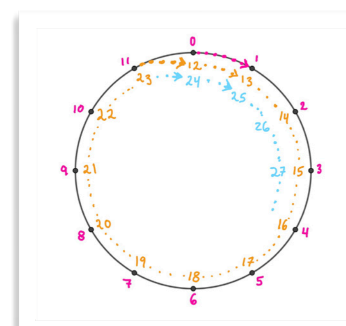


Sophie Germaine and the Mod n Cardioid

Part 2: 'Exploring Her World'

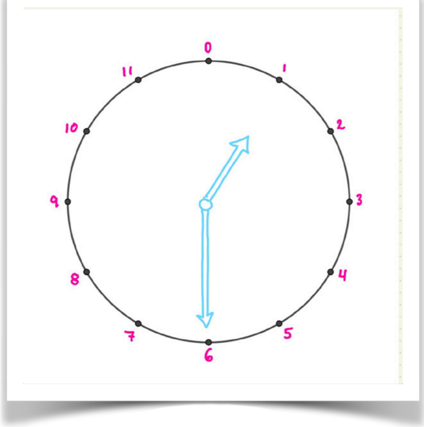
We call this "clock arithmetic" or "Modulo Arithmetic"

- A "clock" that goes from 0 -11 is called "Modulo 12", or "**Mod-12**".
- We could travel around the circle counting 1,2,3,4....etc until we get back to 0. But we don't have to stop there!
- If we continue counting we would count 12 where the "0" is, 13 where the "1" is, 14 where the "2" is and so on.... We could keep going forever,... but maybe you get the idea.
- Count around the circle a few more times and label the numbers on the drawing. Do you notice any patterns?....



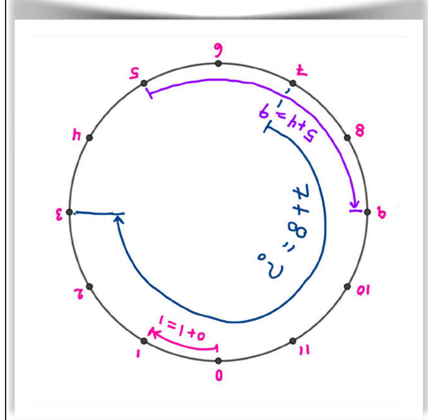
Let's take a break from the story to look at some of this colorful math...

- Have you ever seen numbers arranged in a circle like this before?....
- What does it remind you of?....



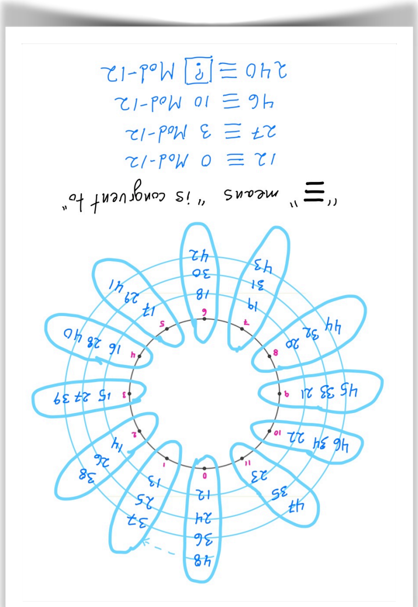
We can add the numbers around the circle just like we add regular counting numbers...

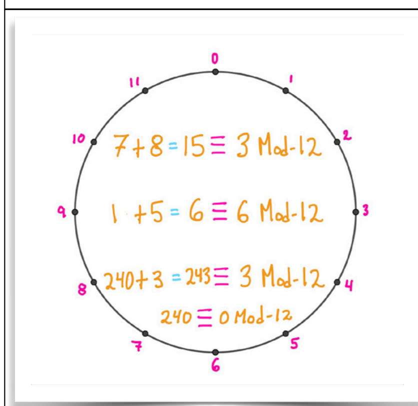
- 1+1 =2, 4+5=9,...etc. Try it!
- What about 7+8? What does 7+8 equal on the clock?...
- 3? 15?...27?
- Think about how you would add 16+24 using our Mod-12 'clock'.... what makes sense in this case?...



In Modular Arithmetic we focus on the inner circle, 'clock numbers', just like when we tell time...

- So 12 becomes 0 Mod-12,
- 13 becomes 1 Mod-12...etc.
- The **infinite family of numbers** that grows out from each inner circle number can all be thought of as equivalent to that single inner circle number.
- So 12, 24, 36... are all equivalent to 0 Mod-12.
- Likewise, 13, 25, 37... can all be thought of as equivalent to 1 Mod-12
- Mathematicians call this family of numbers a **congruence class**.
- We say: 37 "is congruent to" 1 Mod-12





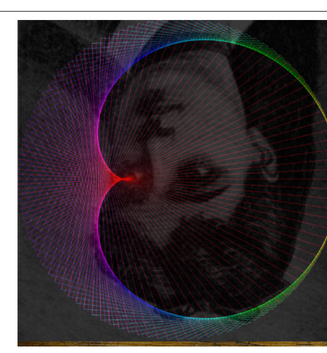
Adding on a clock can be fun, but Modular Multiplying can be beautiful!....

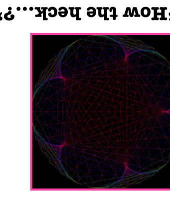
(look for "Part 3: 'Multiplying by Heart'")

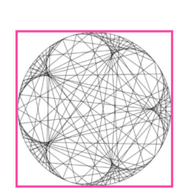
* * *for credits, links, references and questions, email: TheLadyLhi@gmail.com

Sophie Germaine and the Mod n Cardioid

Part 3: 'Multiplying by Heart'







"How the heck...?"....

Likely you have no idea where you are going or how to start.... That's ok.... The first step in every math adventure is to write down what you know so far, and then to stumble about. Mess around. Experiment, until you find something that looks like something....

*Write down and organize your observations.
*Talk to others.
*Can you make any new conclusions with your observations?
*keep messing around....
* "wait, what was I looking for?"... go back and check in with your original question. Does it need revising? Are you getting any closer to an answer?.... Keep messing around!

*** * * Math is MESSY, and you are the one making the map! * * ***

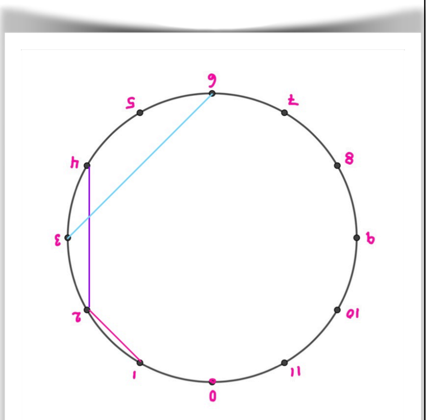
* * *for credits, links, references and questions, email: TheLadyLhi@gmail.com

Let's connect the dots using a simple rule:

- Start at 0 and **multiply by 2.**
- 0 connects to itself because 0*2=0 so we draw no line.
- Next multiply 1*2=2
- **Connect 1—> 2** with a straight line.
- Next multiply 2*2=4 and **connect 2—> 4** with a straight line.
- Now 3*2=6 so **connect 3—> 6**.... And so on...

What happens at 6*2=12?...

- Remember 12 is in the same family as 0
- so 6*2=0 Mod-12
- We connect 6 —> 0 with a straight line.
- Can you see why 7—>2 ?...



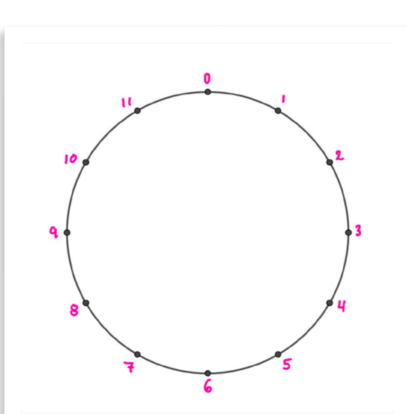
Keep going until you are repeating lines.

* *It may help to keep track of your regular and Mod-12 multiplication in a table like this one, on a separate paper.

#	*2	*2 Mod-12
1	2	2
2	4	4
3	6	6
...	...	...
7	14	2

Now step back and take a look at your lines...

- what do you see? ...
- Any patterns? shapes? symmetries?...
- Discuss your ideas with a friend...
- Write down what you observe....



What do you think will happen with a different number of dots?

- What about 32 dots?...
- Try multiplying each number by 2 again and connecting to the answer Mod-32
- What happens now?
- What is different?
- What is the same?
- Will this always happen?

- Try multiplying by 3 this time.... or 4... Experiment!

- Can you make any predictions?

Why is this happening?...

*You can use the Zine of extra circles to experiment.
or try one of these websites to play around:

https://mathsbot.com/tools/cardioids

https://lengler.dev/TimesTableWebGL/

So... on the previous page we asked "why is this happening?"..... up until now we have presented some math of multiplication. It may have felt challenging, or easy, depending on the knowledge you had going into this. We were practicing multiplication tables...in an unconventional way... and finding beautiful shapes in the process. Nice.

...But something surprising happened! A beautiful shape appeared and we saw symmetry! And repeating patterns!... and maybe you were wondering "why did this happen?" ....
And THIS! This question: "WHY did this happen?", is the START of your adventure!

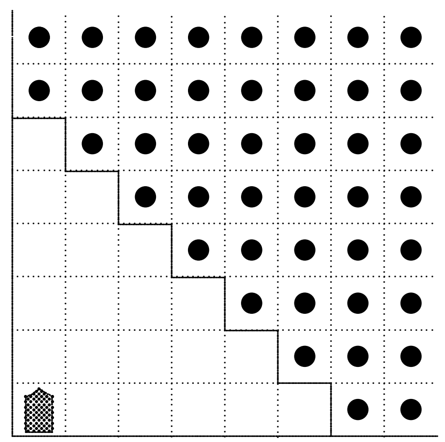
You have seen something beautiful. You can let it alone and just admire it or forget about it. But if you ask "why" and try to answer... **NOW** you are doing mathematics!

Warning! There is no one here to tell you which way to go! You have friends to talk things out with. Perhaps a guide to give you hints and help sometimes when you are really lost.... The true idea here is that the adventure is all yours!

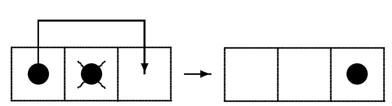
The Great Beetle Escape

Haoran Chen and **Yoshiyuki Kotani**

An evil wizard casts a spell on the beetles and transports them to a world which consists of the first quadrant of the plane divided into unit squares. There is a black magic zone at the southwest corner, consisting of 21 squares separated from the remaining squares by a zigzag fence. There are no beetles inside the zone, but there is one in each square outside. A black tower stands at the southwest corner square. If any beetle can reach the black tower, the spell will be broken and all surviving beetles will be released. In that case, the Great Escape is successful.



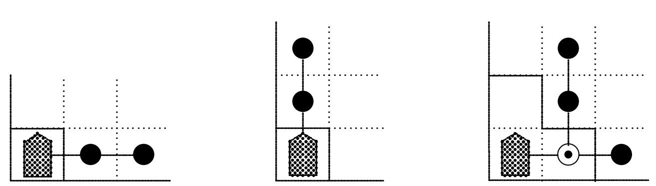
Let us first describe the movement of a beetle. It can only hop over another beetle in an adjacent square in the same row or column, landing on a square beyond which must be vacant at the time. The beetle being hopped over is crushed into the ground, perishes and vanishes. The fence does not hinder hopping.



Section A. Initial Attempts.

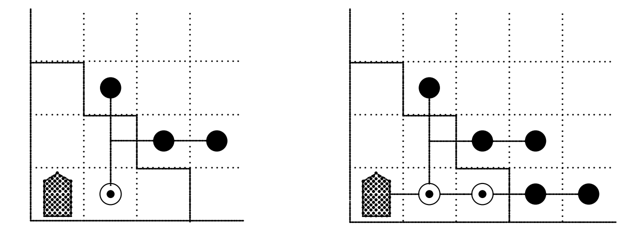
The beetles hold a council of war. The task on hand seems daunting. So they decide to work on simpler cases first, to gain some insight on the problem.

Case 1. The zone has only 1 square.
The diagram below on the left or in the middle shows that two beetles are sufficient to get one of them to the black tower.

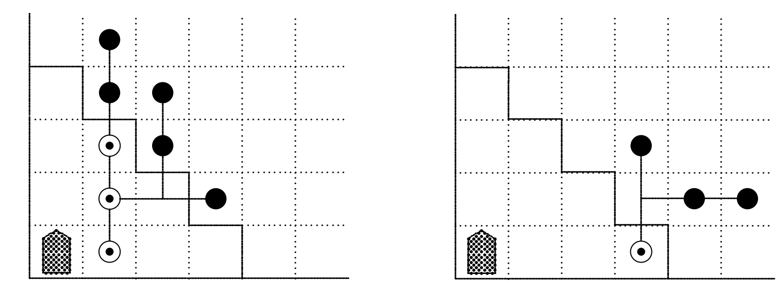


Case 2. The zone has 3 squares.
Copying the solution in the diagram above in the middle will put one beetle in the zone. Copying the solution in the diagram above on the left will get a third beetle to the black tower. Thus three beetles are required, as shown in the diagram above on the right.

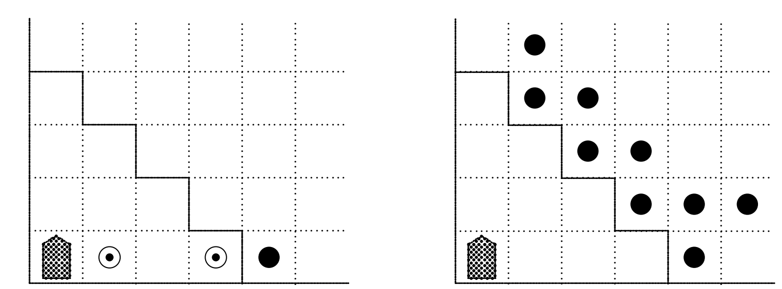
Case 3. The zone has 6 squares.
The first objective is to get a beetle next to the black tower. This can be accomplished using the solution to Case 2, as shown in the diagram below on the left. The next objective is to get a beetle next to the first beetle. This can be accomplished using the solution to Case 1, as shown in the diagram below on the right. Altogether, 5 beetles are involved.



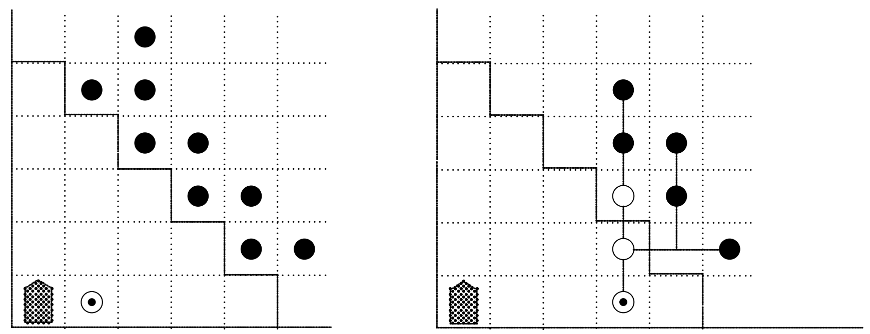
Case 4. The zone has 10 squares.
In the diagram below on the left, the solution to Case 3 is used to put a beetle inside the zone. In the diagram below on the right, the solution to Case 2 is used to put another beetle inside the zone.



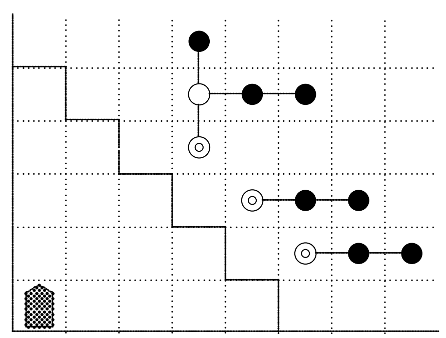
These two beetles provide the stepping stones for another beetle to reach the black tower, as shown in the diagram below on the left. Altogether, 9 beetles are involved. Their starting positions are shown in the diagram below on the right.



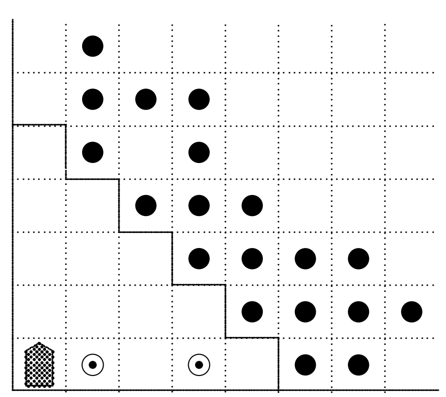
Case 5. The zone has 15 squares.
In the diagram below on the left, the solution to Case 4 is used to put a beetle inside the zone. In the diagram below on the right, the solution to Case 3 is used to put another beetle inside the zone.



There are three beetles which are involved in both sides of the diagram above. They are marked by double circles in the diagram below. They may be duplicated by using the solutions to Cases 1 and 2.



There are now two beetles inside the zone. A third may be added on the bottom row just outside the zone. Together, these three beetles provide the stepping stones for another beetle to reach the black tower. Altogether, 19 beetles are involved. Their starting positions are shown in the diagram below.



Section B: Interlude.

The beetles have now come to Case 6, which is the actual task on hand. However, following the same strategy is getting much harder as the number of "overlapping" beetles becomes too large to manage. The council of war decides to take a break, and consider a theoretical question.

Clearly, if the zone gets larger and larger, there will come a point when the Great Escape is doomed to failure. Would that start with Case 6? Not daring to face the truth right away, the beetles decided to postpone it and consider the next case.

Case 7. The zone has 28 squares.

Suppose the Great Escape succeeds. All beetles that are not involved can be ignored, so that there will only be a single beetle in the black tower at the end. Assign it the value 1. It gets to this position by hopping over another beetle. Assign x to the crushed beetle and y to the hopping beetle before making its move. After the move, the beetle in the new position replaces the other two. It is desired that $x+y=1$, so that the total value of the beetles remains constant. The closer a beetle is to the black tower, the more valuable it is. Hence $1>x>y$. Suppose a beetle with value z crushes the one of value y and becomes one with value x. Then $y+z=x$. Take $y=x^2$ with $x+y=x^2+x=1$ and $z=x^3$. Indeed, $y+z=x^3+x^2=x(x^2+x)=x$.

Clearly, the value of a beetle is determined by its location. So values may be assigned directly to the squares themselves. These are shown in the diagram below. The extension to the squares not shown is obvious.

x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}
x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}
x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}
x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
1	x	x^2	x^3	x^4	x^5	x^6	x^7

The total value of the squares outside the fence is

$$S=8x^7+9x^8+10x^9+11x^{10}+\cdots.$$

Then

$$xS=8x^8+9x^9+10x^{10}+\cdots.$$

Subtracting this equation from the preceding one, we have

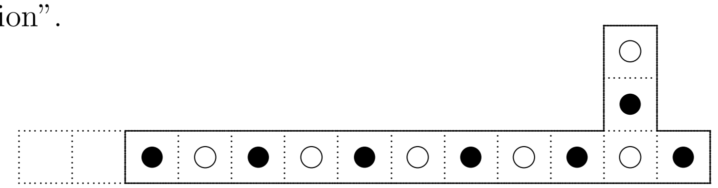
$$(1-x)S=8x^7+x^8+x^9+x^{10}+\cdots=7x^7+\frac{x^7}{1-x}.$$

Recall that $x^2+x=1$, so that $1-x=x^2$. Hence $x^2S=7x^7+x^5$ and $S=7x^5+x^3=37x-22<1$ since $x=\frac{\sqrt{5}-1}{2}<0.62$. Thus there are not enough beetles to reach the black tower.

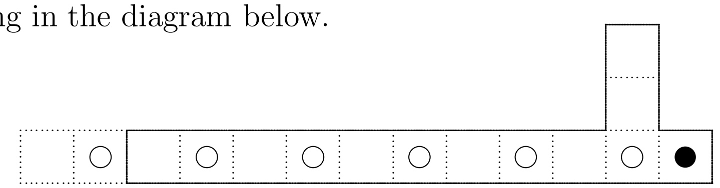
Section C: The Final Assault.

A similar calculation for Case 6 shows there are just enough beetles for the task, though that in itself does not guarantee the success of the Great Escape. An actual algorithm must be developed.

In the solution to Case 2, the beetles' formation resembles the capital letter T. They now stretch one end of the T-bar into a long "handle", as shown in the diagram below. This is called a "generalized T-formation".

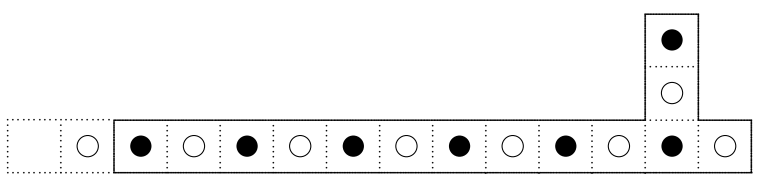


In operation, all beetles marked by white circles hop over their neighbors to the left or immediately beneath, resulting in the diagram below.

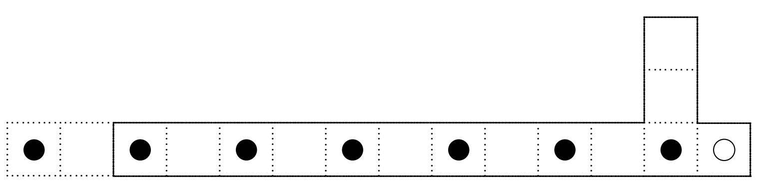


The lone beetle marked by a dark circle can now make a sequence of hops and end up two squares beyond the handle.

Suppose there is a beetle immediately beyond the handle, as shown in the diagram below. It apparently blocks the eventual projection of a beetle two squares beyond the handle.



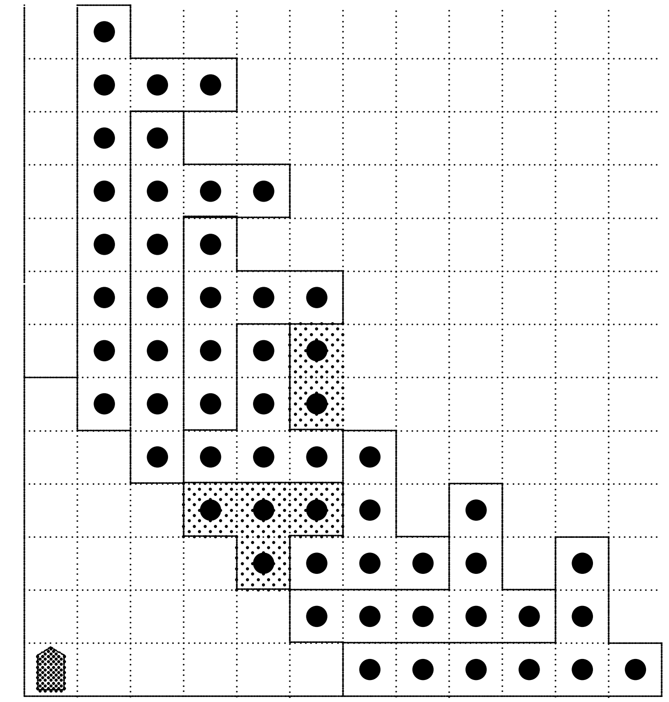
This time, all beetles marked by dark circles hop over their neighbors to the left or immediately beneath, resulting in the diagram below.



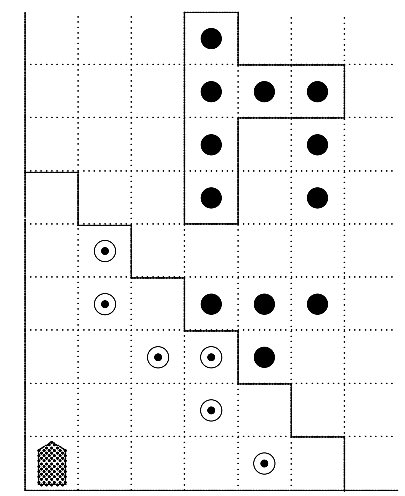
The lone beetle marked by a white circle can now make a sequence of hops and replace the "blocking" beetle.

The advantage of a generalized T-formation is that the length of its handle can be stretched arbitrarily. This is critical in avoiding "overlapping" beetles.

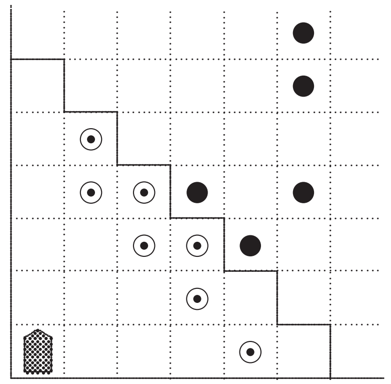
After much work, the beetles come up with a battle plan involving 50 beetles in seven generalized T-formations, plus 6 individual beetles for a total of 56, as shown in the diagram below.



The diagram below shows the result after all but one of the generalized T-formations have projected their beetles into the zone.



We use (i, j) to denote the square in the ith column from the left and the j row from the bottom. Thus the black tower is on (1,1). Now the beetle on (5,4) hops over the beetle on (4,4) and lands inside the zone on (3,4). The beetle on (4,4) is then replaced by the one projected from the remaining generalized T-formation, as shown in the diagram below.



The following chart shows the last 11 hops of this successful Great Escape.

1. (3,4) over (3,3) to (3,2)
2. (4,2) over (3,2) to (2,2)
3. (2,5) over (2,4) to (2,3)
4. (2,3) over (2,2) to (2,1)
5. (4,4) over (4,3) to (4,2)
6. (6,7) over (6,6) to (6,5)
7. (6,5) over (6,4) to (6,3)
8. (6,3) over (5,3) to (4,3)
9. (4,3) over (4,2) to (4,1)
10. (5,1) over (4,1) to (3,1)
11. (3,1) over (2,1) to (1,1)

Color Contrast Contortions

Barry Cipra
bcipra@rconnect.com

Last year (2023) I was reading Josef Albers's classic book *Interaction of Color*, and got to wondering about questions of color contrast: what you perceive when different colors are adjacent. A standard way of studying color contrasts is by juxtaposing squares of different colors. Doing so presents a plethora of possible problems. I went down one particular path. It's likely other directions lead to problems that are equally interesting, if not more so. I'll content myself here with describing what I wound up pondering in playing around with arrangements of colored squares.

The main thing I settled on was to produce arrangements that contain all possible color contrasts—i.e., for any pair of colors, there should be a pair of adjacent squares of those two colors. This condition is vacuous, of course, if there's just one color, and trivial if there are two:

Figure 1. For a two-color contrast, a simple domino suffices.

It's a little less trivial with three colors: putting Red, Blue, and Green in a row only gives two of the three color contrasts; to get the third you need one more square:

Figure 2. For a three-color contrast, four squares are needed.

It's worth pointing out that you actually can get all three contrasts with just three squares if you have a Green square that straddles the Red-Blue pair:

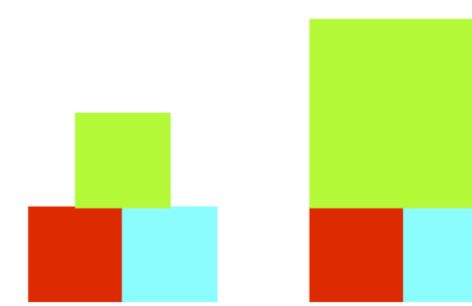


Figure 3. These arrangements of squares also "solve" the problem, but they're not what we have in mind.

But I hope you'll agree it's fair to disallow such shenanigans (though, of course, allowing them may be one of the interesting paths not traveled). In other words, it seems natural to pose the problem in terms of coloring the squares of a *polyomino*.

Similarly, it seems fair to disallow arrangements that have adjacent squares of the same color, so the three-color arrangement below on the left is OK, but not the one on the right:

Figure 4. The arrangement on the left looks fine. Not so much for the one on the right.

With four colors things begin to get a little interesting. Here, for example, is a string of squares with all six color contrasts for Red, Blue, Green and Yellow:

Figure 5. An arrangement with all 6 contrasts among 4 colors. But Blue and Green are next to each other twice.

But note, there is a duplication here of the Blue-Green contrast. I decided to disallow duplications—i.e., I decided to look for arrangements of colored squares that contain all possible color contrast once *and only once* (or, more succinctly, *exactly* once). This became my central criterion.

The "exactly once" criterion for k colors requires there be exactly $\binom{k}{2}$ adjacencies of squares. That's a significant constraint. For one thing, it puts an absolute upper bound on the number of squares that can participate in a complete set of color contrasts, namely $k(k-1)$, which is what you get with $\binom{k}{2}$ disconnected dominoes, each domino being a pair of contrasting colors. If you insist that your arrangements be connected—that is, if you want your arrangement to be an honest-to-god polyomino (which, by convention, is defined to be rookwise connected)—the upper bound (for $k \geq 3$) is less, and this raises the first interesting question:

As a function of k, what is the largest number of squares in a "color-contrast polyomino"—i.e., a connected arrangement in which each color contrast among k colors occurs exactly once?

And, of course, the same question can be posed with "smallest" instead of "largest."

But wait, there's more: Even if you establish proper upper and lower bounds, it's not clear what happens with numbers in between. Could it be, for example, that there is a color-contrast polyomino with $k = 8$ colors of size 22 and one of size 24 but not of size 23? It seems doubtful but who knows?

Another direction I decided to explore was the possibility of "equi-color" color-contrast polyominoes—i.e., arrangements that use each color the same number of times. In this case it's not possible to get connected arrangements with $k = 3$ or $k = 4$ colors, but it is possible with $k = 5$, 6, and 7 (and it's always trivially possible if you don't require connected solutions, simply using dominoes).

Here are some arrangements with five colors, using two squares of each color:

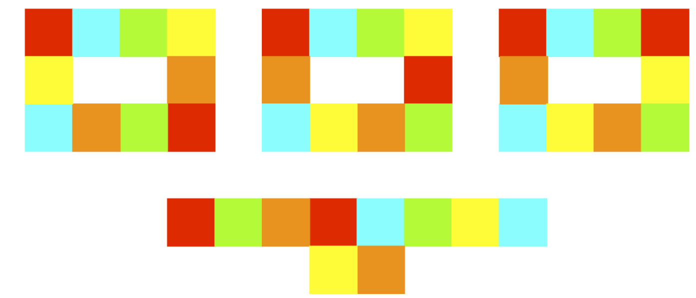


Figure 6. Four arrangements of five colors, with two copies of each color. Are the top three the "same" or "different"?

The top three arrangements here may look similar, but they are really quite different. In the one at left, there is a pair of squares of the same color—red—that are separated by four squares (both clockwise and counterclockwise); in the one at right each colored square is within three squares of its twin; and neither of these is true for the middle one. Consequently it's not possible to turn one arrangement into another by simply permuting the colors, or by advancing each square's color some number of squares clockwise (which you can think of essentially as turning the rectangle into a circle, rotating the circle, and then turning it back into a rectangle).

The colors can also be permuted in the fourth arrangement in Figure 6, of course, but there are other simple actions one can take to produce arrangements that differ from but are arguably the "same" as the fourth arrangement:

Figure 7. Two more arrangements of five colors with two copies of each color. Do they really differ from what's in Figure 6?

The arrangement at left can be thought of as swapping the red square at the left end of the fourth arrangement in Figure 6 and the yellow/blue domino from the right end, both of which are attached to green squares. The one at right moves the red square back below the leftmost green square and the leftmost blue square below the other yellow square. The reader can probably see many other simple rearrangements of such ilk, raising the question of how to say, mathematically, when two arrangements are "essentially the same" and when they're "distinctly different." For example, the two 9-color arrangements with four copies of each color in Figure 8 below are pretty clearly the same, but what's a succinct way to describe the steps that turn one into the other?

Figure 8. How would you describe, mathematically, the steps that turn the top arrangement into the bottom (in a way that might apply in other cases)?

It's worth noting that whenever the number of colors k is of the form $k = 4h+1$, then one can create rectangular arrangements using $2h = (k-1)/2$ copies of each color, such as the ones in Figures 8 and 6 for $k = 9$ and 5, respectively.

Clearly the number of copies of each color in any equi-color color-contrast polyomino must increase with the number of colors: With 9 colors, for example, there must be at least two squares of each color, since each color must appear next to the other eight colors but no square has more than four neighbors. In fact, since there are always squares on the outer edge that have fewer than four neighbors, the minimum with 9 colors is necessarily at least three copies of each color. But is there a three-copy arrangement with 9 colors? The reader is invited to find out.

All questions with regard to the existence of color-contrast polyominoes with various properties such as minimality or maximality of the numbers of squares are of NP complexity: If an arrangement exists, it is straightforward to verify, but proving arrangements (satisfying some additional constraints) don't exist can be a challenge.

It's easy to experiment forming color-contrast polyominoes with squares cut from colored construction paper. (Indeed, Albers recommends using colored paper rather than pigment and paint for the experiments he has in mind, because it "permits a repeated use of precisely the same color without the slightest change in tone, light, or surface quality.") But the "straightforward" process of verifying that some arrangement of colored squares is a color-contrast polyomino can be a bit of a pain. Even with just five colors I often found I was either omitting a contrast or duplicating one (and sometimes both), so anytime I had what I thought was a solution, I had to systematically check my work, which is kind of tedious; if you don't believe me, I urge you to make sure that the rectangles in Figure 8 really do exhibit all 36 color contrasts among the nine colors once and only once—don't just take my word for it!

It occurred to me that the verification process could be done at a glance if the colored squares all had notches cut into their four sides, and there were a batch of two-color diamonds to fill the holes created when squares are set adjacent:

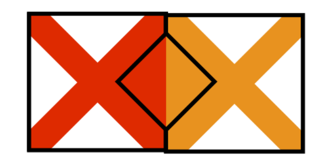


Figure 9. Filling the hole in adjacent notched squares with a two-colored diamond makes it easy to verify you have a legitimate color-contrast polyomino.

In this way, given a complete collection of $\binom{k}{2}$ diamonds and a set of notched squares, it's easy to see that you've used all the diamonds (and all the notched squares, if that's important to you), and any duplicate color contrasts will stand out as unfilled potholes in the arrangement. The final page here is a complete set of 15 diamonds for the color contrasts on six colors, with two notched squares of each color; the reader is invited to print it out (stiff paper works best, the stiffer the better) and play with it.

I showed this idea to my friend Loren Larson, who came back the next day with a beautiful version using different woods to represent the colors. Figure 10 shows an arrangement that uses all ten diamonds for five different woods. Loren's version extends to six colors with an additional five diamonds and two more notched squares, and to seven colors with a total of 21 diamonds and three notched squares for each type of wood.

Figure 10. Loren Larson's wooden version of the Color-Contrast Puzzle with five types of wood.

In addition to being a tactile pleasure, the rigidity of wooden pieces solves a problem that the reader may notice when working with paper, namely that even relatively stiff paper is hard to keep in place; wood does a much better job of staying put.

Loren, who is an accomplished mathematician in addition to being an expert woodworker, also answered the question asked earlier about the maximum number of squares in a color-contrast polyomino, showing the maximum to be $\binom{k}{2}+1$. Indeed, he showed that any polyomino with N squares and A adjacencies satisfies the equation

$$N = A + 1 - (I + C)$$

where I counts the number of "Interior" vertices of the polyomino and C counts the number of "Cavities"—i.e., regions that are completely surrounded by squares of the polyomino—with definitions of I and C suggested by the example in Figure 11, which has $N = 24$ squares, $A = \binom{8}{2} = 28$ adjacencies, $I = 4$ interior points, and $C = 1$ cavity. (The white "holes" in the Figure 11 polyomino are not cavities, because they are not completely surrounded by colored squares. Note, you can think of an interior point as a kind of cavity, because it's completely surrounded by the four squares for which it's a common vertex.) Since $I + C$ is necessarily non-negative, $\binom{k}{2} + 1$ is an upper bound on the number of squares in a k-color color-contrast polyomino. Loren (and I) then found various ways to achieve the upper bound.

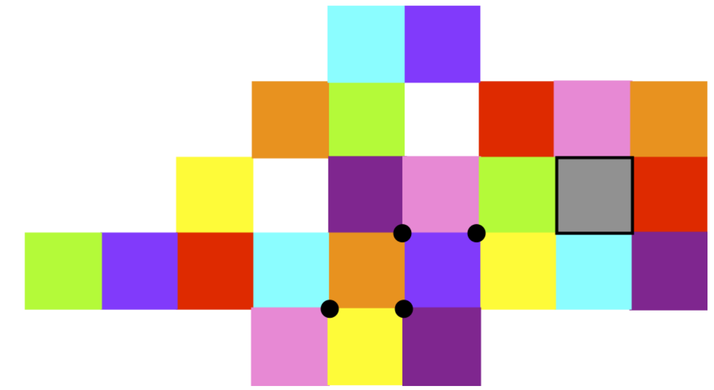


Figure 11. An 8-color color-contrast polyomino with $N = 24$ squares, $I = 4$ interior vertices (black dots) and $C = 1$ Cavity (gray square).

One way to achieve the upper bound, if k is odd, is to follow an Euler circuit on the complete graph on k vertices, with each vertex correponding to one of the k colors, producing a long string of colored squares whose colors correspond to the sequence of vertices visited in the circuit, as indicated for $k = 5$ in Figure 12. This approach adapts to even values of k (as indicated for $k = 6$ in Figure 12), by deleting one edge per vertex of the complete graph, producing a string of squares for an Euler circuit on the reduced graph (whose vertices now all have even degree), and then studding that string on top and/or bottom with squares to account for the missing contrasts, which correspond to the edges that were removed from the complete graph.

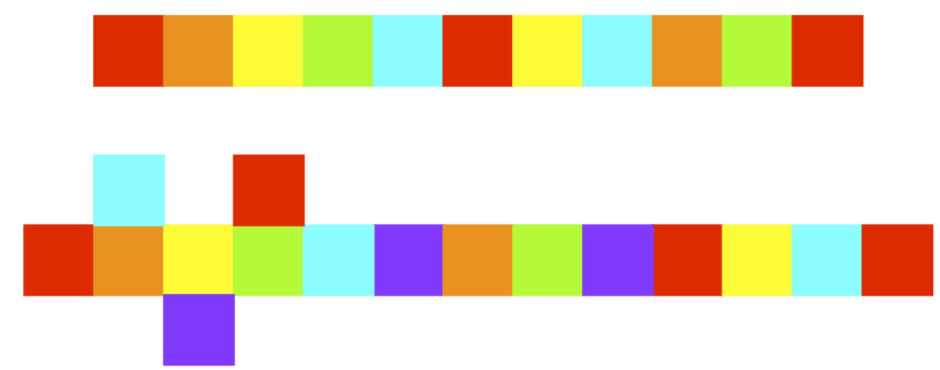


Figure 12. Maximal color-contrast polyominoes for 5 and 6 colors, respectively.

The "minimal" version of the problem seems to be a good deal more difficult. It's not hard to show that the minimum number of squares for $k = 2, 3, 4, 5,$ and 6 colors are $2, 4, 6, 9,$ and 12. A little more work gives the value 15 for $k = 7$. (Loren's formula implies that a polyomino with 14 squares and $\binom{7}{2} = 21$ adjacencies would have to have 8 interior points and/or cavities; colored or not, no such polyomino exists.)

These values led me to the sequence A278299 in the Online Encyclopedia of Integer Sequences (oeis.org), which was contributed in 2016 by Alec Jones and Peter Kagey. Their sequence, which continues with $19, 24, 30$, and 34 for $k = 8, 9, 10$, and 11, is defined to be the smallest number of squares in a polyomino that contains each color contrast at least once, rather than exactly once, so for now at least it's only a lower bound for our sequence. It's plausible, perhaps even likely, that the lower bound is sharp—Jones's and Kagey's OEIS entry includes an example with 10 colors that includes duplicate color contrasts (though Loren noticed an easy way to move two of its squares so as to eliminate the duplicates)—but it's also plausible that our no-duplicate constraint is strict enough to necessitate some additional squares.

Regarding the question of determining whether the squares of a given polyomino can be colored so that each color contrast occurs, either at least once or exactly once, Lily Chung at MIT pointed out a connection with "complete" and "exact" colorings of graphs—i.e., determining when the vertices of a graph can be colored so that for each color pair there is "at least" or "exactly" one edge connecting vertices of those two colors (and, as usual, there are no edges connecting vertices of the same color). The graphs here simply amount to using the centers of a polyomino's squares as the vertices, with edges connecting the centers of adjacent squares—what we might call "grid graphs."

According to the Wikipedia article on exact coloring, Keith Edwards has shown that the problem is NP-complete in general, even when restricted to graphs that are trees, but is solvable in polynomial time on graphs of bounded degree. It's unclear (to me at least) what happens with grid graphs, which are certainly of bounded degree (no vertex has more than four neighbors) but are not trees (except when $I + C = 0$).

The difficulty of determining whether a polyomino can be "exactly" colored (i.e., turned into a color-contrast polyomino) is nicely illustrated by an example that Loren discovered: Each of the polyominoes in Figure 13 has 15 squares and 21 adjacencies, so each is a possible color-contrast polyomino on $k = 7$ colors. Two of them actually can be exactly colored, while one cannot, and of the two that can, one can be exactly colored in just one way (aside from permuting the colors) while the other can be exactly colored in two different ways. The reader is invited to figure out which is which.

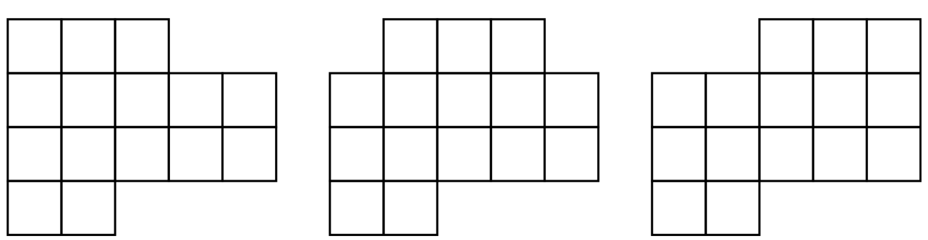


Figure 13. How many different ways can each of these be turned into a color-contrast polyomino?

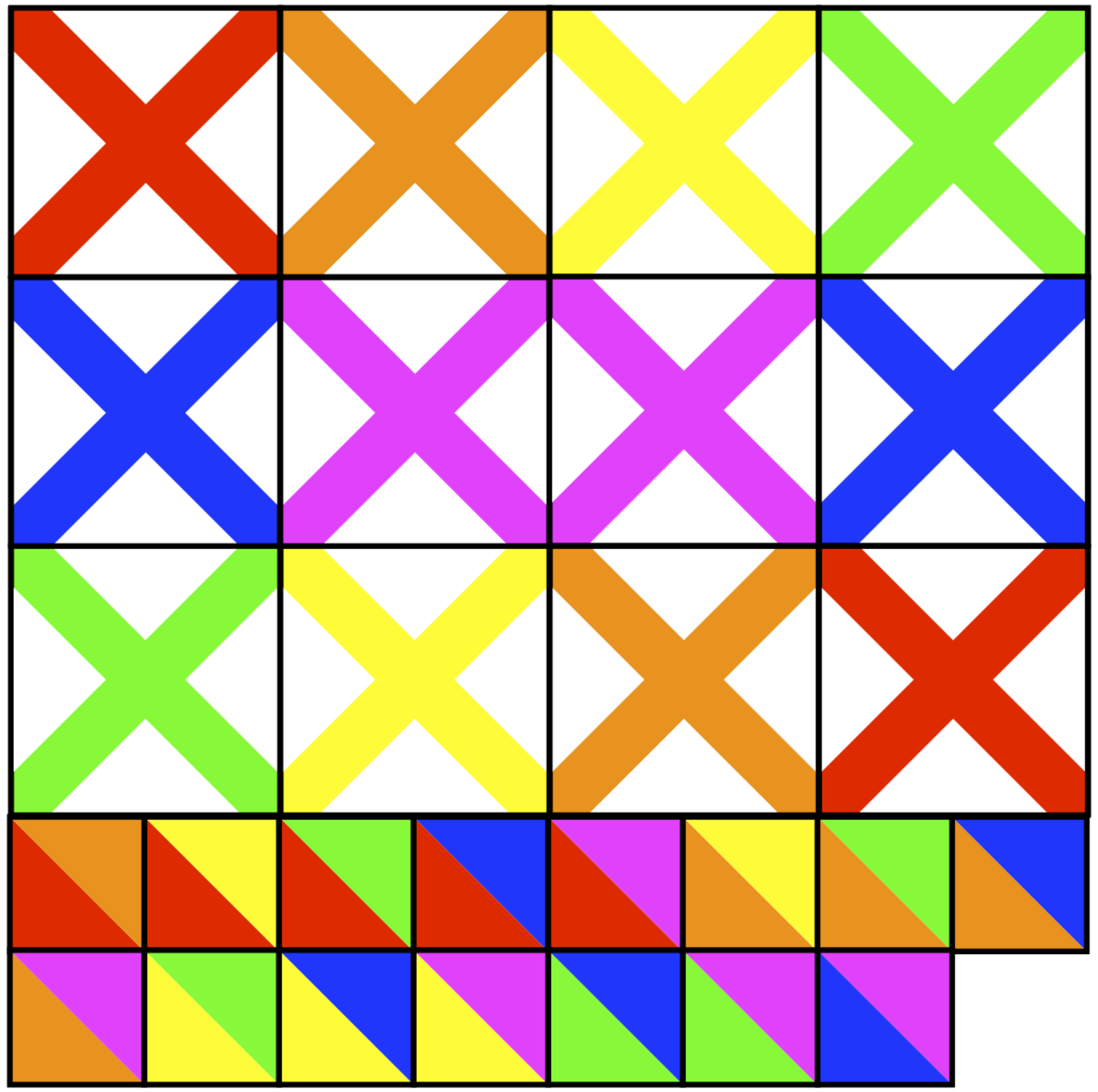
The status of the equi-colored color-contrast problem is similarly uncertain. Loren's equation tells us something about the number of interior points and cavities: if m copies of k colors are used, we must have

$$m = \frac{k-1}{2} + \frac{1-I-C}{k}$$

which in particular implies (for $k > 2$) that there will be at least one interior point or cavity (and, for odd k, a positive number congruent to 1 mod k). On the other hand, we must have $k - 1 \leq 4m$. In fact, since polyominoes always have "corner" squares with only two neighbors, we must have $k - 1 \leq 4m - 2$. Together these tell us

$$\left\lceil \frac{k+1}{4} \right\rceil \leq m \leq \left\lfloor \frac{k+1}{2} + \frac{1}{k} \right\rfloor$$

In general this only gives a range of possible values for m. But for $k = 8$, it pins things down precisely: We must have $m = 3$ copies of each color, and any equi-color color-contrast polyomino on 8 colors must have $I + C = 5$ interior points and/or cavities. The reader is invited to see if such color-contrast polyominoes actually exist.



G4G15 Gift Exchange Paper

Hiding in Plain Sight: Secret Messages in Vyshyvanka Embroidery

Maria Droujkova and math friends
NaturalMath.com

Vyshyvanka is an embroidered Ukrainian shirt. Each region has its own traditions for vyshyvanka colors and designs. The artists who cross-stitch their vyshyvankas often hide names or inspirational messages in their creations.

This paper is a flyer you can use for a math circle (ages 5+) or a personal exploration. Experience the mathematics of the vyshyvanka tradition with paper cutting, software, drawing, or actual embroidery. People who love art as well as those who can only draw a stick figure can generate these appealing designs and explore geometric transformations and dihedral groups in an accessible way.

The activity is festive enough for math days, math festivals, and other community events—such as Vyshyvanka Day, the international holiday that celebrates vyshyvankas on the third Thursday of May.

Family Math Circle, January 2024, Chapel Hill, North Carolina

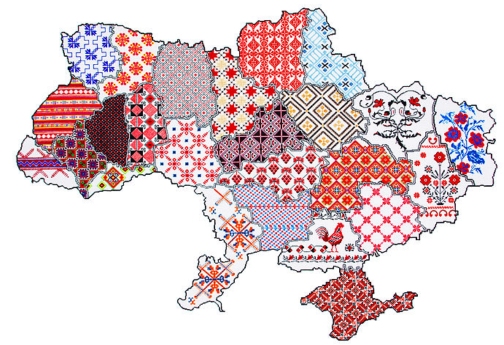
Secret Messages in Vyshyvanka Embroidery

PATHWAYS
#20 Group Theory and Generalizations
#51 Geometry

Family Math Circle

Pathways name our math inspirations using Mathematics Subject Classification. Mathematicians around the world label their work with library codes from MSC, https://msc2020.org/ Let us help students feel happy familiarity with each subject area!

Family Math Circle is an informal learning space where participants make advanced mathematics accessible to everyone in kind ways.



Embroidered Ukrainian Map by Qypchak on Wikipedia

What Will Everyone Make?

Roots and wings: Make models. Make connections.

Fold and cut paper to create snowflakes and other symmetric art, and connect it to algebra and group theory. Model symmetry with vyshyvanka designs. Compare and contrast paper-folded and vyshyvanka symmetries.

MATH IS WHAT YOU MAKE IT!

Words With Math Friends

Tell friends and family all about your math creations. Use these terms +"math" to find images, videos, and articles on the web.

- fold, edge, hole, notch, paper model
- to reflect, line symmetry, to rotate, radial symmetry
- group, group of symmetries, dihedral group, wallpaper group, subgroup

Add your family's own words to the Research Journal. Use these personal terms along with the standard terms.

Interesting Choices

Mathematicians do many different things. What kind of math person will you be today?

How many times will you fold your paper to make your snowflake? What letters or words will you hide in your symmetric designs? Will you choose colors that make your message harder or easier to read? Will you use rotational symmetry, mirror symmetry, or both?

These are starter choices. You will come up with more ideas. Add them to the Research Journal.

Toolbox

Physical (gray=optional):
Paper, graph paper, colored pencils, scissors, embroidery supplies.

Virtual:
Symmetry painting http://weavesilk.com/
Wallpaper groups explorer https://eschersket.ch/
Online vyshyvanka maker https://www.ornament.name/creator

https://naturalmath.com/circles/

Extra activities, videos, math connections, books, and other resources for math circle leaders.

Photo: Vyshyvanka Day, by Vladimir Yaitskiy on Wikipedia

Secret Messages in Vyshyvanka Embroidery

0. Warm Up: Live Mirrors Improv Game

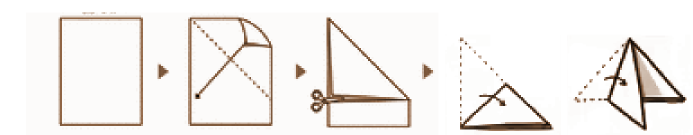
Stand in front of each other. Choose how to move and make faces. Copy each other's moves. That's it!

You are exploring mirror symmetry.

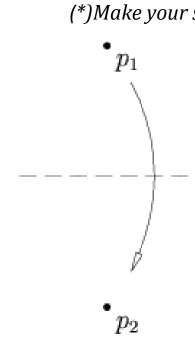
Images: Natural Math, My Little Pony, Spiderman

1. Letter Snowflakes

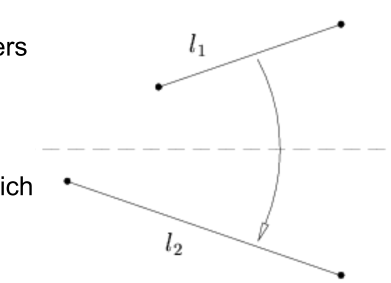
Let us play with paper snowflakes. That helps us to "touch the symmetry" and explore it by hand. Have paper and scissors ready! Fold a square(*) sheet of paper several times, with the fold line always going through the paper's center:



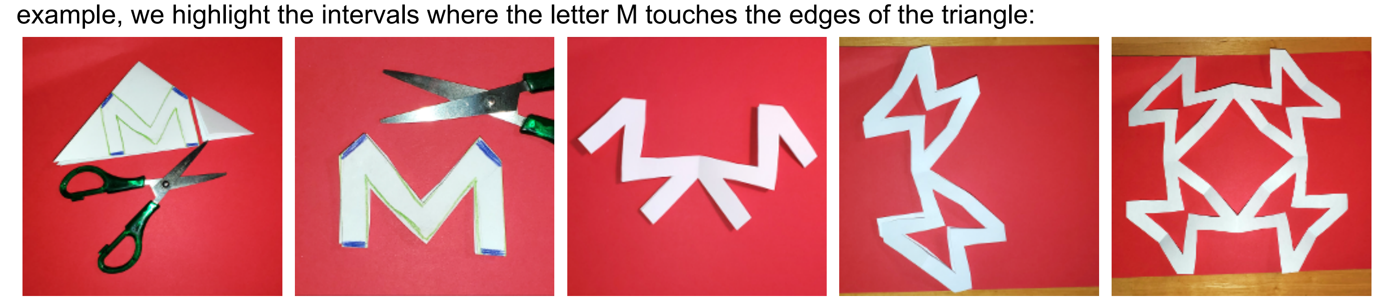
(*)Make your sheet square first, if it isn't already! Images by education.com, twinkle.com, and https://en.wikipedia.org/wiki/Huzita-Hatori_axioms



To fold nicer and to notice more math, play with edges and corners as you go. Edges=lines, corners=points. Before folding, picture each fold in your mind. Which edge must fold on top of which edge? Trace the two edges with your hands before folding them together, to get a good feel. Which corner must fold on top of which corner or edge? Poke the corners to feel them, too.



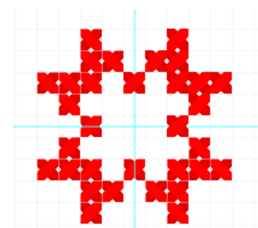
How many triangles does your snowflake fold create? That number, divided by two for the mirroring, names your symmetry: here, 4-fold symmetry. Now, draw a chunky letter that touches all three edges of your triangle. It must touch not only at a point, but along some interval. In this example, we highlight the intervals where the letter M touches the edges of the triangle:



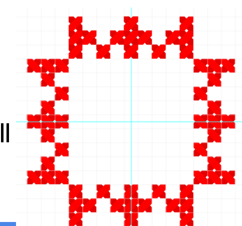
Cut out your letter. Then unfold your letter snowflake. Reflections and rotations will hide your letter in plain sight. These paper snowflakes are beautiful to many of us because humans are attracted to symmetry. There's much symmetry in snowflakes.

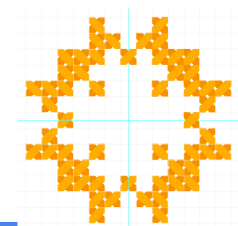
The 4-fold snowflake in our example has 4 lines of symmetry (unlike the ice snowflakes that can only be 3-, 6-, or 12-fold). Do you see the 4 symmetry lines? With your finger, trace the symmetry lines where two copies of your letters mirror (reflect) one another. That's where the letter you drew touched the edges earlier. Ask your friends and family to guess what your unfolded snowflake spells. That's surprisingly hard! Fold the snowflake back, one step at a time, and ask them to guess again at each step. After decoding a few different letters, people grow their math eyes and get much better at seeing the hidden writing.

2. Vyshyvanka Codes



Vyshyvanka is an embroidered Ukrainian shirt. Each region has its own traditions for vyshyvanka colors and designs; you can see some of them in this map of Ukraine "embroidered" region by region. The artists who cross-stitch their vyshyvankas often hide names or inspirational messages in their designs. Here is the same letter **M** (for math!) hiding in several cross-stitch patterns. Can you find each copy of M? There are many ways to use symmetry to hide the same letter!





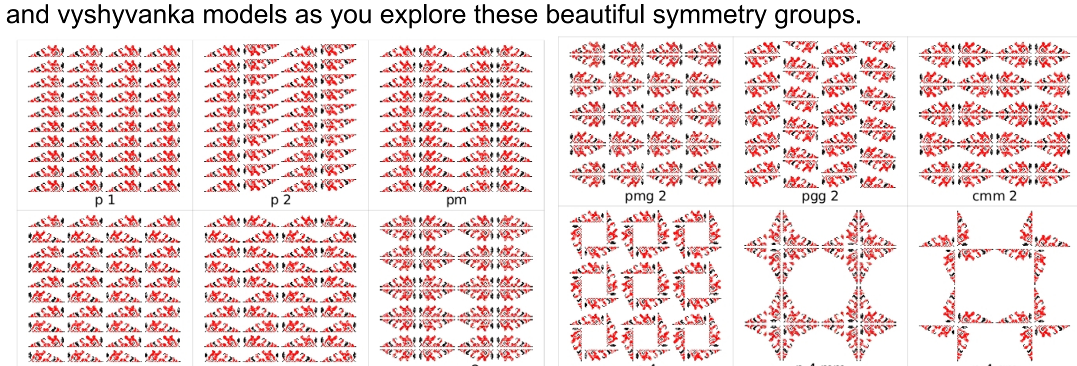
Design your own vyshyvanka secret messages. First, decide how to "pixelate" your letters. Color squares on physical or virtual graph paper to make letters. Second, draw 4 or 8 symmetry lines and reflect your letters. This online designer tool comes with pre-made letters you can use, as well as simple elements for making letters from scratch: https://www.ornament.name/creator

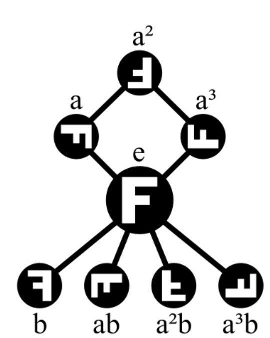
3. Folding vs. Drawing

Go on a scavenger hunt for words, creatures, plants, and human creations that resemble vyshyvanka designs or letter snowflakes. Out of the five images above, one is impossible to model with paper-cutting. Which one? Why?

4. Grow Your Math Eyes: Groups vs. Groups

Some embroidery designs can repeat again and again and again in all directions. In other words, the designs can cover any amount of fabric. That takes certain kinds of symmetry. Turns out, there are exactly seventeen types of these designs, called the seventeen wallpaper groups. The area of math studying these beautiful objects is called group theory. A Ukrainian researcher, Iryna Zasornova, found these twelve wallpaper groups in traditional vyshyvankas. The groups we model with paper snowflakes are called dihedral groups. You could use your snowflake and vyshyvanka models as you explore these beautiful symmetry groups.





IS WHAT YOU MAKE IT!

▸▸▸ Name _ _ _ _ _ _ _ _ _ _ _ _ _ _ ◂◂◂

~ Words ~ Examples ~ Questions ~ Problems ~ Stories ~ Conjectures ~ Models ~ Art ~ Formulas ~ Graphs ~

▸▸▸ Research Journal ◂◂◂

Math Circle design by you and math friends

The Statistics of the I Ching*

Jim Guinn

Professor of Physics and Astronomy

Georgia State University Perimeter College

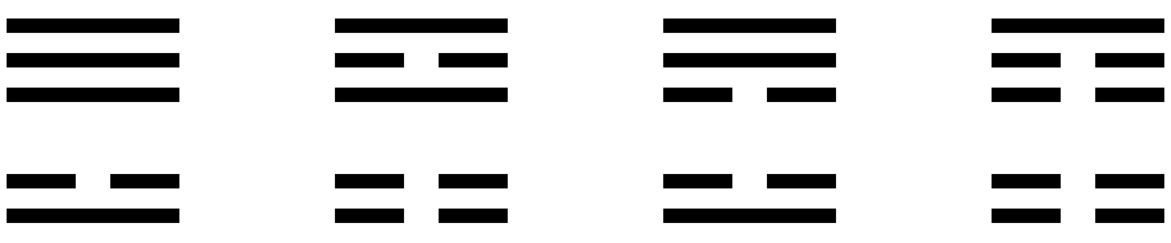
The I Ching (pronounced “ee ching”), or Book of Changes, is one of the great classics of Chinese Confucian and Taoist philosophy. It is thought to be between two and three thousand years old. The main body of the text contains interpretations of hexagrams created through numerical processes that can be used as oracles or a source of wisdom . The method that the oracle reader (henceforth referred to as “reader”) uses to determine which hexagrams are to be used, is very mathematical and I believe to be an amazing combination of seemingly random choices to generate a series of hexagrams with over four thousand possible outcomes with predictable probabilities. There are two approaches for determining the hexagrams to be used, the traditional method using yarrow stalks, and another using coins. In this paper I will discuss the yarrow stalk method.

History: It seems that the development of the I Ching began with just two possible outcomes for the oracle. These were represented by a solid line

which represented yes or masculine (yang), and a broken line

which represented no or feminine (yin). As more variation in the oracle was needed, a second line was added, above the first, yielding four possibilities,

and then a third line above the first two

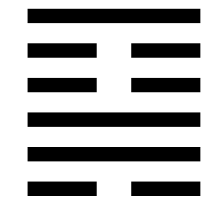


yielding eight trigrams. Each trigram has a name, an attribute, an image, and a family relation associated with it. For example,

* The information about the I Ching in this paper comes from “The I Ching or Book of Changes”, translated from Chinese into German by Richard Wilhelm, and translated from German into English by Cary F. Baynes, Bollingen Series XIX, Princeton University Press, 1950.

is named Sun, or the Gentle, is considered penetrating, is represented by wind or wood, and is the first daughter of the family. Two trigrams were then stacked, one atop the other, to form sixty-four possible hexagrams, and each hexagram had an interpretation depending on the trigram below and the trigram above.

For example,



is the hexagram Ku, with the trigram Sun below and the trigram Kên above. A further variation was added to include a second hexagram developed from the first hexagram. The first hexagram (first generation hexagram) represents the present state for the oracle requester and the second hexagram (second generation hexagram) represents the future state for the requester. The main body of the I Ching goes through the interpretation of each hexagram (as a first-generation or second-generation hexagram).

Each line of the hexagram was taken to have two forms. The solid masculine yang line could be either an old yang (old man) or a young yang (young man), and the broken feminine yin line could be either an old yin (old woman) or a young yin (young woman). Each type of line, solid or broken, in the first-generation hexagram could remain the same or change into a different line, in the second-generation hexagram depending on whether it is old or young in the first-generation hexagram. Old yang dies and is reborn as young yin, young yin grows up into old yin, old yin dies and is reborn as young yang, and young yang grows up into old yang. This meant that any hexagram in the present (first generation) could become any other hexagram in the future (second generation), yielding 64 x 64 = 4096 different possible oracular interpretations. The question was then which hexagrams to use for a requester’s oracle.

The yarrow-stalk method is one of the two ways (the other is the coin method) of choosing the hexagrams for an individual. This method begins with fifty stalks, but one is put aside and is not used for the remainder of the procedure. (Although I have no evidence for this, I like to think that the fiftieth stalk started out as a suggestion to have one extra, in case a stalk gets broken or lost, and that over the years, the optional stalk turned into a required stalk.) The oracle requester then divides the forty-nine stalks into two groups in proportions of their choosing, one group placed on the left side and one placed on the right side. The oracle reader then does the following: one stalk from the right-side group is removed and placed between the oracle reader’s little finger and ring finger of the left hand. The left-side group then has bundles of stalks removed four at a time and put aside until one, two, three, or four stalks remain. These are placed between the ring finger and middle finger of the oracle reader’s left hand. The right-side group then has bundles of stalks removed four at a time until one, two, three, or four stalks remain; these are then placed between the middle finger and index finger of the oracle reader’s left hand. The total number of stalks in the reader’s left hand will now be either five or nine. Assuming a relatively equal separation of the initial group of forty-nine stalks, five stalks will remain with a probability of 0.75, and nine stalks will remain with a probability of 0.25. A remainder of five stalks count as three “points” while a remainder of nine stalks count as two “points”. These five/nine stalks are then put aside and the remaining forty-

* The information about the I Ching in this paper comes from “The I Ching or Book of Changes”, translated from Chinese into German by Richard Wilhelm, and translated from German into English by Cary F. Baynes, Bollingen Series XIX, Princeton University Press, 1950.

four/forty stalks (respectively) are divided by the oracle requester into two groups again and the previous procedure is repeated. Again, assuming relatively equal separation of the new group, the possible remainders now are eight, with a probability of 0.5, or four, with a probability of 0.5. The four stalks count as three “points” while the eight stalks again count as two “points”. The four/eight stalks are again set aside and the remaining thirty-two/thirty-six/forty stalks are used for a repetition of the procedure for a third time and final time. The possible final remainders are again eight, with a probability of 0.5 , or four, with a probability of 0.5. Again, eight stalks count as two “points” and four stalks count as three “points”.

The point totals then determine the first line for the first-generation hexagram. The possible final point totals are six, with a probability of 0.3125, which forms the old yin line, a total of seven, with a probability of 0.4375, which forms the young yang line, a total of eight, with a probability of 0.1875, which forms the young yin line, and a total of nine, with a probability of 0.0625, which forms the old yang line. These are summarized in Table #1 below. Notice that the probability of forming a yang line (young yang + old yang = 0.4375 + 0.0625) is 0.5 and the probability of forming a yin line (young yin + old yin = 0.1875 + 0.3125) is 0.5. It is equally likely that a first-generation line is a yang line or yin line.

Table #1

Point Total	First Generation Line	Probability
6	Old Yin	0.3125 (= 5/16)
7	Young Yang	0.4375 (= 7/16)
8	Young Yin	0.1875 (= 3/16)
9	Old Yang	0.0625 (= 1/16)

For the second-generation line, old yin becomes young yang (yin → yang), young yang becomes old yang (yang → yang), young yin becomes old yin (yin → yin), and old yang becomes young yin (yang → yin). The probabilities for the second-generation lines are summarized below in Table #2.

Table #2

Point Total	Second Generation Line	Probability
6	Young Yang	0.3125 (= 5/16)
7	Old Yang	0.4375 (= 7/16)
8	Old Yin	0.1875 (= 3/16)
9	Young Yin	0.0625 (= 1/16)

Notice that for the second generation, the probability of having a yang line (young yang + old yang = 0.3125 + 0.4375) is 0.75 , while the probability of having a yin line (young yin + old yin = 0.0625 + 0.1875) is 0.25 . It is three times more likely for a second-generation line to be a yang line than a yin line.

The procedure described above is then done five more times, for a total of six lines, the first line appearing at the bottom of the hexagram, and the sixth line at the top of the hexagram. The hexagram formed with the first-generation lines represents the present state of the individual’s life, while the hexagram formed with the second-generation lines represents the future state of the individual’s life.

* The information about the I Ching in this paper comes from “The I Ching or Book of Changes”, translated from Chinese into German by Richard Wilhelm, and translated from German into English by Cary F. Baynes, Bollingen Series XIX, Princeton University Press, 1950.

With the probabilities given above, the most probable hexagram (probability ≈ 0.092) is one formed with three young yang lines, two old yin lines, one young yin line, and zero old yang lines. (This high probability is due in part to the probability of the individual lines, and in part to the large number of permutations of this arrangement.) This then is a first-generation hexagram with three yang lines and three yin lines. The most probable second-generation hexagram then has three old yang lines, two young yang lines, one old yin line, and zero young yin lines. This then is a second-generation hexagram with five yang lines and one yin line.

The highest probability of yang/yin lines, regardless of the age of the line, is also three yang and three yin, with a total probability of 0.3125.

The first-generation hexagram with the lowest probability is one formed with six old yang lines (probability ≈ 5.96×10^{-8}). The lowest probability of yang/yin lines, regardless of age, are either a hexagram with six yang or six yin lines (each of which have a probability ≈ 0.016).

I do not know whether the early developers of the yarrow-stalk method of choosing the hexagram lines understood the probabilities of the yarrow-stalk outcomes, but I believe that anyone who cast the oracle many times would eventually notice that some of the lines and hexagrams occurred more often than others. The early users of the yarrow stalks may have developed the method to yield, what was to them, an acceptable distribution of outcomes. Personally, I find the whole approach beautiful and fascinating. The procedure seems on the surface, to be rather complex, and yet yields a simple set of probabilities with some yin/yang symmetries and other asymmetries.

* The information about the I Ching in this paper comes from “The I Ching or Book of Changes”, translated from Chinese into German by Richard Wilhelm, and translated from German into English by Cary F. Baynes, Bollingen Series XIX, Princeton University Press, 1950.

Can the Number of Pieces in a Rectangular Jigsaw Puzzle be a Multiple of the Number of Edge pieces?

By D. Scott Hewitt 2/2/2022

Scott496@comcast.net

The short answer is yes. There are an infinite number of solutions. However, some interesting results appear in connection to this question. I occasionally do Jigsaw puzzles, and I usually work on the edge pieces first. That's how I became interested in this topic.

Note: If you are new to jigsaws, there are quite a few options for doing Jigsaw puzzles. There are many available in stores or online. More choices are available online when you do an internet search for software. Some are free and some require the purchase of a license. I like the option of creating my own puzzles from jpg files, which some Jigsaw companies offer.

Let's start with some definitions and formulas:

Let x = the number of pieces on the shorter side.

let y = the number of pieces on the longer side.

let e = the number of edge pieces.

let f = the ratio of total pieces to edge pieces.

Let d = y – x (the difference between the long side and the short side).

Now let t = x y = total pieces.

Number of edge pieces = e = 2x + 2y – 4 (true for any rectangular jigsaw puzzle)

Then f = $\frac{t}{e}$

We are interested in positive integer solutions for f.

We can assume that x > 1 because if x = 1 we just get a single row of pieces which is trivial.

Before talking about solutions, we have a few theorems in order to hopefully eliminate some possibilities: Most of the theorems are fairly basic and the proofs are short.

Theorem 1 is somewhat trivial and concerns jigsaw puzzles with only 2 rows of pieces.

Theorem 1 If and only if x = 2 and y is any positive integer, then f = 1 and t = e

Proof: t = 2y and e = 2*2 +2y -4 = 2y

So f = $\frac{t}{e} = \frac{2y}{2y}$ = 1, and every piece in the puzzle is an edge piece.

Note: If x > 2, then of course t > e.

Theorem 2 If x = 3, There is no solution for integer f.

Proof: t = 3y, and f = $\frac{t}{e} = \frac{3y}{6+2y-4} = \frac{3y}{2y+2}$

Since y ≥3, this fraction f takes on values on the interval $(\frac{9}{8}, \frac{3}{2})$ as y takes on values from 3 to ∞.

The next possible integer value of f is 2, so there is no solution for x = 3.

Theorem 3 If x = 4, there is no solution for integer f.

Proof: t = 4y, and f = $\frac{t}{e} = \frac{4y}{2(4+y)-4} = \frac{4y}{2y+4} = \frac{2y}{y+2}$

Since y ≥ 4, this fraction f takes on values on the interval $[\frac{4}{3}, 2)$ without reaching 2, as y takes on values from 4 to ∞.

The next possible integer value of f is 2, so there is no solution for x = 4.

Theorem 4 x and y cannot both be odd.

Proof: If x and y are both odd then t is odd. We have f = $\frac{t}{2x+2y-4}$

The numerator is odd and the denominator is even, so x and y both odd is not possible.

Theorem 5 x and y cannot be equal, so square jigsaw puzzles are not possible with integer f.

Proof: t = x^2

f = $\frac{t}{e} = \frac{x^\wedge 2}{4x-4}$

Case I

In the case of an even square where x = 2m, we have $\frac{4m^\wedge 2}{8m-4} = \frac{m^\wedge 2}{2m-1}$

If this fraction is equivalent to an integer, then k(2m-1) = m^2

We have: m^2 -2km+k = 0 so m = k+$\sqrt{k(k-1)}$

For m to be an integer k(k-1) must be a perfect square.

However, consecutive integers do not share any factors. Therefore, k and k-1 must be consecutive perfect squares which is impossible.

Case II

In the case where x = 2m + 1, we have f = $\frac{4m^2+4m+1}{8m}$, but this is also impossible since an even number cannot divide an odd number. Therefore, x and y cannot be equal.

Theorem 6 y cannot be a multiple of x.

Proof: Suppose y = ax, then t = xy = ax^2

e = 2(x+ax) – 4 or e = x(2+2a) – 4

We now have f = $\frac{x*x*a}{x(2+2a)-4}$ and we know that x ≥ 5 from previous work.

If x = 5 we have f = $\frac{25a}{5(2+2a)-4}$ and a number of the form 25r or 5r cannot be divisible by a number of the form 5r - 4.

If x = 6 we have f = $\frac{36a}{6(2+2a)-4}$ and again, a number of the form 36r or 6r cannot be divisible by a number of the form 6r – 4

For x ≥ 7 let us look at the general case f = $\frac{x*x*a}{x(2+2a)-4}$ The numerator is of the form xr and the denominator is of the form xr – 4 . Numbers of the form xr – 4 can never divide numbers of the form xr with x ≥ 7.

Therefore, for all x ≥ 5, f cannot be an integer. This proves that y cannot be a multiple of x.

Now let us look at a condition that does yield solutions. Suppose x and y have a difference of 2.

Theorem 7 If d = y - x = 2 there are an infinite number of solutions.

Proof: If y - x = 2 then t = x(x+2) and e = 2x +2(x+2) – 4 = 4x

We have f = $\frac{x(x+2)}{4x} = \frac{x+2}{4}$ and we can solve for any f.

Example A: Suppose we wish the number of edge pieces to be exactly half the total number of pieces.

So, f is 2 and we have $\frac{x+2}{4} = 2$

X+2 = 8

X = 6 and therefore y =8

t = 48 and e = 24

Example B: suppose we wish f to be 3.

We have $\frac{x+2}{4} = 3$

X + 2 = 12

X = 10 and therefore y = 12 giving us t = 120 and e = 40

Example C: suppose we wish f to be 73.

We have $\frac{x+2}{4} = 73$

X + 2 = 292

X = 290 and y =292 giving us t = 84680 and e = 1160

The following table gives a solution for f =2 through 20 where y - x = 2. Incidentally, these are also the jigsaws of smallest area with those f values. Other solutions exist for these f values when y – x $>$ 2.

Table 1

f = t/e	x = width	y = length	t = total pieces	e = edge pieces
2	6	8	48	24
3	10	12	120	40
4	14	16	224	56
5	18	20	360	72
6	22	24	528	88
7	26	28	728	104
8	30	32	960	120
9	34	36	1224	136
10	38	40	1520	152
11	42	44	1848	168
12	46	48	2208	184
13	50	52	2600	200
14	54	56	3024	216
15	58	60	3480	232
16	62	64	3968	248
17	66	68	4488	264
18	70	72	5040	280
19	74	76	5624	296
20	78	80	6240	312

Table 2 gives the solutions for jigsaw puzzles of minimum area with consecutive x values where x = width. Notice that when x is a prime such as 31 or 43, or when x is an odd square like 49; y is sometimes quite large compared to neighboring y values due to the more difficult task of finding a solution with integer f.

Table 2

X = width	Y= length	t = x y	e = edge	f = t/e
5	12	60	30	2
6	8	48	24	2
7	30	210	70	3
8	18	144	48	3
9	14	126	42	3
10	12	120	40	3
11	24	264	66	4
12	20	240	60	4
13	132	1716	286	6
14	16	224	56	4
15	26	390	78	5
16	42	672	112	6
17	36	612	102	6
18	20	360	72	5
19	306	5814	646	9
20	27	540	90	6
21	38	798	114	7
22	24	528	88	6
23	48	1104	138	8
24	44	1056	132	8
25	92	2300	230	10
26	28	728	104	7
27	50	1350	150	9
28	65	1820	182	10
29	60	1740	174	10
30	32	960	120	8
31	870	26970	1798	15
32	50	1600	160	10
33	62	2046	186	11
34	36	1224	136	9
35	44	1540	154	10
36	68	2448	204	12
37	150	5550	370	15
38	40	1520	152	10
39	74	2886	222	13
40	57	2280	190	12
41	84	3444	246	14
42	44	1848	168	11
43	1722	74046	3526	21
44	90	3960	264	15
45	86	3870	258	15
46	48	2208	184	12
47	96	4512	282	16
48	92	4416	276	16
49	282	13818	658	21
50	52	2600	200	13

From table 1 and table 2, we see that every x value greater than or equal to 5 yields at least one solution, and every f value greater than or equal to 2 yields multiple solutions.

Tables 3 and 4 are just to show that perfect squares and cubes are possible for the number of edge pieces.

Table 3 lists very specific cases where e is a perfect square and x $\leq$ 100. Although there appears to be an infinite number of solutions, there are only 5 solutions for x below or equal to 100. If we were to go a bit further, there are 18 solutions for x below 500.

Table 3

x	y	t = x y	e	f = t/e
18	56	1008	144	7
50	152	7600	400	19
98	296	29008	784	37
98	1472	144256	3136	46
100	2352	235200	4900	48

Table 4 lists cases where e is a perfect cube and x is less than 1000.

Table 4

x	y	t = x y	e	f = t/e
54	56	3024	216	14
162	704	114048	1728	66
250	252	63000	1000	63
686	688	471968	2744	172

Triangular numbers are numbers of the form T(n) = $\frac{n(n+1)}{2}$. If we ask whether both x and y can be triangular numbers, the answer is yes; but solutions seem to be somewhat scarce. Table 5 gives the only 10 solutions I could find with the restriction that both x and y are less than 100,000. I was unable to find a formula to calculate these directly. It's interesting that both T88 and T168 show up twice.

Table 5

x	y	t=x*y	e=edge	f	Triangular Index for x	Triangular Index for y
78	171	13338	494	27	T12	T18
903	1378	1244334	4558	273	T42	T52
1770	24310	43028700	52156	825	T59	T220
2850	3916	11160600	13528	825	T75	T88
3916	5253	20570748	18334	1122	T88	T102
11325	14196	160769700	51038	3150	T150	T168
14196	17578	249537288	63544	3927	T168	T187
26106	31375	819075750	114958	7125	T228	T250
52003	60726	3157934178	225454	14007	T322	T348
81406	93528	7613740368	349864	21762	T403	T432

Now let us examine differences between y and x. Using a computer program, I did a search and found solutions for every difference up to d = 250 with the exception of d = 1, 3, 4, and 6.

Table 6 gives minimum solutions for all known consecutive values of d = y – x up to 25.

Table 6

d = y-x	x = short side	y = long side	t = total pieces	e = edge pieces	f = $\frac{t}{e}$
2	6	8	48	24	2
5	9	14	126	42	3
7	5	12	60	30	2
8	12	20	240	60	4
9	35	44	1540	154	10
10	8	18	144	48	3
11	15	26	390	78	5
12	30	42	1260	140	9
13	11	24	264	66	4
14	18	32	576	96	6
15	104	119	12376	442	28
16	14	30	420	84	5
17	21	38	798	114	7
18	32	50	1600	160	10
19	17	36	612	102	6
20	24	44	1056	132	8
21	209	230	48070	874	55
22	10	32	320	80	4
23	7	30	210	70	3
24	132	156	20592	572	36
25	23	48	1104	138	8

I call d values which do not yield a solution "difference outlaws".

Hypothesis(A): d = 1, 3, 4, and 6 are difference outlaws.

Hypothesis(B): d = 1, 3, 4, and 6 are the only outlaws. Therefore, all other differences are possible. Some differences yield x values that are quite large. For example, d = 225 yields the solution x = 25199, y = 25424, and f = 6328 with no smaller solution.

If someone would like to work on hypothesis A or B, I can get you started:

d = 1 → f = $\frac{x^2+x}{4x-2}$ → $x^2 - 4xf + x + 2f = 0$

d= 3 → f = $\frac{x^2+3x}{4x+2}$ → $x^2 - 4xf + 3x - 2f = 0$

d = 4 → f = $\frac{x^2+4x}{4x+4}$ → $x^2 - 4xf + 4x - 4f = 0$

d = 6 → f = $\frac{x^2+6x}{4x+8}$ → $x^2 - 4xf + 6x - 8f = 0$

The fractions for f are equivalent to the Diophantine equations on the right that we would like to prove have no solution for x and f as positive integers. My feeling is that these 4 Diophantine equations probably have integer solutions but that the solutions involve negative integers. Incidentally, these equations represent hyperbolic curves.

Let's take a last look at one of the outlaws: d = 1. This looks like it may be easier than d = 3, 4, or 6.

<u>Theorem 8</u>

X and y cannot be consecutive integers. In other words, $d \neq 1$.

Proof: Suppose d = 1, then we have $f = \frac{x(x+1)}{4x-2}$

Suppose we treat this as a quadratic equation and solve it for x. We have

x^2 +(1-4f)x + 2f = 0

Using the Quadratic Formula: $x = \frac{(4f-1) \pm \sqrt{(1-4f)^2 - 8f}}{2}$

To have a solution, the discriminant must be a perfect square: $(1-4f)^2 - 8f$ = n^2

We have 1 – 16f + 16f^2 = n^2 and n must be odd for x to be a positive integer.

The left side of the equation can be written as $16(f^2\text{-}f) + 1$ and this is equivalent to $8(2f^2 - 2f) + 1$

This looks encouraging since all odd squares must be of the form 8n+1. In fact, all odd squares are of the form 8T + 1 where T is a triangular number of the form $\frac{r(r+1)}{2}$

The question now is: Can $2f^2 - 2f$ ever be a triangular number?

If so, we have $2f^2 - 2f = \frac{r(r+1)}{2}$ (standard form for triangular numbers)

$4f^2 - 4f - r(r+1) = 0$ and we must again resort to the quadratic formula.

$$f = \frac{(4) \pm \sqrt{4^2 + 16r(r-1)}}{8} = \frac{4 \pm 4\sqrt{r(r+1)+1}}{8}$$

Looking at this fraction, r(r+1) must be equivalent to $8T = 8\,\frac{v(v+1)}{2} = 4v(v+1)$, and we are missing a factor of 4. We have encountered a contradiction, so jigsaw puzzles with integer f are not possible if y – x = 1.

Feel free to send me an email if you have a comment about my paper or if you have made progress on Hypothesis A or B that you would like to share.

<u>In conclusion, I would like to wish you an enjoyable time working on your next jigsaw puzzle, regardless of whether or not the number of pieces is a multiple of the number of edge pieces! Have fun!</u>

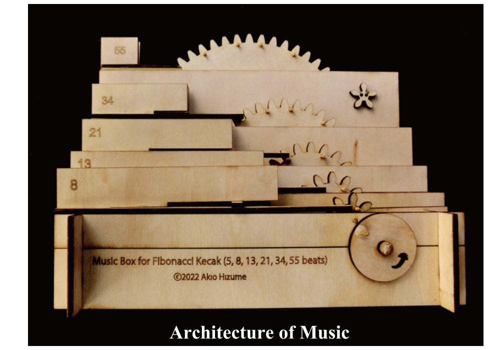
La caja de música de arena

Music Box of the Fibonacci Kecak

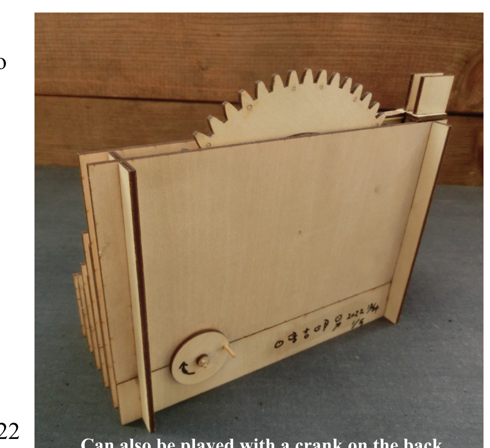
Fibonacci Kecak was published in 1995. (Commonly known as "TTKTK")

Abstruct
Real Keak System
http://starcage.org/papers/realkecaksystem.pdf

Audio Source
A Consert of the Fibonacci Kecak
http://starcage.org/music/ttktk.mp3



Architecture of Music


Can also be played with a crank on the back

Although computerized and live performances were realized, its expression as a music box had been pending for 27 years.
The music box has now finally been completed.

The configuration of the number of gear teeth is as follows;
"T" is rests and "K" is notes.

5 TTKTK
8 TTKTTKTK
13 TTKTTKTK TTKTK
21 TTKTTKTK TTKTTKTK TTKTK
34 TTKTTKTK TTKTTKTK TTKTK TTKTTKTK TTKTK
55 TTKTTKTK TTKTTKTK TTKTK TTKTTKTK TTKTTKTK TTKTK TTKTTKTK TTKTK
and so on....

This music box is structurally incapable of rotating backwards.
In other words, we have no choice but to move forward into the future.

The LCM of 5,8,13,21,34,55 is 2,042,040.
For a 6 gears, the period of music played by this music box is 2,042,040 beats.
Assuming that one second is spent per crank revolution (0.2 seconds per beat), the period is 4 days + 17 hours.
This means that to hear the passage you just heard again, you would have to keep turning the crank for nearly five days without sleep or rest.

Jorge Luis Borges' short story "El libro de arene (The Book of Sand)" (1954) contains the following passage.
An antiquarian book seller puts it this way.
"Sir, the page you just watched, you'll never see it again" .
This is a Cantor-like novel about the uncountable infinity of real numbers.

The idea for this music box was conceived in 1995 and the name had been decided on as "La caja de música de arena (The Music Box of Sand)".
So I would like to say this.
"Sir, this phrase you have just heard, you will never hear it again".

24th December 2022
Akio Hizume

Special Thanks: Morita Lab. Ryukoku Univ.

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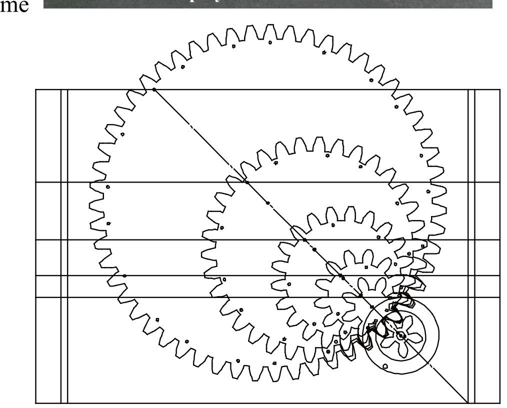
The Fibonacci Kecak concept of 1995 was originally intended to be a music box, but since there was no laser cutter nearby, it was developed as an interactive application software.The percussion ensemble "TTKTK" was also very popular, but it was also an alternative to the music box.It was a deeply moving experience to realize an idea that had been 27 years in the making in just one week.

"The Music Box of Sand" Video Source
https://youtu.be/Oiz_wB5Ei3E


PLAN

Almost orthogonal polyhedra in general, and this one in particular

Robin Houston

G4G15

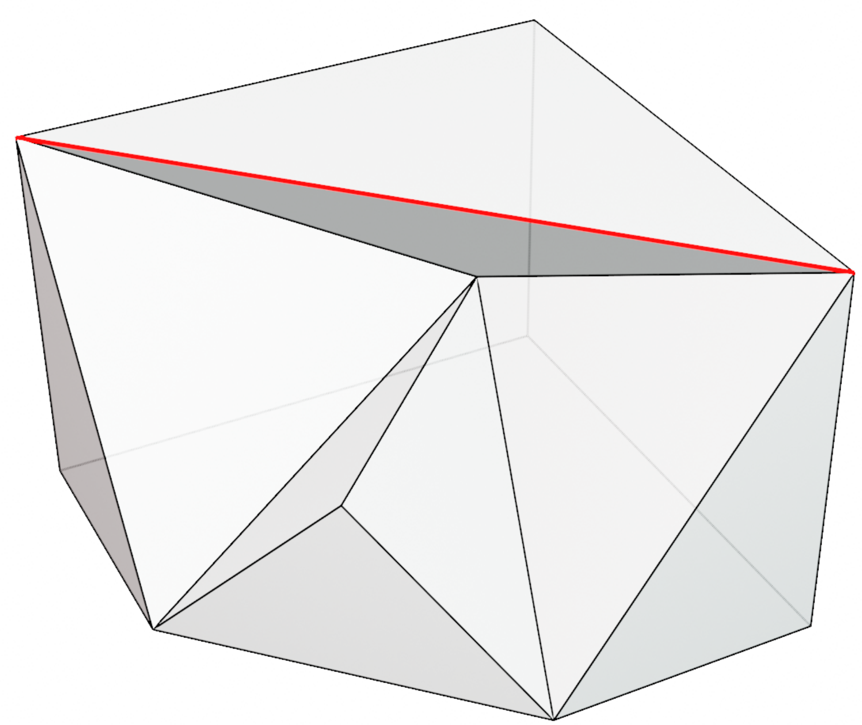


Figure 1

If everything has gone to plan, you will find among your gift exchange delights a smallish white plastic non-convex polyhedron that has twelve faces, and looks like Figure 1.

The intention of this booklet is to answer some of the questions you may have about this object. In particular, it aims to answer the three primary questions: *What?*, *How?*, and *Why?*.

If you have other questions, you are cordially invited to buttonhole me, or to email me at robin.houston@gmail.com. I may not know the answers, but I expect I shall enjoy thinking about your questions.

What is it?

It is an *almost orthogonal* polyhedron: a polyhedron whose adjacent faces are orthogonal to each other, *except on one edge*. On this special edge, drawn in red in Figure 1, the faces meet at 45°.

It is the simplest such polyhedron that is known to exist, or at any rate it is the simplest that *I* know.

If the special edge is permitted to have some other dihedral angle, not necessarily 45°, then I know a simpler one, illustrated in Figure 2 below. It has eight triangular faces. Its two largest faces meet at an angle of arccos(-1/3) ≈ 109.5° on the blue edge, and all the other edges have a dihedral angle of 90° or 270°.

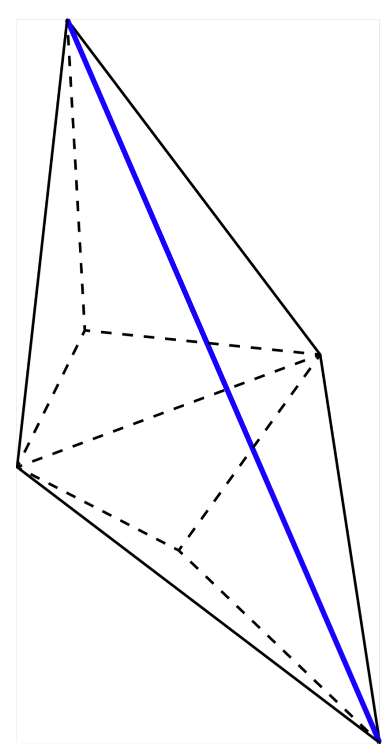


Figure 2. An almost orthogonal octahedron

How did I construct it?

I began with the hexahedron shown in Figure 3. All its dihedral angles are right, with two exceptions: the red edge has a dihedral angle of 135°, and the blue edge has a dihedral angle of arccos(-1/3) ≈ 109.5°. The red and blue edges are orthogonal to each other.

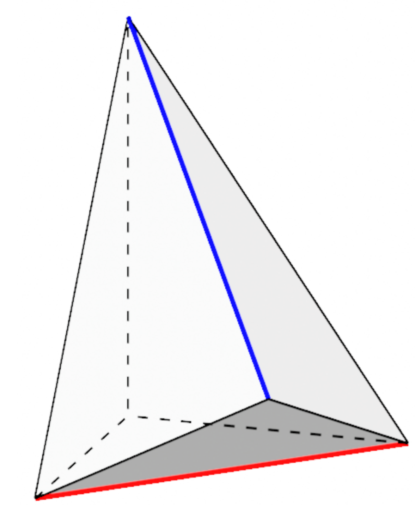


Figure 3. This hexahedron is due to Sydler, of which more below.

Take three copies of this hexahedron. Glue two of them together, so that their red edges coincide and their blue edges are collinear and share an endpoint: you will obtain the octahedron of Figure 2.

Scale down your octahedron by a factor of 2, so that its blue edge is the same length as the blue edge of your remaining hexahedron.

Overlay the two objects, the scaled-down octahedron and the remaining hexahedron, so their blue edges coincide and the faces that meet at those edges are coplanar. You will notice that, in this arrangement, the interior of the octahedron is a strict subset of the interior of the hexahedron. Subtract the octahedron from the hexahedron: this has the effect of removing the edge whose dihedral angle is arccos(-1/3). The result is shown in Figure 4: an almost orthogonal decahedron whose non-right dihedral angle is 135° (marked in red on the diagram).

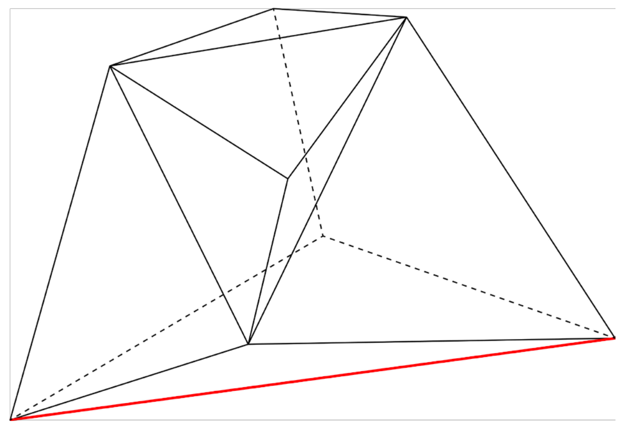


Figure 4. An almost orthogonal decahedron whose non-right dihedral angle is 135°.

This construction is not easy to visualise from diagrams alone. I have a set of 3D-printed models that may help: I can show them to you if you ask me.

The final step is to take the complement of this decahedron with respect to a suitably-aligned box, as shown in Figure 5.

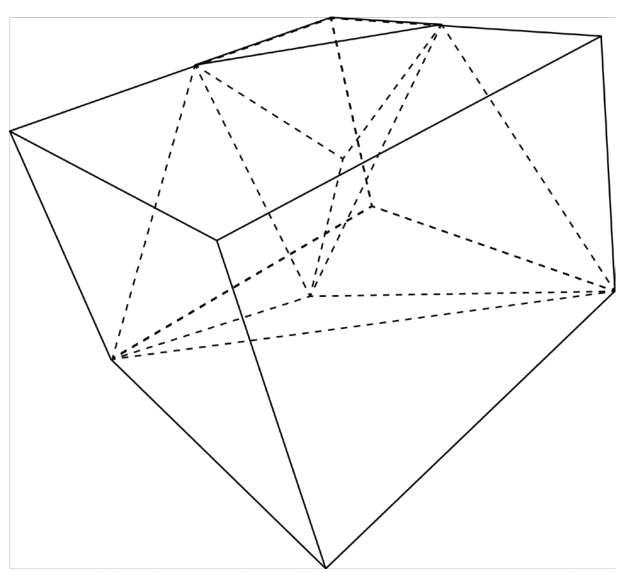


Figure 5

The complement is the almost-orthogonal dodecahedron of Figure 1.

There are other ways to obtain a 45° almost-orthogonal polyhedron starting from the 135° almost-orthogonal polyhedron of Figure 4. For example, we could attach an isosceles right-triangular prism as shown in Figure 6.

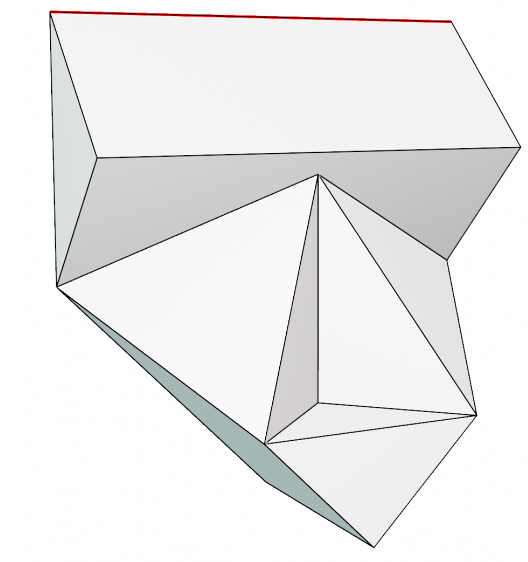


Figure 6

Or we could attach it as shown in Figure 7, which would make an imposing statue.

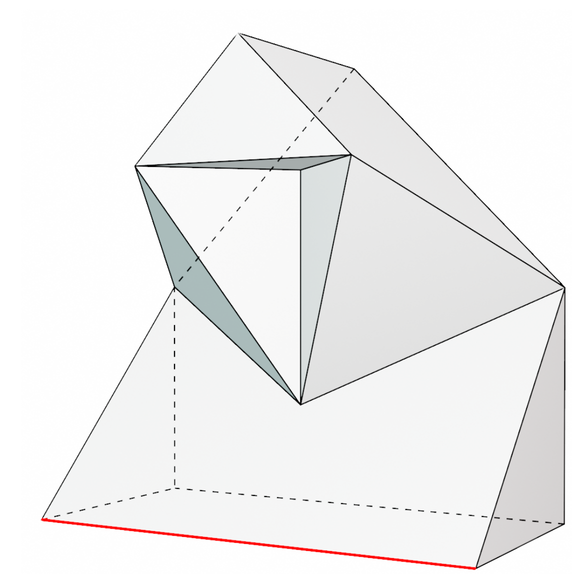


Figure 7

In both cases, the back face is pentagonal, combining a rectangular face of the prism and a triangular face of the decahedron.

It is interesting to deconstruct the decahedron of Figure 4 into three pieces: an isosceles right-triangular prism, and two pieces that are mirror images of each other and have rotational symmetry of order 3. These pieces have seven faces, and we will refer to their shape as the *fundamental heptahedron*. This convex polyhedron has three edges whose dihedral angles are not right, drawn in red in Figure 8. These are mutually orthogonal, and each has a dihedral angle of 135°.

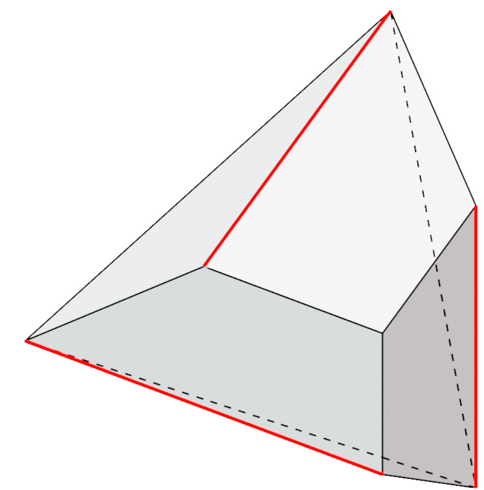
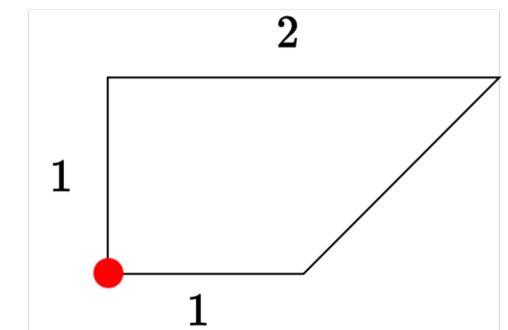


Figure 8. The fundamental heptahedron



The fundamental heptahedron is the convex hull of three mutually orthogonal copies of this quadrilateral, coinciding at the marked point

Eight of these – four in each orientation – can be assembled into Jessen's orthogonal icosahedron[1], illustrated in Figure 9.

[1] Børge Jessen. Orthogonal Icosahedra, Nordisk Matematisk Tidskrift 15, no. 2/3 (1967) pp 90–96

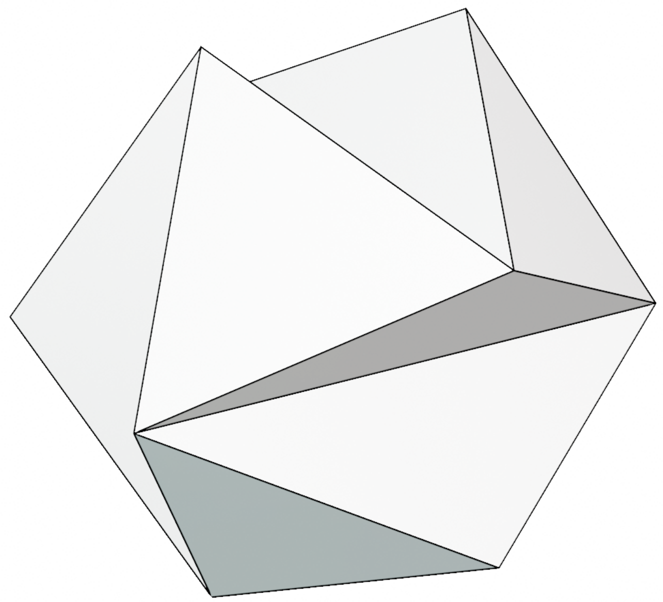
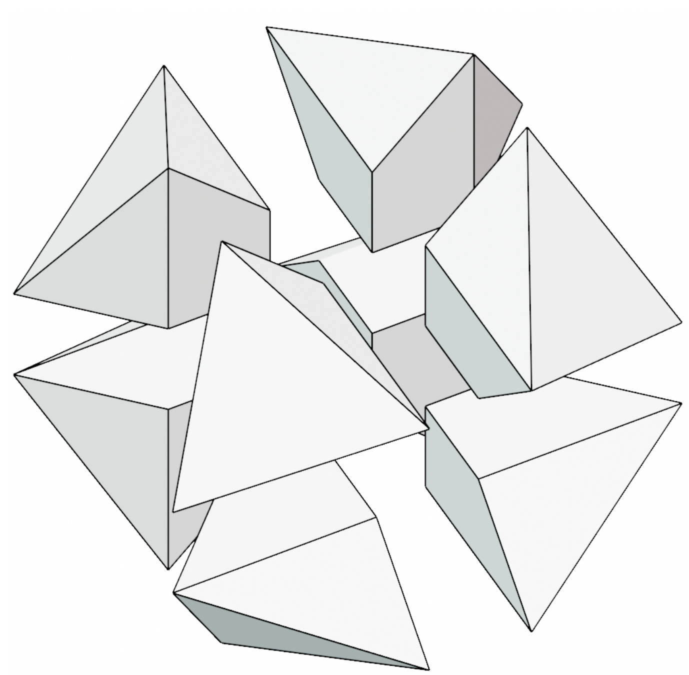


Figure 9. Jessen's orthogonal icosahedron. It may be dissected into eight copies of the fundamental heptahedron, four in each orientation.



Why?

In 1900, David Hilbert proposed a list of 23 then-unsolved mathematical problems that he regarded as important. These problems exerted a strong influence on mathematics well into the second half of the 20th century.

Hilbert's Third Problem asked whether, given two polyhedra of the same volume, it is always possible to cut one of them into a finite number of polyhedral pieces and reassemble those pieces into the other.

It was the first of Hilbert's problems to be solved. Max Dehn showed that, in addition to volume, there is another quantity, determined by the polyhedron's edges – now known as the *Dehn Invariant* – that always stays the same even when a polyhedron is cut into a finite number of polyhedral pieces and reassembled. Since a cube and a regular tetrahedron of the same volume have different Dehn invariants, it is impossible to cut up a cube and reassemble the pieces into a regular tetrahedron, or vice versa.

Dehn's rapid triumph suggested a more difficult question: is that the only obstruction? If two polyhedra have the same volume *and the same Dehn invariant*, then is it always possible to cut one of them into a finite number of polyhedral pieces and reassemble those pieces into the other?

This harder question was eventually answered in the affirmative in 1965, by Jean-Pierre Sydler[2], a Swiss librarian who had studied the problem as a PhD student in the 1950s, and afterwards continued to work on it in his spare time.

For one part of his argument (in Chapitre 1), Sydler needed to construct a family of polyhedra with the property that we're interested in here:

[2] Sydler, J.-P.. "Conditions nécessaires et suffisantes pour l'équivalence des polyèdres de l'espace euclidien à trois dimensions.." Commentarii mathematici Helvetici 40 (1965/66): 43–80.

adjacent faces are orthogonal to each other, except for one pair of adjacent faces that are at 45° to each other.

The hexahedron of Figure 3 belongs to a family of hexahedra that Sydler constructs in Chapitre 1, and which play a key role in his construction: they are *almost* almost-orthogonal, i.e. they have *two* edges whose dihedral angles are not right. Those two non-right dihedral angles α and β are related by the equation

$$(3 + \cos 2\alpha)(1 - \cos \beta) = 4$$

where $90° < \alpha, \beta < 180°$; letting $\alpha = 135°$ gives $\cos \beta = -1/3$.

Although Sydler's construction is effective, he does not dwell on the particulars of the polyhedra that result. Much more recently (in 2017) Matthias Goerner[3] went through Sydler's construction step-by-step and created a 3D model of the resulting polyhedron, which proved to be hilariously complicated:

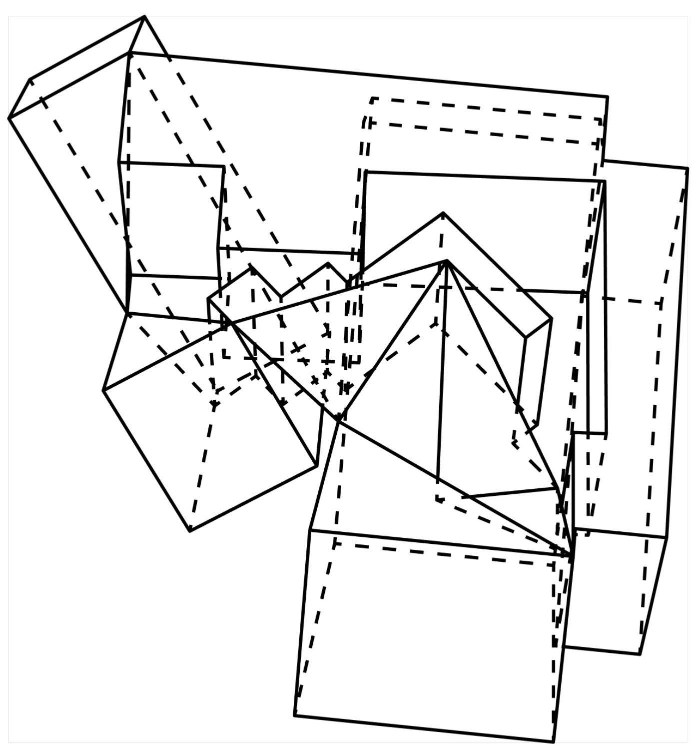


Figure 10. You are not expected to understand this diagram, but to marvel at its complexity.

[3] https://www.unhyperbolic.org/sydler.html

In July of 2022, Henry Segerman published a video[4] demonstrating a 3D print of Goerner's model, which inspired me to look for a simpler polyhedron with the same property. You've seen various results of that effort: my favourite is shown in Figure 1; Figures 6 and 7 show related alternatives.

It follows from work of Jessen[5] that there exist almost-orthogonal polyhedra whose non-right dihedral angle is any *algebraic angle*, i.e. any angle whose cosine (or equivalently sine) is an algebraic number.

However, Jessen's proof is not effective: it does not yield an actual construction for any such angle. So we are left with an interesting geometric challenge: for any given algebraic angle α, to construct an almost-orthogonal polyhedron whose remaining dihedral angle is α.

Might there even be a general method to construct, for *any* algebraic angle, an almost-orthogonal polyhedron with two faces meeting at that angle?

[4] https://www.youtube.com/watch?v=tH6vLXMaCwQ

[5] Børge Jessen, The algebra of polyhedra and Sydler's theorem, Math Scand 22 (1968) pp 241–256

Spotted It! – Strategies to Win Dobble

Gathering 4 Gardner Conference (G4G15)
Atlanta, Georgia from February 21–25, 2024

Christian Hummert,
Agentur für Innovation in der Cybersicherheit
"'Innovation for Cybersecurity"
Große-Steinstraße 19, D-06108 Halle (Saale), Germany
Email: hummert@cyberagentur.de

Bernd Krause,
Anhalt University of Applied Sciences,
Department Computer Science and Languages
Lohmannstraße 23, D-06366 Köthen, Germany
Email: bernd.krause@hs-anhalt.de

***Abstract*— Dobble or Spot It! is a game created by Blue Orange Games for two up to eight players. In the game the players have to find symbols in common between cards. The mathematics of the game is surprisingly interesting and several papers have been published on it. However, the works concentrate on the creation of dobble decks. This paper describes a psychological effect while playing and a statistical strategy to win the game.**

I. Introduction

Dobble or Spot It! Is a game created by Blue Orange Games for two up to eight players. In the game the players have to find symbols in common between cards. The game contains a deck of 55 cards with eight symbols on each card. Each player receives a pile with the same number of cards. One card remains in the middle of the table. Now each player compares the top card of their individual pile with the card in the middle of the table. As soon as they find a matching symbol, they may discard the card, which now becomes the new card in the middle of the table. The first player to discard all cards wins the round.

The game is particularly interesting because there is no chance. There is no dice and since all pairs of cards have exactly the same number of matching symbols, which is one, so there are no better or worse cards. In fact, at first glance it seems difficult to develop strategies for the game. Rather, it seems as if it is primarily about a talent or ability to find such pairs quickly.

The idea of the game goes back to the 19th century. In 1850, Thomas Penyngton Kirkmanan Anglican clergyman, submitted a puzzle to *The Ladies and Gentleman's Diary*, an annual recreational mathematics magazine that took content from both amateurs and professional mathematicians. The question read, *Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.* Dobble is a variation on this Kirkman's Schoolgirl problem.[1]

This paper will show that there are some psychological aspects that have an influence on quickly finding the right matching symbol. Thus, the game is not completely independent of luck. On the other hand, there is at least one theoretical strategy to gain advantage in the game: counting cards.

II. Dobble Mathematics

A complete Dobble game would have to consist of 57 cards and in this work all calculations will be made on a deck of 57 cards. At first, the number 57 seems quite unusual as it corresponds to $3 \cdot 19$ and the connection is initially unclear. The reasoning is as follows: First, any Dobble card is designed: it contains 8 symbols. each card in the game must match exactly one of these symbols. The rest of the deck is then divided into eight rows corresponding to the symbols of the first card. Suppose the first symbol on the first card corresponds to an armadillo. Since all cards in the first row must contain an armadillo, they must each contain 7 other unique symbols.

If a row contains k cards, then there are $8+7(k-1)$ different symbols in total (8 for the first card, and 7 new ones for each additional card in the row). For $m = 8$, that would be 57 symbols. However, the number m must also equal 8, as every card that is not in the armadillo row must share exactly one symbol with every card in this row. However, there are exactly 8 symbols on each card, so such a card can only match 8 cards in the armadillo row. The largest possible deck for cards with 8 symbols therefore consists of 57 cards.

In the US, there exists a variation of the game called Spot It! Junior, which only has 6 symbols per card. Consequently, the game consists of $6 + 5(6 - 1) = 31$ cards [2]. Interestingly, Heemstra showed in 2014 that it is impossible to create a Dobble for $k = 7$ with 43 cards [3]. Games with 11 and 12 symbols per card have been calculated, whether a game with 13 symbols per card is possible is still unknown [4]. Another strange fact about Dobble is, that geometry can be used, too, to answer these questions. In fact, Dobble has almost exactly the same structure as a geometrical object called a finite projective plane. This connection was described by Donna Dietz in 2013 [5]. However, there are also Dobble games that are not a finite projective plane [6]. A very nice mathematical description of Dobble is provided by the thesis of Christian Kathrein from the University of Graz [7].

In fact, the games on the market contain two cards less than a complete Dobble. A comment by Guillaume Gille-Naves, a former member of the development team for the French version of Dobble, explains that the game has 55 cards (instead of 57) for marketing purposes and to ensure that the rules of the game could be printed on extra cards without exceeding the manufacture limitation of 60 cards [8].

III. Evaluatung Dobble Games

To find starting strategies for Dobble games, I let two of my children play Dobble against each other. I wrote down each symbol that was called out loud. The Hungarian version of Dobble with 57 cards was used for the games, which only contains animals as symbols. I noted down a total of $k = 10$ games so that I could do some statistics. The rows of noted symbols are not the same length, as the game ends as soon as one of the children wins.

Table I shows the called out symbols of an example game. This game ended with the victory of one of the children after the 38th symbol was called out

TABLE I
NOTED SYMBOLS OF AN EXAMPLE GAME

No.	Symbol	No.	Symbol
1	Panda	20	Sheep
2	**Frog**	21	Hedgehog
3	**Frog**	22	Icebear
4	Zebra	23	**Scorpion**
5	Kangaroo	24	**Scorpion**
6	Camel	25	**Scorpion**
7	Donkey	26	Camel
8	Jellyfish	27	Raccoon
9	Seahorse	28	Eagle
10	Lion	29	Owl
11	Grasshopper	30	Goldfish
12	Horse	31	Sloth
13	Seahorse	32	Icebear
14	Chicken	33	**Dolphin**
15	Sloth	34	**Dolphin**
16	Crocodile	35	Pelican
17	**Bat**	36	Turtle
18	**Bat**	37	Buffalo
19	**Bat**	38	Pidgeon

A Dobble play ended after an average of 45.8 symbols called. It is immediately noticeable that in the example game it occurs several times that the same symbols are called out several times in direct succession and thus form tuples in the table (highlighted in bold here).

We consider a sequence of three cards. If the same symbol is named twice, this is called an *event*. If there are exactly two symbols that are called one after the other and are followed by another symbol, then this is a *double*. Accordingly, we call three identical symbols in a row a *triple* (This corresponds to two consecutive events.) and four symbols in a row a *quadruple* (This corresponds even to three consecutive events.). Table II shows the statistics for the events, doubles etc. in the ten games observed.

For each game the *Events per Sequences of 3 Cards* was calculated by dividing the *No. of events* by the *Total Symbols Called* -1. The last entry in the table is the rate of events in the total $n = 448$ $(458-10)$ inspected pairs of called symbols.

We want to investigate how high the expected value is for such events in a game. For the beginning of the game, however, it is clear how to calculate the probability of an event. Let's assume that the first match of the first hand card and the card in the middle was the armadillo. There are eight cards in the game showing the armadillo, two are used up. There are 55 cards left in the deck, so the chance is $p = 6/55$, about 0.109, that the next card is one with armadillo. However, this is not the probability of a double since we have to ensure that the third symbol called is not the armadillo.

TABLE II
STATISTICS OF THE TEN OBSERVED GAMES

Game	Total Symbols Called	No. of Doubles	No. of Triples	No. of Quadruples	No. of of Events	Events per Sequences of 3 Cards
1	46	4	0	0	4	0.089
2	48	7	0	0	7	0.149
3	48	5	0	0	5	0.106
4	45	2	0	0	2	0.045
5	38	2	2	0	6	0.162
6	43	6	1	0	8	0.190
7	45	6	0	0	6	0.136
8	41	3	2	0	7	0.175
9	53	6	0	0	6	0.115
10	51	6	0	3	15	0.300
Total	458	47	5	3	66	0.147

In the same way we can calculate the probability of calling three times the same symbol in a sequence of four cards. In our example there are 54 cards left in the pack and only five armadillo cards. The probability is therefore $6/55 \cdot 5/54 = 0.0099$. Accordingly, the probability of calling four times the same symbol in a sequence of five cards is $6/55 \cdot 5/54 \cdot 4/53 = 0.00073$. Extending this idea up to sequences of eight cards we have the probability of $6/55 \cdot 5/54 \cdot 4/53 \cdot 3/52 \cdot 2/51 \cdot 1/50 = 1.72 \cdot 10^{-6}$ for a septuple.

There are other interesting questions.

- After how many turns will a game be over?
  If only one player at a time finds the matching symbol, the game ends after $n = 28$ moves (in a two-player game). If the players are equally good up to their very last call, the game ends after $2n - 1 = 55$ moves. What is the distribution of the random number M of moves of a game, $28 \leq M \leq 55$? Assuming that $q > 1 - q$ is the probability that a player names the symbol before his opponent, then the distribution of the random number M can be calculated as follows
  $$P(M = n+k) = \binom{n+k-1}{k} \left(q^n(1-q)^k + (1-q)^n q^k\right)$$
  for $k = 0, 1, \ldots, n-1$. Based at the observed average, the most likely value is $q = 0.607$.
  If one assumes that on average both players can name the matching symbols in the same time, i.e. $q = 1/2$, then the mathematical expectation of the number of moves until a player wins would be $\mathrm{E}M = \mu_M = 50.05$. The standard deviation is $\sigma_M = 22.72$. This means that the average value of 45.8 for the game length calculated from the 10 observed games is within the associated 95%-confidence interval $\mu_M \pm 1.96 \cdot \sigma_M/\sqrt{10} = 50 \pm 14$.
- How many different symbols are mentioned during a game?

As there are a total of 57 different symbols in the game, a maximum of 55 different symbols can be named (one different symbol in each turn). As each symbol can be named a maximum of seven times in a game, at least 28 divided by seven, this is four different symbols that must be named. Table III shows the number of different symbols that were called in the ten observed games. The question asked is analogous to the question: what is the expected value of the number of different numbers drawn in r rounds of roulette? If symbols were actually randomly determined 55 times in a row from the existing 57 symbols by drawing and replacing them, then the average would be

$$57 \cdot \left(1 - (1 - 1/57)^{55}\right) = 35.5$$

different symbols drawn. However, since on average only 45.8 symbols were mentioned, only 31.7 symbols would be expected

TABLE III

NUMBER OF DIFFERENT SYMBOLS CALLED IN THE TEN OBSERVED GAMES

Game	Total Symbols Called	Different Symbols Called
1	46	34
2	48	33
3	48	37
4	45	35
5	38	29
6	43	28
7	45	31
8	41	30
9	53	32
10	51	29
Means	45.8	31.6

But can we assume that the probability of events occurring changes over the course of the game? The cards are evenly distributed and the symbols on the cards are also evenly distributed. The cards are drawn evenly without putting them back.

Since the probability $p = 6/55$ for an event calculated for the beginning of the game does not change over the entire game, one should expect to see $(m-2)p$ events in a game with m moves. The observed games show a slight accumulation of events compared to the calculated probability p.

The author assumes that the explanation for this lies in a psychological effect. There are many indications in research that what we humans see depends on what we hear, i.e. that the senses influence each other [9], [10]. This means that when I hear a certain symbol pronounced, such as armadillo, my visual cortex is conditioned to recognize armadillos and I will detect an armadillo much faster than if the symbol has not just been named.

Now a player can be lucky enough to have one of his cards in hand show a symbol just mentioned more often than the opponent's deck does. However, this psychological effect then gives him an advantage in the game. This means that the game is not completely free of luck.

IV. Statistical Evidence

The results of the ten observed games are already sufficient to reject the hypothesis of a purely random sequence of symbols and thus to prove that repeating symbols are actually recognized more often.

As the population of our statistical model, we consider all (equally probable) sequences of three different cards. As already explained, the probability that two identical animals are named (event A) is $p = P(A) = 6/55$. The random number X of events in n inspections is a binomially distributed random variable with mathematical expectation $\mathrm{E}(X) = n \cdot p$ and variance $\mathrm{Var}(X) = n \cdot p \cdot (1-p) = \sigma^2$. Thus, the number X of events occurring in the total number $n = 448$ of inspections in the observed ten games has the expectation $\mathrm{E}(X) = 448 \cdot 6/55 = 48.9$ with $\mathrm{Var}(X) = \sigma^2 = 448 \cdot 6/55 \cdot 49/55 = 43.5$ or $\sigma = 6,6$. The corresponding confidence interval to the significance level of $\alpha = 0.05$ is $np \pm 1.96\sigma$. With a probability of more than 95% X must therefore satisfy $35 \le X \le 62$. Because this obviously did not occur in the series of games observed, as can been seen from Table II, $X = 66$, it can be assumed that there is actually a higher event rate.

This can be explained by a modified calculation model for the event probability p'. In order to do this, sequences of four consecutive cards are considered. As before, the first two cards determine the event symbol (armadillo), while the following two cards belong to the two players. Assuming that the player who has the armadillo on his card always says this prior to the other player, then the value of p increases almost twice. More precisely, the probability that neither player can continue with the armadillo is $1 - p' = 49/55 \cdot 48/55$. That means $p' = 1 - 49/55 \cdot 48/55 = 0.208$.

If the player who can continue with armadillo is only able to use this advantage with probability λ and name the matching symbol prior to his opponent, the event probability scales linearly in λ,

$$p(\lambda) = p + \lambda(p' - p).$$

Based on this and the estimation $p(\lambda) = 0.147$ from Table II we get the most probable value of λ as $\hat{\lambda} = 0.386$.

V. Dobble Strategies

At first glance, there are no real strategies for winning a game of Dobble. The game seems to depend on individual talent. However, it is possible to improve the chances of winning, at least statistically, through strategy.

Let us assume that the human brain has to compare the symbols sequentially, and there is at least some evidence that this is a common strategy [11]. This means that I choose a symbol from my hand card, for example the armadillo, and look for it on the card in the middle. If I don't find a match, I choose the next symbol, check whether there is a match, and continue like this.

As a direct result of the above-mentioned probabilities, it is an advantage to first compare symbols that have not yet been mentioned or that have not yet or rarely been seen on previous cards (However, it is very difficult to count all the symbols. It can be assumed that all 57 symbols are seen at least once during a game. The player would therefore have to count 57 values in his head at the same time and would hardly have time to do so.). Since it is very difficult oder not possible to quickly record and count which symbols can be seen on all cards, we concentrate on the symbols that have been named by a player.

Since it is still difficult for the human brain to count how many times a maximum of 57 symbols have already been named (In the previous chapter, we showed that there are 31.6 symbols on average.), let's concentrate on the symbols that are actually named together. According to the statistics in the previous chapter, this occurs in most games, and the average of these events is 6.6 times.

A single symbol can be called a maximum of seven times in a game. However, it is not irrelevant whether the mentions are connected, i.e. whether they occur directly one after the other, or whether they are not connected, i.e. whether a different symbol is mentioned between each mention. In the case of related entries, the maximum is seven.

This means that according to the psychological effect that we humans see what we hear. The human mind automatically checks whether the symbol we have just heard leads to another match. Whether this is the case or not is a matter of luck. However, if this is not the case, symbols that have not yet been heard as doubles, triples or even quadruples should first be searched for sequentially on the other card. Since at least two and a maximum of nine of these events occurred in the games examined, it is not too difficult to memorize these events. For the symbols that have not yet been mentioned in an event, the chance that there will be a match is increased.

It is interesting to note that the proposed strategy does not work for computers. A computer always needs eight comparisons to find a match between the two cards. This means that when two computers play Dobble against each other, the question of whether one of them wins depends only on the order of the symbols on the cards. In each round, one of the two computers is lucky that its symbol is higher up in the list of eight comparisons.

With the proposed strategy, however, the computer gains nothing, but loses time, as it requires an extended comparison with the stored events. It does not save any comparison, but has to compare more. with humans, however, it is different, the comparison with the memorized symbols takes much less time than searching for a symbol on the other card. This is where the strategy makes sense.

Nevertheless, it would be interesting to have two PCs play Dobble against each other. A lot could certainly be learned from these games and the statistics could be improved.

VI. Conclusion

It's obvious that the math behind the game Dobble is exciting. However, most of the papers published so far relate to the creation of Dobble sets. I am not aware of any paper that deals with the gameplay itself.

The game is especially interesting at first sight because luck does not seem to be a component of the game. However, in this paper we have shown that the game is not completely free of luck. This is due to the effect anchored in the human psyche that we recognize a symbol that we hear more quickly because the visual and auditory cortex are strongly connected. This effect could already be statistically verified, on a small sample.

On the other hand, this work has proposed a strategy for Dobble games, the counting of cards. It does not have a great influence on the course of the game, but slightly improves the statistical chances of finding a pair first.

Dobble is a fascinating game that even very young children can learn well. The game is also well suited to introducing pupils or students to mathematical problems. In the past, it has already been used in language lessons [12], [13], but it could definitely be used in math courses.

In this sense – Let's play Dobble!

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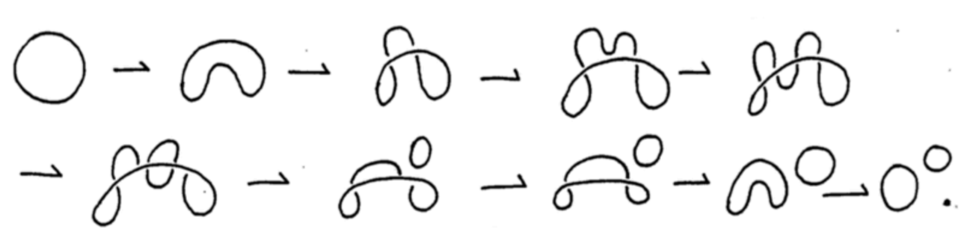
Note: Spot It! is a registered trademark of Asmodee Group and Dobble is a trademark of Asmodee Group.

15 MATHEMAGICS for G4G15 -

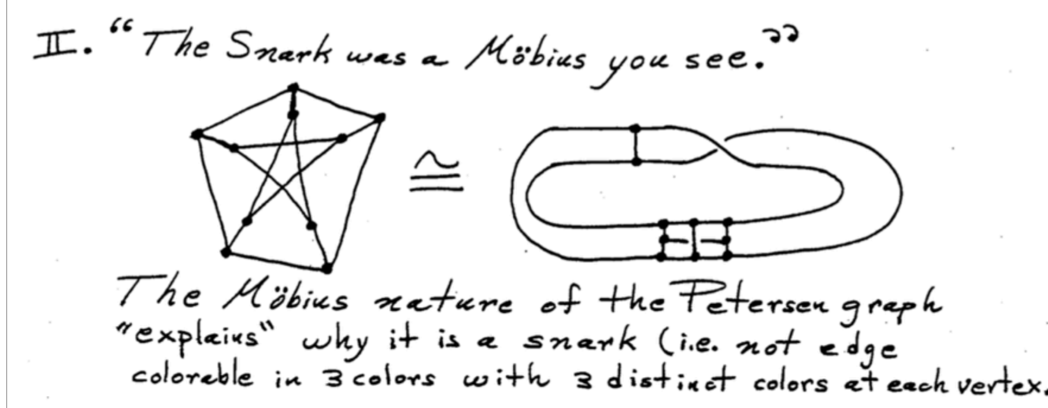
In-Between Magic and Topology

Louis H Kauffman
Math UIC
Chicago, IL
loukau@gmail.com
www.math.uic.edu/~kauffman

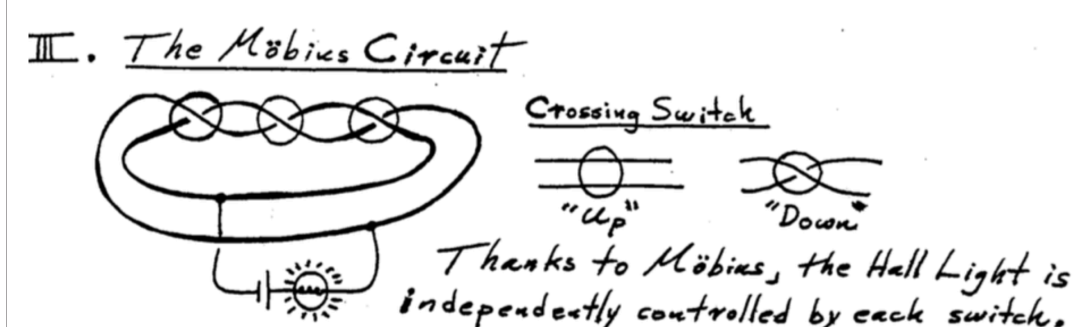
I. A Self-Reproducing Loop
(Courtesy of Kurt Reidemeister and
Sam Lloyd)



This shows how one loop could become two loops in a series of actions that almost looks topological.

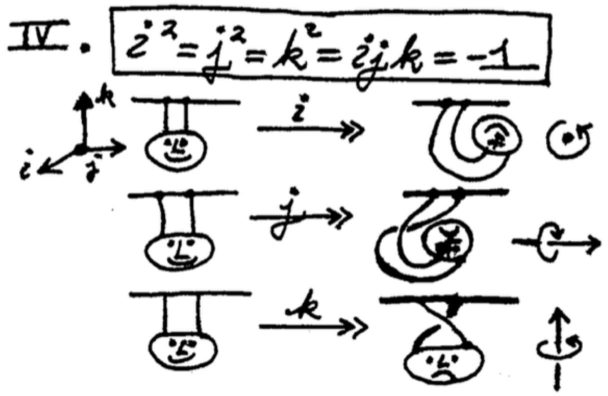


The famous Petersen graph is on the left in its usual incarnation, but really the Petersen is just another appearance of the Mobius strip.

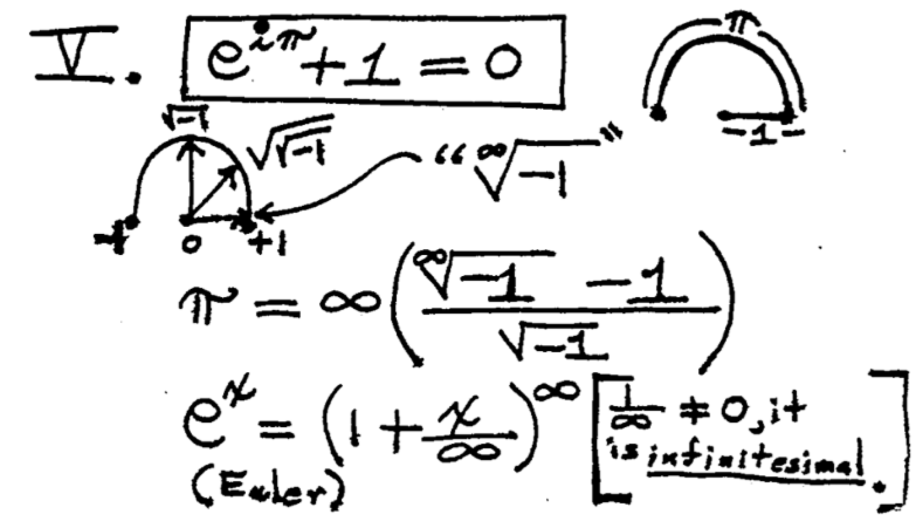


We hope that this is self-explanatory, and that you will go home and use the Mobius band to design a circuit to control the light at your front door from switches in every room in your house.

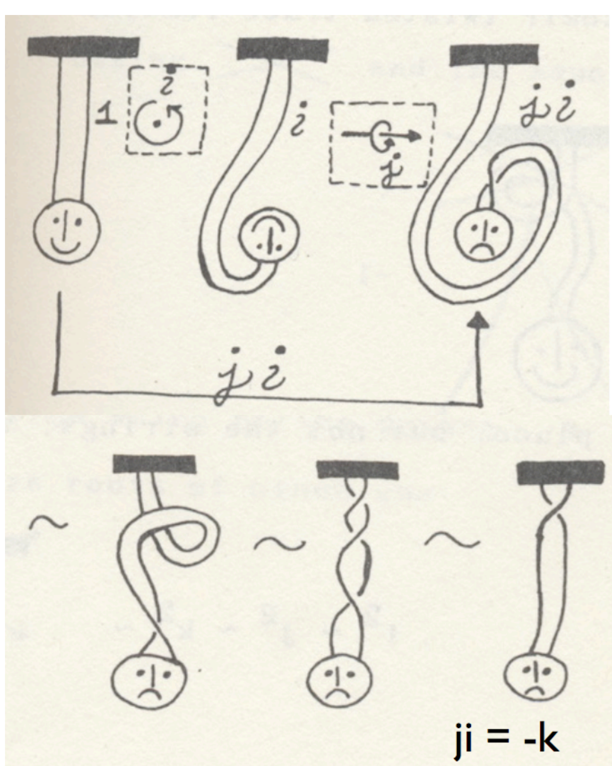
The Quaternions Personified

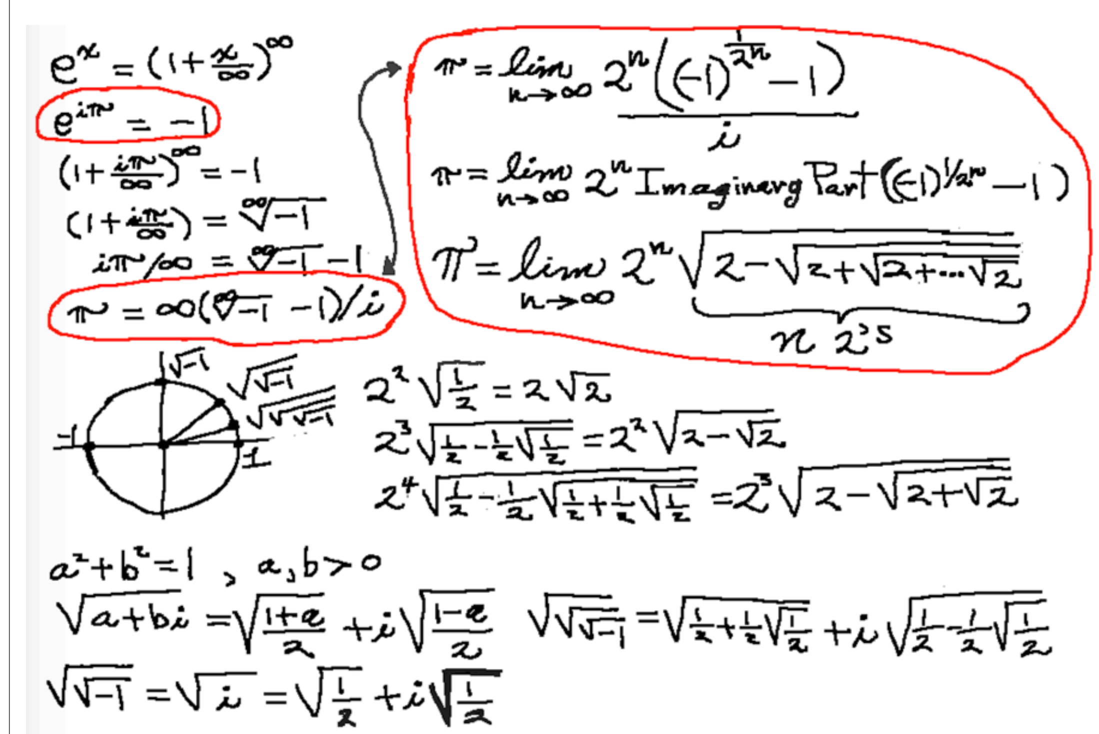


Yes. There they are the quaternions i, j and k. And they can be understood as the topological symmetries of a little face attached by puppet strings to the ceiling.



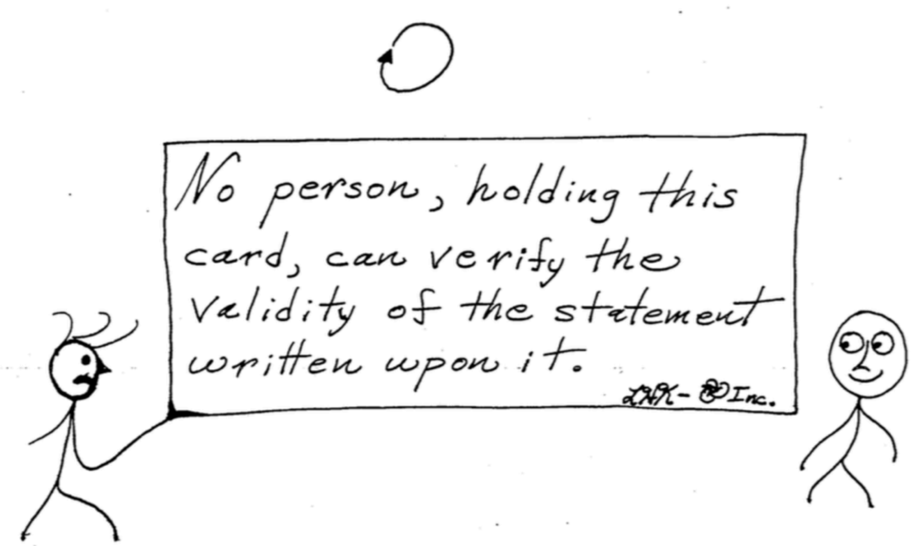
Here we have Euler's beautiful formula and an iconoclastic formula for Pi that is obtained by solving for Pi in Euler's formula. The formula for Pi is correct!



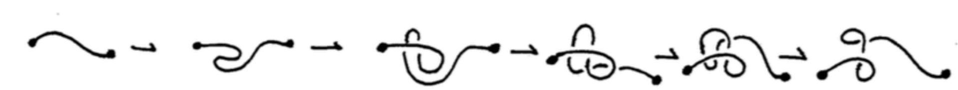


$$\pi = \infty\left(\frac{(-1)^{1/\infty} - 1}{\sqrt{-1}}\right)$$

VI.



"I'm holding the card.
But if I was not holding the card, then I could easily see that anyone who does hold the card is prevented from asserting the validity of the statement on the card. Thus it certainly is correct — what is written on the card.
But I am holding the card.
Therefore I am prevented from doing what I have just done!"



My Favorite Math Jokes

Tanya Khovanova

February 29, 2024

Abstract

For many years, I have been collecting math jokes and posting them on my website. I have more than 400 jokes there. In this paper, which is an extended version of my talk at the G4G15, I would like to present 66 of them.

1 Mathematicians and Humor

Mathematicians are very logical and precise. Here is a joke about that.

> A guy is complaining to his mathematician friend:
> — I have a problem. I have difficulty waking up in the morning.
> — Logically, counting sheep backwards should help.

Did I mention that mathematicians are precise? Once, during a lecture, a professor said the following.

> Assume, for the sake of clarity, that this yellow cube is a blue sphere.

Mathematicians are so focused on their abstractions that they ignore the people around them. I met a lot of introverted mathematicians. Here is their favorite joke.

> —What is the difference between an extroverted and an introverted mathematician?
> —An introverted mathematician will look at his own shoes when he talks to you. An extroverted one will look at your shoes.

I was around mathematicians all my life. I even married a mathematician when I was very young. I didn't learn from my mistakes and married a mathematician again, and again.

> —Honey, we are like two parallel lines.
> —Why do you say that?
> —The intersection of our life paths was a mistake.

I like humor, but I didn't have enough of it around me. So I started collecting jokes and posting them on my website [1].

2 My Own Joke

Currently, I have more than 400 jokes. Here is a classic one.

> A topologist is someone who doesn't know the difference between a coffee cup and a doughnut.

I usually do not invent jokes but rather collect them. Here is one of the exception based on the previous joke.

> A topologist walks into a cafe:
> — Can I have a doughnut of coffee, please?

Of course, I also told math jokes to my children when they were young; and they too got creative.

3 Homework Excuses

My kids heard the standard excuse: my dog ate my homework, and decided to invent more intelligent excuses. They started with a simple variation.

> My biology homework ate my math homework!

Then, they made their excuses more mathematical.

> I did part of the homework; the part I have left to do, is 0.999999999...

Here is why they didn't do the physics homework.

> I tried to build a black hole in my bedroom when my homework suddenly disappeared.

Then, they got interested in computers.

> I accidentally divided by zero and my computer burst into flames.

Later, one of them became a programmer.

> My mother redefined my doTheHomework() method with doTheDishes() method.

And, as all nerds, they loved "The Lord of the Rings".

My homework was consumed by the power of the One Ring, and it no longer submitted to my will.

4 My Grandson's Joke

My family consists of nerds. If I asked them, 'What's up?', I always get one of two replies, depending on whether we are inside or outside: 'the sky' or 'the ceiling'. If I ask them to say something, they reply 'something'. Now, I tell math jokes to my grandchildren. For example, this famous one.

A logician rides an elevator. The door opens and someone asks:
—Are you going up or down?
—Yes.

My grandson invented his own silly follow-up for the logicians joke.

A logician rides an elevator. The door opens and someone asks:
—Are you going up or down?
—No, replies the logician and walks out.

5 My Teaching

You won't be surprised to know that the homework I give to my students starts with a math joke. For example, my homework on prime numbers includes the following one.

Two is the oddest prime.

My recent homework on Fibonacci numbers had this joke.

Fibonacci salad: For today's salad, mix yesterday's leftover salad with that of the day before.

Here is one for geometry homework.

Without geometry, life would be pointless.

And another one for algebra.

—Why was algebra so easy for the Romans?
—X was always 10.

My topology homework starts with this joke.

—Why did the chicken cross the Möbius strip?
—To get to the other ... er, um ...

I teach a lot of number theory, so I need many jokes about numbers. Here is one of them.

—Do you know what's odd?
—Every other number.

Though my students are in middle school, sometimes we dive into calculus.

—What did the student say about the calculus equation she couldn't solve?
—This is derive-ing me crazy!

Here is a joke for when I cover statistics.

—Did you hear about the statistician who drowned crossing a river?
—It was three feet deep, on average.

I like geometry, and I might give a long lecture about triangles. This is a triangle joke.

—Why is the obtuse triangle always upset?
—It is never right.

6 My Students

Once, part of the homework I gave was to invent a math joke. Here is what one of my student submitted.

—Why is Bob scared of the square root of 2?
—Because he has irrational fears.

Here is another student's creation.

Everyone envies the circle. It is well-rounded, and highly educated: after all, it has 360 degrees.

Yet another student invented a joke on that week's topic: sorting algorithms.

—Which sorting algorithm is the most relaxing?
—The bubble bath sort.

7 Submitted to Me

Not only do I get new jokes from my family and students, I also receive them from friends and even strangers. This is my friend's daughter talking to her teacher.

Teacher: What are whole numbers?
Student: Like 0, 6, 8, 9.
Teacher: And what about 10?
Student: It is half-whole, 1 doesn't have a hole.

Here is another one from the same source.

Teacher: Solve the equation: $x + x + x = 9$.
Student: $x = 3$, 3, and 3.

This one was received from another friend.

Quantum entanglement is simple: when you have a pair of socks and you put one of them on your left foot, the other one becomes the "right sock", no matter where it is located in the universe.

This joke was submitted by a stranger.

You have to be odd to be number ONE.

8 Funny Theorems

Many people laugh at mathematicians. But do mathematician work on funny research? You bet! My coauthor Joel Lewis invented two theorems [2].

The Big Point Theorem. Any three lines intersect at a point, provided that the point is big enough.

The Thick Line Theorem. Any three points lie on the same line, provided that the line is thick enough.

These ideas can be used for the following famous puzzle. You are given nine dots arranged in three rows and three columns. Connect the dots by drawing four straight, continuous lines that pass through each of the dots without lifting your pencil from the paper. The solution to this puzzle is the most famous example of thinking outside the box. However, there is a humorous solution [3] where you can connect the dots using only three lines. Can you figure it out?

9 Physics

Mathematics is close to physics. So I have a few physics jokes in my collection. The following joke combines math and physics.

Einstein-Pythagoras equation: $E = m(a^2 + b^2)$.

The next joke is about pure physics.

Looking for energy?
Multiply time by power!

Now, a joke about time.

The barman says, "We don't serve time travelers in here."
A time traveler walks into a bar.

The next joke might be considered to be about time too, but it is about the famous physicist, Werner Heisenberg, and his work.

Heisenberg gets pulled over on the highway.
Cop: "Do you know how fast you were going, sir?"
Heisenberg: "No, but I know exactly where I am."

After my talk at the G4G15, David Albert sent me a sequel to this joke.

Heisenberg gets pulled over on the highway.
Cop: "Do you know how fast you were going, sir?"
Heisenberg: "No, but I know exactly where I am."
Cop: "You were going 85 miles per hour".
Heisenberg: "Oh great—now I'm lost!"

10 Computer Science

Mathematics is close to computer science too. Not to mention that one of my children is a computer scientist. So I have a section for computers and programming.

Humanity invented the decimal system, because people have 10 fingers. And they invented 32-bit computers, because people have 32 teeth.

Here is another joke about computers.

— Why is your disc drive so noisy?
— It is reading a disc.
— Aloud?

And another one about people interacting with computers.

My computer always beats me in chess. As revenge, I always beat it in a boxing match.

The next joke is about computer science's influence on people.

I saw our system administrator's shopping list. The first line was tomatoes.zip for ketchup.

The next joke compares famous modern websites.

Wikipedia: I know everything.
Google: I can find anything.
Facebook: I know everyone.
Internet: You are nothing without me.
Electricity: Shut up, jerks.

Here is a recent joke about AI.

My Roomba has just devoured a piece of cheese I wanted to pick up and eat. The war between humans and robots is already here.

11 Puns

Because English is my second language, it took me some time to appreciate puns. I started adding them to my collection only recently. Let me start with a self-referencing pun.

Not all math puns are terrible. Just sum.

Speaking about sums. . . .

—Why did the two 4's skip lunch?
—They already 8!

Another pun about numbers.

—Why shouldn't you argue with a decimal?
—Decimals always have a point.

Now we move to geometry.

—Who invented the Round Table?
—Sir Cumference.

Here is another related one to circumference.

—How many bakers does it take to bake a pi?
—3.14.

12 Teacher Puns

One category for which I receive a lot of puns is math teachers and schooling. Let's start with multiplication.

—Why did the student do multiplication problems on the floor?
—The teacher told him not to use tables.

Now we go to counting.

—Why do cheapskates make good math teachers?
—Because they make every penny count.

Next, we have a geometry teacher joke.

I saw my math teacher with a piece of graph paper yesterday. I think he must be plotting something.

Here is one combining calculus with geometry.

—Why was math class so long?
—The teacher kept going off on a tangent.

Here is a joke about behavior of math teachers.

—What does a hungry math teacher like to eat?
—A square meal.

I live in Massachusetts, so the next joke is dear to me.

—What state has the most math teachers?
—Math-achusetts.

13 More about Mathematicians

Now back to mathematicians.

—Did you hear about the mathematician who's afraid of negative numbers?
—He'll stop at nothing to avoid them.

Here is a pun about mathematicians.

—What do mathematicians and the Air Force have in common?
—They both use pi-lots.

And another one.

—Where do mathematicians like to go?
—Times Square.

14 Kinds of People

Some jokes go in series. One of my favorite series starts as, "There are X kinds of people."

There are three kinds of people in the world: those who can count and those who can't.

The next joke is about binary system.

There are 10 kinds of people in the world, those who understand binary, and those who don't.

Here is a rare fractal joke.

There are two kinds of people: those who know nothing about fractals and those who think that there are two kinds of people: those who know nothing about fractals and those who think that there are two kinds people. . . .

Data science is adjacent to mathematics. So I include a joke about data here.

There are two kinds of people in this world: Those that can extrapolate from incomplete data. . . .

15 Miscellaneous

Here are some extra jokes for desert.

—Did you know that the human brain uses only one third of its capacity?
—Hmm, what does the other third do?

—Why was the equal sign so humble?
—Because she knew she wasn't greater than or less than anyone else.

—What are ten things you can always count on?
—Your fingers.

50% of marriages end with divorce. The other 50% end with death.

If a man tries to fail and succeeds, which did he do?

References

[1] Tanya Khovanova, Math Humor, available at `http://tanyakhovanova.com/Jokes/jokes.html`.

[2] Tanya Khovanova, The Big Point Theorem, available at `https://blog.tanyakhovanova.com/2022/08/the-big-point-theorem/`

[3] Tanya Khovanova, Thinking Inside and Outside the Box, math:GM arXiv:1605.02232, 2016.

The Hilbert Curve ⇔ Thue-Morse Sequence

Douglas M. McKenna

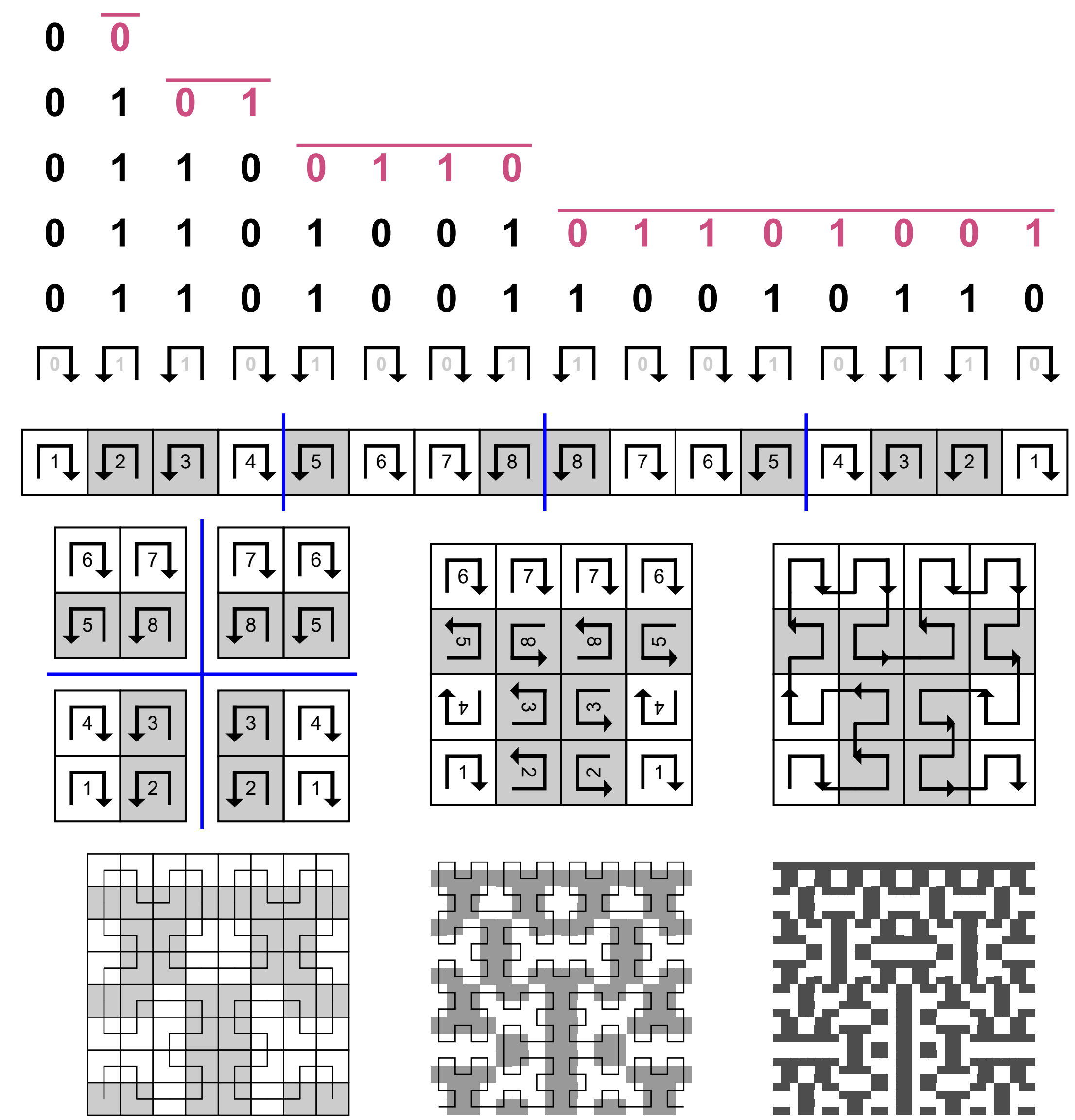
Department of Hierarchæology
Mathemæsthetics, Inc.
Boulder, Colorado USA

doug@mathemaesthetics.com

G4G15 Gift

Abstract

The Hilbert Curve comprises multiple, smaller-scale Hilbert Curves. At each scale, each sub-curve is oriented either horizontally or vertically, all piecewise-connected into a continuous traversal of the square. We show (without words) that the sequence of these orientations is the same as the Thue-Morse sequence.

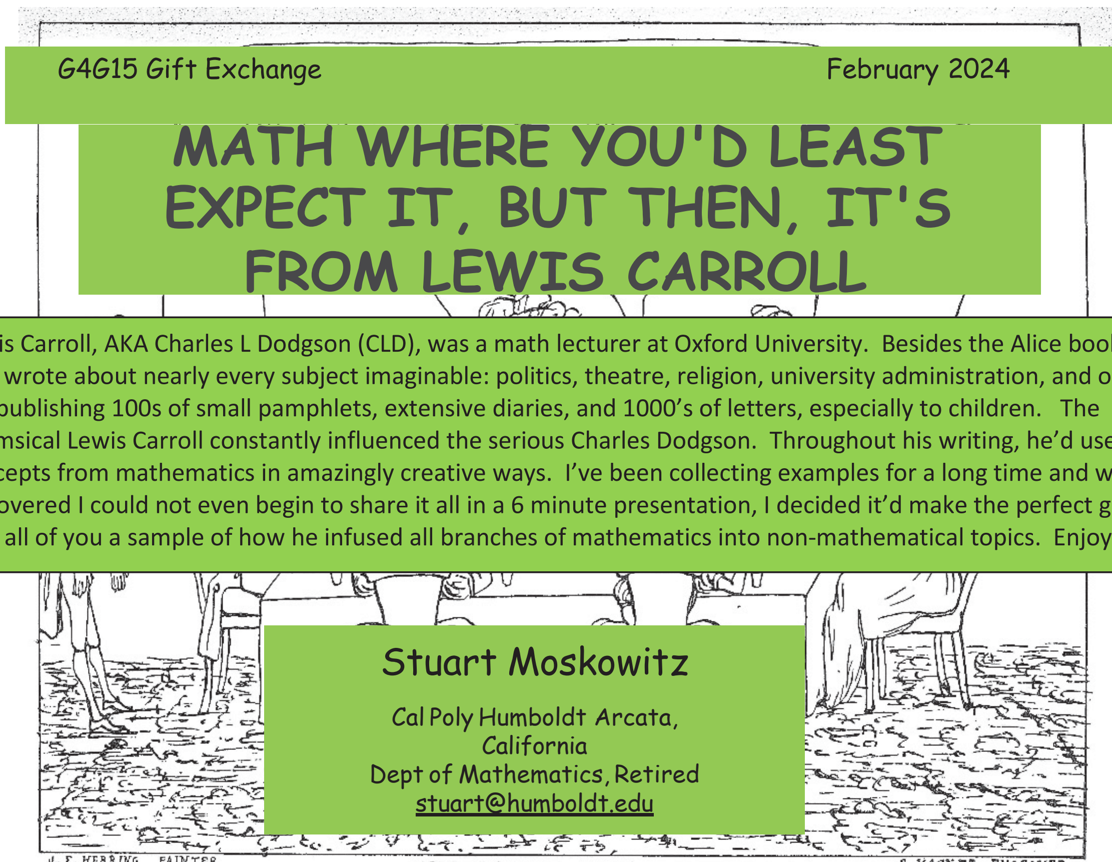


G4G15 Gift Exchange February 2024

MATH WHERE YOU'D LEAST EXPECT IT, BUT THEN, IT'S FROM LEWIS CARROLL

Lewis Carroll, AKA Charles L Dodgson (CLD), was a math lecturer at Oxford University. Besides the Alice books, he also wrote about nearly every subject imaginable: politics, theatre, religion, university administration, and on and on, publishing 100s of small pamphlets, extensive diaries, and 1000's of letters, especially to children. The whimsical Lewis Carroll constantly influenced the serious Charles Dodgson. Throughout his writing, he'd use concepts from mathematics in amazingly creative ways. I've been collecting examples for a long time and when I discovered I could not even begin to share it all in a 6 minute presentation, I decided it'd make the perfect gift, to give all of you a sample of how he infused all branches of mathematics into non-mathematical topics. Enjoy!!

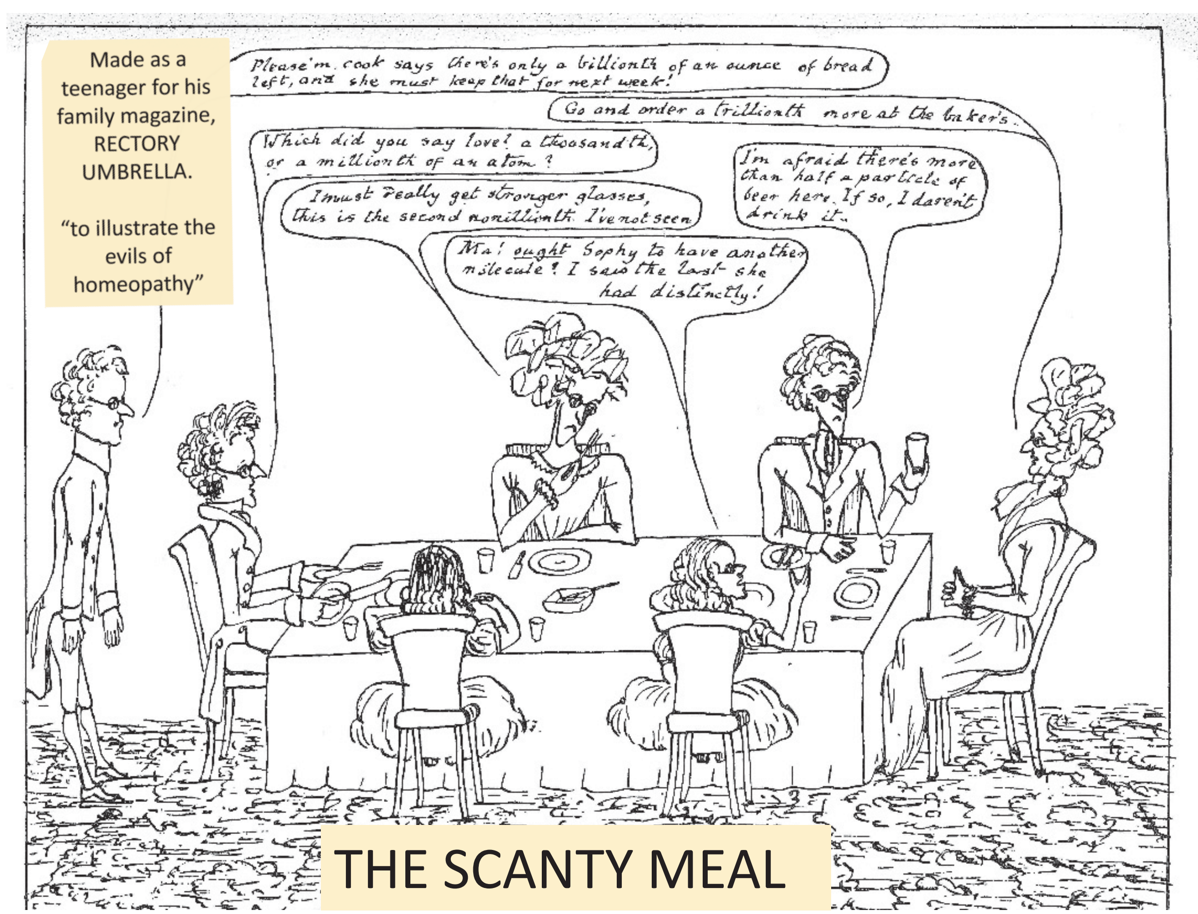
Stuart Moskowitz
Cal Poly Humboldt Arcata, California
Dept of Mathematics, Retired
stuart@humboldt.edu



THE SCANTY MEAL.
from the picture in the Vernon Gallery.

Made as a teenager for his family magazine, RECTORY UMBRELLA.

"to illustrate the evils of homeopathy"



THE SCANTY MEAL

Fractions, Decimals, and Big Numbers

"Children are three-fourths of my life"

as quoted by Isa Bowman in *The Story of Lewis Carroll,* by Isa Bowman

Here he uses repeating decimals seemingly just to get Dorothy's attention. In this case, .99999… = 1 is counterintuitive, but I'd guess that Dorothy will never forget it after this!

Nov 11, '96

There my dear Dorothy,

……..I write to ask whether you are disengaged for next Saturday evening, and if so, whether I may fetch you at $6\frac{1}{2}$, to one of my grand dinner - parties.

Do not be alarmed at the *number* of the guests: it will be .99999&c. It *looks* alarming, I grant: but circulating decimals lose much of their grandeur when reduced to … fractions!

$.99999\&c = .\overline{9} = 1$

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He wrote so many letters that he invented a case for organizing stamps. And he included a little booklet with advice for writing letters. Here's three of the "wise words".

RULE #2: Don't fill more than a page and a half with apologies for not having written sooner.

RULE #3: *Don't repeat yourself*. When once you have said your say, fully and clearly, on a certain point, and have failed to convince your friend, *drop that subject*: to repeat your arguments, all over again, will simply lead to his doing the same; and so you will go on, like a Circulating Decimal. *Did you ever know a Circulating Decimal come to an end?*

RULE #5: If, in picking a quarrel, each party declined to go more than *three-eighths* of the way, and if, in making friends, each was ready to go *five-eighths* of the way, —why, there would be more reconciliations of quarrel!

July 21, 1876

My Dear Gertrude,

.....I send you seven kisses (to last a week) and remain

Your loving friend

C. L. Dodgson.

My Dear Amy,

.... Also I send two kisses and a half, for you to divide with Agnes, Emily, and Godfrey. Mind you divide them fairly.

Yours affectionately,

C. L. Dodgson

April 19, 1878

My Dear Gertrude,

Your loving friend

C. L. Dodgson.

I send you $4\frac{3}{4}$ kisses.

April 13, 1878

My Dear Gertrude,

.... I send you 10,000,000 kisses, and remain

Your loving friend

C. L. Dodgson

August 15, 1892

My Dear Alice,

.........I send my best love, for you to divide with your brother: and I would advise you to give two-thirds to him, and take three-quarters for yourself

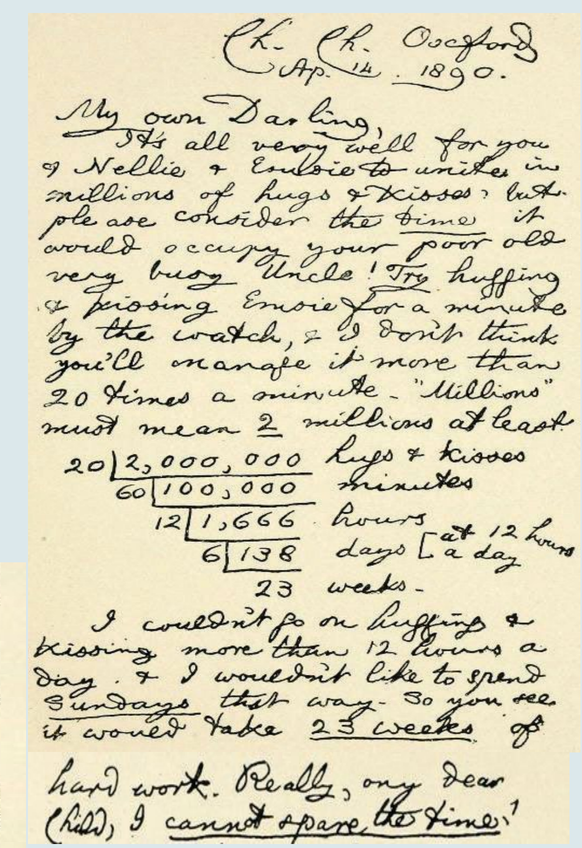
Yours affectionately,

C. L. Dodgson

Any form of exaggeration generally called from him a reproof, though he was sometimes content to make fun. For instance, my sisters and I had sent him "millions of kisses" in a letter. Below you will find the letter that he wrote in return.

Isa Bowman

I have included nearly the whole of his response to Isa because of how clearly he's able to explain why "millions of kisses" is impossible.

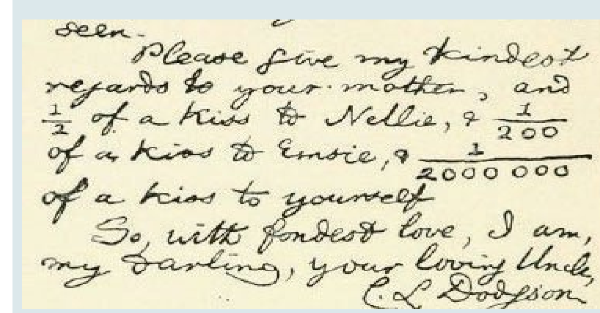


Ch. Ch. Oxford
Ap. 14. 1890.

My own Darling,
It's all very well for you & Nellie & Emsie to unite in millions of hugs & kisses: but please consider the *time* it would occupy your poor old very busy Uncle! Try hugging & kissing Emsie for a minute by the watch, & I don't think you'll manage it more than 20 times a minute. "Millions" must mean 2 millions at least.

20 | 2,000,000 hugs & kisses
60 | 100,000 minutes
12 | 1,666 hours
6 | 138 days [at 12 hours a day
23 weeks.

I couldn't go on hugging & kissing more than 12 hours a day: & I wouldn't like to spend Sundays that way. So you see it would take 23 weeks of hard work. Really, my dear child, I cannot spare the time.



seen.
Please give my kindest regards to your mother, and $\frac{1}{2}$ of a kiss to Nellie, & $\frac{1}{200}$ of a kiss to Emsie, & $\frac{1}{2000000}$ of a kiss to yourself.
So, with fondest love, I am, my darling, your loving Uncle,
C. L. Dodgson

CLD was the 3rd of 11 children, a big family. We know that he was somewhat lonely and was bullied by other children. Not much is known about his mother. The letter below (I've included the whole letter), which she wrote to him while she was away caring for a sick relative, was something he treasured for his entire life. And it might explain what motivated him to sign his letters with so many different quantities of kisses. To keep his sisters from taking the letter, he immediately put a rather sad response on the back and treasured the letter his whole life.

My Dearest Charlie,

I have used you rather ill in not having written to you sooner, but I know you will forgive me, as your Grandpapa has liked to have me with him so much and I could not write and talk to him comfortably. All your notes have delighted me, my precious children, and show me you have not quite forgotten me. I am always thinking of you, and longing to have you all round me again more than words can tell. God grant that we may find you all well and happy on Friday evening. I am happy to say your dearest Papa is quite well—his cough is rather *tickling,* but is of no consequence. It delights me, my darling Charlie, to hear that you are getting on so well with you Latin, and that you make so few mistakes with your Exercises. You will be happy to hear that your dearest Grandpapa is going on nicely—indeed I hope he will soon be quite well again. He talks a great deal and most kindly about you all. I hope my sweetest Will says "Mama" sometimes, and thatyour precious Tish has not forgotten. Give them all my other treasures including yourself, 1,000,000,000 kisses from me, with my most affectionate love. I am sending you a shabby note, but I cannot help it. Give my kindest love to Aunt Dar, and believe me, my own dearest Charlie, to be your sincerely affectionate

MAMA

on the back he wrote:

"No one is to touch this note, for it belongs to C. L. D.
Covered with slimy pitch so that they will wet their fingers."

MEASUREMENTS….

In a letter to Edith Denman:

….There is a rashness, which I can only deplore, in your assertion that I cannot be as fond of figure-drawing as yourself! The point cannot be settled till we have measured the two fondnesses by the same unit. Now the unit of pleasure (which I suggested years ago, and which Society hasn't yet adopted!) is "the pleasure felt in eating one penny-bun in one minute." Please to estimate the pleasure which *you* get from an hour of figure drawing, using that as a unit, and then we can compare numbers: my number is 235. Trying to settle it without a unit is like arguing about two rooms, each saying, "I'm sure *this* room is the hottest!" without ever referring to a thermometer

From no units to mixing up units:

"And on the dead level our pace is---?" the younger suggested; for he was weak in statistics, and left all such details to his aged companion. "Four miles in the hour," the other wearily replied. " Not an ounce more; he added, with that love of metaphor so common in old age, " and not a farthing less!"

Tangled Tale, Knot 1

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Playing With Infinity

'The more noise you make, the less jam you will have, and *and vice versa'.*
And I thought they wouldn't know what 'vice versa' meant: so I explained it to them. I said 'If you make an infinite noise, you'll get no jam: and if you make no noise, you'll get an infinite lot of jam.'

A Tangled Tale, Knot 5

"Come to me, my little gentleman," said our hostess, lifting Bruno into her lap, "and tell me everything." "I ca'n't" said Bruno. "There wouldn't be time. Besides, I don't know everything."

Sylvie and Bruno, Concluded

Using Division by zero
in his report as keeper of the Oxford club's wine cellar:

The consumption of Madeira has been during the past year, zero. After careful calculation I estimate that, if this rate of consumption be steadily maintained, our present stock will last us an infinite number of years. …. (We) may yet cheer ourselves with the thought of how economically it can be done.

Twelve Months in a Curatorship. 1884

PLAYING WITH TIME

November 30, 1879

I have been awfully busy, and I've had to write heaps of letters- wheelbarrows full, almost. And it tires me so that generally I go to bed again the next minute after I get up: and sometimes I go to bed agan a before I get up! Do you ever hear of any one being so tired as that?

Your loving friend,
C. L. Dodgson

PROOF BY CONTRADICTION

You are quite correct in saying it is a long time since you have heard from me: in fact, I find that I have not written to you since the 13th of last November. But what of that? You have access to the daily papers. Surely you can find out negatively, that I am all right! Go carefully through the list of bankruptcies ; then run your eye down the police cases ; and, if you fail to find my name anywhere, you can say to your mother in a tone of calm satisfaction, " Mr. Dodgson is going on well."

ON USING STATISTICS

Long and painful experience has taught me one great principle in managing business for other people, viz., if you want to inspire confidence, *give plenty of statistics.* It does not matter that they should be accurate or even intelligible, so long as there is enough of them. A Curator who contents himself with simply *doing* the business of a Common Room, and who puts out no statistics, is sure to be distrusted. "He keeps us in the dark!" men will say. "He publishes no figures. What does it mean? Is he assisting himself?" But, only circulate some abstruse tables of figures, particularly if printed in lines and columns, so that ordinary readers can make nothing of them, and all is changed at once. "Oh, go on, go on!" they cry, stiated with facts. "Manage things as you like! We traust you entirely!"

Three Years in a Curatorship. 1886.

GEOMETRY

My Dear Ethel,
…….it was very nice of you to bring my dear old friend to see me, and when she had vanished from my gaze what had I but mathematical considerations to console me? "She may be limited and superficial," I said to my myself. "She may even be without depth. But she is at least equilateral and equiangular—in one word, what is she but a Polygon?"!

Using Euclid to make fun of the 1865 Parliamentary elections

The Dynamics of a Parti-cle

CHAPTER 1: DEFINITIONS

I. PLAIN SUPERFICIALITY is the character of a speech, in which any two points being taken, the speaker is found to lie wholly with regard to those two points.

II. PLAIN ANGER is the inclination of two voters to one another, who meet together, but whose views are not in the same direction.

III. When a Proctor, meeting another Proctor, makes the votes on one side equal to those on the other, the feeling entertained by each side is called Right Anger.

IV. When two parties, coming together, feel a Right Anger, each is said to be COMPLEMENTARY to the other (though strictly speaking, this is very seldom the case.

V. OBTUSE ANGER is that which is greater than Right Anger.

He used mathematics to explain nearly everything…

- On Saturday Isa had a Music Lesson, and learned to play on an American Orguinette. It is not a very handle round and round: so she did it nicely. You put a long piece of paper in, and it goes through the machine, and the holes in the paper make different notes play. They put one in wrong end first, and had a tune backwards, and soon found themselves in the day before yesterday: so they dared not go on, for fear of making Isa so young she would not be able to talk. The A.A.M. does not like visitors who only howl, and get red in the face, from morning to night.

  - *from The Story of Lewis Carroll by Isa Bowman*

THE MONEY ACT

…The Professor brightened up again. "The Emperor started the thing, he said. "He wanted to make everybody in Outland twice as rich as he was before-just to make the new Government popular. Only there wasn't nearly enough money in the Treasury to do it. So I suggested that he might do it by doubling the value of every coin and banknote in Outland. It's the simplest thing possible. I wonder nobody ever thought of it before! And you never saw such universal joy. The shops are full from morning to night. Everybody's buying everything….

From Sylvie and Bruno. 1889.

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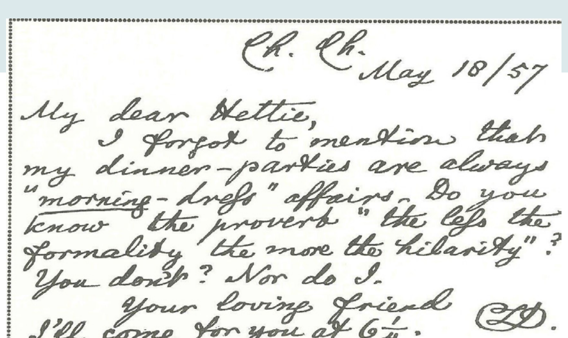
LOGIC

- My Dear Ella,
- …..Now when will you come? We have a college meeting at 1 ½ today, but not likely it will be over by 3, so you might take your chance if you happen to be walking this way. If it is over by 3, it will be; and if it isn't, it won't be. That's logic.

CHRIST CHURCH, OXFORD,   Dec. 9, 1875.

Mv DEAR GERTRUDE,-This really will not do, you know, sending one more kiss every time by post : the parcel gets so heavy it is quite expensive. When the postman brought in the last letter, he looked quite grave. "Two pounds to pay, sir!" he said. "*Extra weight, sir!*" (I think he cheats a little, by the way. He often makes me pay two pounds, when I think it should be pence). '' Oh, if you please, Mr. Postman!" I said, going down gracefully on one knee (I wish you could see me go down on one knee to a postman -it's a very pretty sight), "do excuse me just this once! It's only from a little girl!"
"Only from a little girl!" he growled. "What are little girls made of?" "Sugar and spice," I began to say, "and all that's ni-'' but he interrupted me. "No! I don't mean that. I mean, what's the good of little girls, when they send such heavy letters?" "Well, they're not much good, certainly," I said, rather sadly.
"Mind you don't get any more such letters," he said, "at least, not from that particular little girl. I know her well, and she's a regular bad one ! " That's not true, is it? I don't believe he ever saw you, and you're not a bad one, are you? However, I promised him we would send each other very few more letters-" Only two thousand four hundred and seventy, or so," I said. "Oh ! " he said, "a little number like that doesn't signify. What I meant is, you mustn't send many."
So you see we must keep count now, and when we get to two thousand four hundred and seventy, we mustn't write any more, unless the postman gives us leave.
I sometimes wish I was back on the shore at Sandown ; don't you?

Your loving friend,
Lewis Carroll


Ch. Ch. May 18/57
My dear Hettie,
I forgot to mention that my dinner-parties are always "morning-dress" affairs. Do you know the proverb "the less the formality the more the hilarity"? You don't? Nor do I.
Your loving friend,
CLD.
I'll come for you at 6¼.

Commutativity

My dear Mary,
……why, Oxford is as near to London as London is to Oxford. If your geography-book doesn't tell you that, it must be a wretched affair….
Your loving friend,
Charles L. Dodgson

***Using Experiences from Math Class to reach children during a rare public address in church*….**

…As you rub out the sums on your slate that so leave behind the disobedience, or selfishness, or ill-temper of last week, and begin quite fresh to try your very best, every day, to do what you can towards fulfilling God's law of love.

Address by the Rev. C. L. Dodgson, St Mary Magdalen Church, Nov 1897

Stuart Moskowitz.CalPolyHumboldt g4g15 GiftExchange

The Offer of the Clarendon Trustees

In 1868 he proposed a Mathematical Institute for Oxford University

Dear Senior Censor,

…….It may be sufficient for the present to enumerate the following requisites: others might be added as funds permitted………

A. A very large room for calculating Greatest Common Measure. To this a small one might be attached for Least Common Multiple.

B. A piece of open ground for keeping Roots and practising their extraction: it would be advisable to keep Square Roots by themselves, as their corners are apt to damage others.

C. A room for reducing Fractions to their Lowest Terms. This should be provided with a cellar for keeping the Lowest Terms.

D. A large room, which might be darkened, and fitted up with a magic lantern, for the purpose of exhibiting Circulating Decimals in the act of circulation.

E. A narrow strip of ground, railed off and carefully levelled, for investigating the properties of Asymptotes, and testing practically whether Parallel Lines meet or not: for this purpose it should reach, to use the expressive language of Euclid, "ever so far."

May I trust that you will give your immediate attention to this most important subject?

Believe me, Sincerely yours, MATHEMATICUS.

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From Art to Math

The Polymorphisms of the Polymorphic Elastegrity

Eleftherios Pavlides,[1] David Adesina[2], Ryan Danby,[1] [1]Roger Williams University, [2]Brown University

Introduction

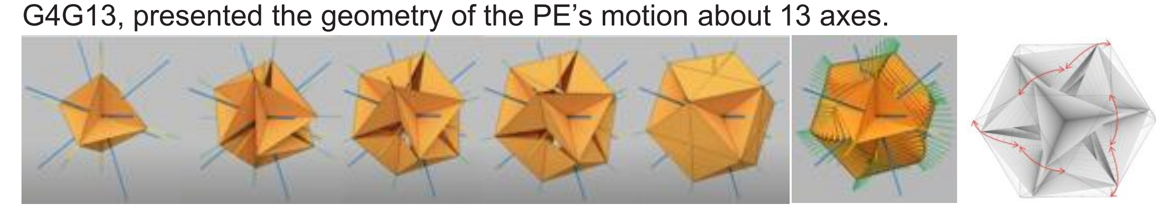
Summary of previous presentations

The Polymorphic Elastegrity (PE) is an art object created in 1982. Named elastegrity because it maintains integrity of form through elasticity,[1] aspects of it were reported at G4G12,[2] G4G13,[3] and G4G14.[4] At G4G15, we will take a closer look at the art processes, which account for its discovery and finding its diverse polymorphisms, which is why it is named "polymorphic."[5]

(a) (b) (c) (d) (e) (f)

Fig 1. (a) Polymorphic elastegrity, is composed with 8 asymmetrical tetrahedra "floating" on 12 pairs of elastically hinged right triangles; (b) Discovered monododecahedron, defined as having twelve congruent but not regular faces; (c) Brown University Mathematics Professor Thomas Banchoff (left) entered coordinates and discovered a path of monododecahedra; (d) at one end of the path rhombic faces are degenerate pentagons with one side of the pentagon equal to zero; (e) regular dodecahedron; (f) at the other end of the path is a cube with rectangle faces, degenerate pentagons with three one angle equal to 180º;

G4G13, presented the geometry of the PE's motion about 13 axes.



(a) (b) (c) (d) (e) (f) (g)

Fig 2. (a), (b), (.c), (d), & (e) 13 axes: 4 tetrahedral (blue) and 3 orthogonal (yellow) do not move; (f) 6 diameters (green), vertices move along ¼ of ellipses; 8 asymmetrical tetrahedra, each with three hinge triangles attached to it, rotate around the tetrahedral axes (blue) as they translate towards the center moving in sync. 4 tetrahedra rotate clockwise and 4 counterclockwise, expanding into a cuboctahedron or contracting into an octahedron; (c) at dihedral angles 90º the 12 vertices of the structure outline a regular icosahedron; (g) the 12 vertices of the structure move along ¼ of an ellipse.

G4G14, a tetradecahedron, a polyhedron with 14 faces, was outlined as the void between the Weaire-Phelan monododecahedron along the path described at G4G12. The Weaire-Phelan combination of two polyhedra approximates the minimum tension surface (bubbles).

. (a) (b) (c) (d) (e) (f)

Fig 3: G4G14 (a) Bubbles, minimum tension surface; (b) Weaire Phelan polyhedral approximation of bubbles with monododecahedron (yellow) and tetradecahedron (translucent blue); (c) PE folded into a WP monododecahedron supported by tetradecahedral skeleton; (d) 3D printed Weaire Phelan monododecahedra with tetradecahedral void clearly showing simple regularity of Weaire Phelan matrix; (e) "Fly-through" tetradecahedral void surrounded by WP monododecahedra; (f) Beijing Olympics "Water Cube" based on Weaire Phelan pattern sliced regular pattern at an angle to create interest.

Art skill-sets and the discovery process.

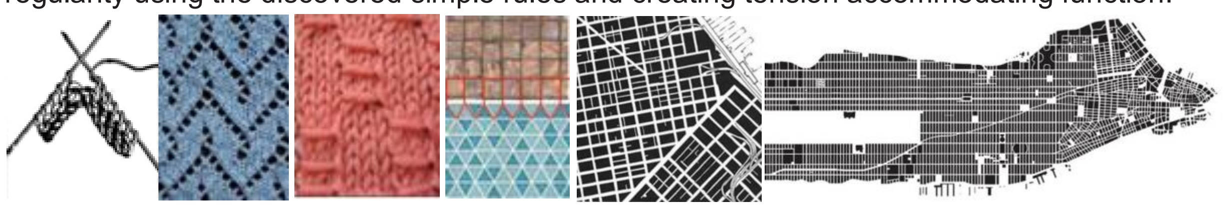
The Polymorphic Elastegrity resulted from what we termed in past G4G presentations dactylognosis (from *dactyl* = finger and *gnosis* = revealed knowledge.) As the fingertips crease, fold, and weave, it leads to discovery. It is the kind of exploration that sculptors, musicians, jugglers, and all artists engage in when creating. What is possible is found with no preconceptions or intention to simulate known forms. In previous G4G we mentioned that two Bauhaus design exercises provided the context for problem-solving that led to the invention of the PE. In G4G15, we will examine the exact art processes that led to problem-solving. *Techne (art in Greek),* where the word technology comes from, had its first meaning being "skillfully made." We will examine how the skill set of *techne* (art) addresses the question: How did a class of structures that includes fifteen polymorphisms of the PE arose from two Bauhaus exercises?

Investigated initially in two art projects at the Yale School of Architecture, it was followed by exploring structure, symmetry, and material through the decade that followed. In architecture school, we quickly learn never to use subjective terms such as "beautiful" in design criticism. We are not interested in how a design looks. Instead, we learn to critically look at what we skillfully make to interpret and evaluate its potential for solving problems. As Picasso said in a 1956 interview, "I have a horror of people who speak about the beautiful. What is the beautiful? One must speak of problems in painting! Paintings are but research and experiment."[6] This is even more true in architecture.

Two Art Bauhaus Exercises Give Rise to the Polymorphic Elastegrity[7]

Art Exercise 1: Creating Interest by Colliding A-B and A-B-C Sphere Closepacking

The 1971 assignment explored how to use regularity to create complexity in design. The premise was that design should always start by understanding simple rules that allow a process to create interest, exploiting the "tension" from colliding regularities. Design should never attempt to become interesting or beautiful. Instead, interest should arise by establishing regularity using the discovered simple rules and creating tension accommodating function.

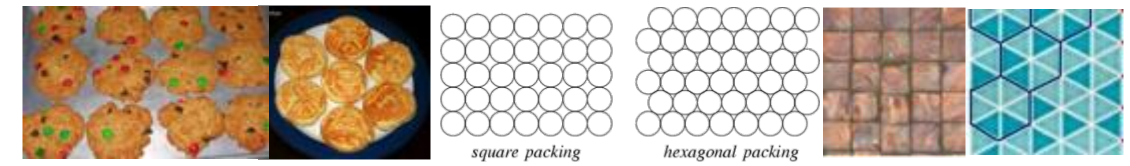


(a) (b) (c) (d) (e) (f)

Fig 4. Simple rules give rise to complexity: (a) The simple rules in knitting: yarn can only be placed over or under a needle; (b) & (c) generate an enormous variety of patterns through knitting; (d) a pentagonal interface mitigates the collision of square and triangular patterns; (e) San Francisco's colliding grids and (f) Broadway Avenue slicing and colliding through Manhattan's regular grid, were given as examples of creating hierarchy that supports commercial activity.

The assignment cited Anni Albers and referenced knitting as an example where simple rules give rise to interest in design. The simple rule of placing yarn over or under one or the other knitting needles leads to a large number of intricate patterns by varying the rhythm of the simple rules Fig. 4(a), (b), & (c). San Francisco's colliding grids and Broadway Avenue slicing through Manhattan's regular grid were given as examples of how disrupting established regularity gave rise to formal hierarchy, creating unique places that supported commercial activity Fig. 4(e) & (f). However, to interrupt regularity, one must first establish order. Interrupting regularities to adjust form to accommodate functions creates interest.

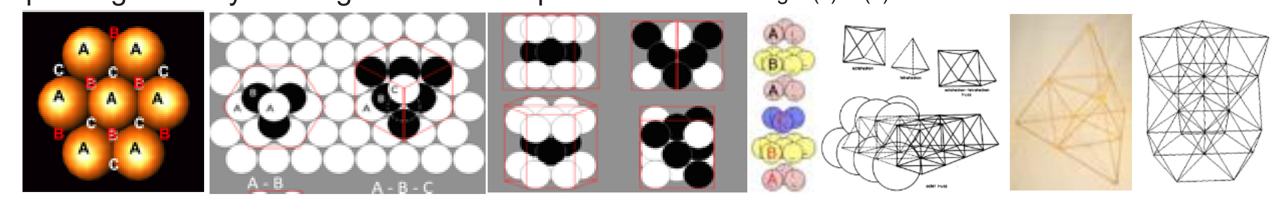
The assignment asked students to become familiar with the simple rules by which crystals close-pack in nature. First conjectured by Kepler, we were told that 100% of crystals in the periodic table crystalize in an A-B or A-B-C sphere close-packing pattern. We were to use these two equally dense, regular patterns found in nature for designing. Close-packing was first introduced as two-dimensional tiling. There are only two ways to tile with regular tiles leaving no gaps: square or triangle/hexagonal, Fig 5(e) & (f). Staggering cookies in a pan allows the most or most closely packed cookies since the space between cookies is the smallest Fig 5(b). Connecting the centers of the close-packed cookies, with straight lines creates hexagonal/ trigonal tiling Fig 5(f).



(a) (b) (c) (d) (e) (f)

Fig 5 Close-packing in two dimensions: (a) compared to (b) and (c) to (d), more cookies fit in the pan staggered into a hexagonal packing because square packing leaves more space not covered by cookies.

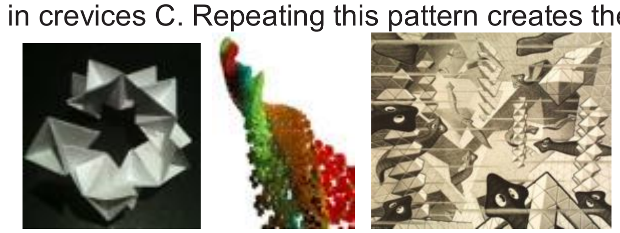
The two distinct ways of close-packing spheres were presented as an extension of close-packing in 2D by thinking of circles as sphere diameters Fig 5 (c) & (d).



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Fig. 6 Sphere and crystal close-packing: (a) Each sphere A in the first layer is surrounded by 3 crevices B and 3 crevices C. Spheres placed in crevices B on the second layer touch each other and cantilever over crevices C, eliminating them from being used in the second layer. On the third layer, there is a choice to use the crevices over C or A of the first layer; (b), (d), & (g) placing them over crevice A results in repetition over every second layer forming a linear structure growing along an axis; (c), (e), & (h) placing over C results in repetition every third layer forming a structure growing around a center.

Each sphere touches six surrounding spheres, creating six crevices where a sphere can be placed on the level above Fig. 6(a). Name the spheres on the first layer, "A," and those on the second layer, "B." Of the six empty crevices surrounding a sphere on level A, only three can be used to place spheres on the second layer B because spheres cantilever over their surrounding crevices, thus eliminating them as locations for placing spheres. We can only utilize half of the available crevices on every level. On the third level, we have a choice to place spheres in crevices over spheres A on the first level, leaving the crevices C, which were not used on the second layer, unused. Proceeding like this, spheres close-pack in an A-B pattern, always leaving the crevices C unused, Fig 6(f-top). Alternatively, on the third layer, we can place spheres in crevices C. Repeating this pattern creates the A-B-C array equally dense as A-B Fig 6(f-bottom).



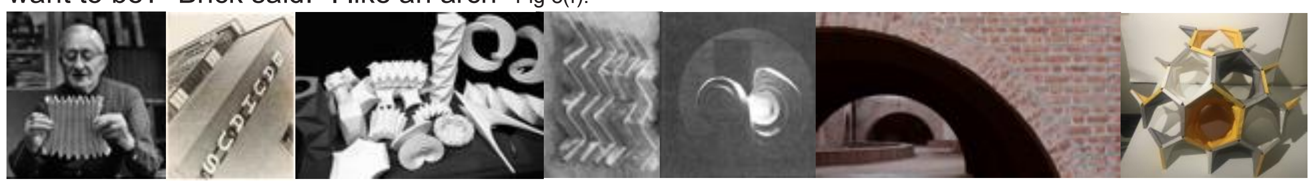
(a) (b) (c)

Fig. 7 (a) paper octahedra connected along faces and edges A-B-A-B-C recreates the 1971 helix; (b) 2014 replication of 1971 helix with branch helices; (g) M.C. Esher Flat Worms is an example of how regularity of the A-B-C close-packing creates pattern diversity.

Connecting sphere centers with straight lines turn sphere close-packing into a tetrahedral/ octahedral lattice Fig. 6(g). The assignment was to use applicator sticks to create designs using tetrahedral-octahedral close-packed lattices Fig 6(h), (I)). An experiment to see what happens when alternating A-B and A-B-C gave rise to a helix of tetrahedra and octahedra coiling around a pentagonal spatial axis. Along the length of the helical trajectory, grooves were formed to grow identical branch helices. In 2013, the 1971 finding was recreated with paper octahedra, Fig 7(a). In 2015, students alternating A-B and A-B-C electronically created a helix with identical helices branches, Fig 7(b). M.C. Escher's flatworms, when first encountered, it was marveled for its design's complexity. We wondered if this was a solution to the 1971 design problem. Upon close scrutiny, we realized Escher had only used the A-B-C close-packing array because all octahedra bordered octahedra along edges, never along faces Fig 7(c). If A-B had been introduced in the design, there would have been a rupture in the continuity of the close-packing space.

Bauhaus Exercise 2: Discovering the Properties of Materials to Create Form

In 1972, the second assignment was based on a well-known Bauhaus exercise to allow material and structure to dictate form Fig 8(a), (b), (c), (d) & (e). Josef Albers challenged his students to "investigate the intrinsic properties of materials, exploiting their structural possibilities and limitations." Working with a wide range of materials–wire mesh, metal, paper, etc.–students were encouraged to examine the latent form possibilities of these substances. These exercises defined what Albers called "learning to see both statically and dynamically."[8] In the spirit of materials dictating form, Louis Kahn, the legendary architect, asked Brick: "Brick, what do you want to be?" Brick said: "I like an arch" Fig 8(f).[9]



(a) (b) (c) (d) (e) (f) (g)

Fig 8 (a) Josef Gropius 1968 demonstrating Negative Poisson Ratio of zigzag fold expanding or contracting in two directions, invented in 1927-8 (b) Bauhaus German art school and design 1919-1933; (c) 1927-9 Gropius paper folding studio; (d) zigzag fold 1928; (e) saddle resulting from concentric creases 1928; (f) Brick to Luis Kahn: "I like an arch"; (g) Paper hyperbolic parabolas lattice (2015).

In that sense, the Bauhaus paper exercise is the opposite of origami, the elegant simulations of natural and geometric forms. Instead, the Bauhaus exercise explores material properties rather than simulating form with paper. It is precisely Frank Lloyd Wright's idea of organic architecture centered around notions of a form's function based on the interpretation of nature's principles rather than the replication of nature's appearance.[10]

In response to the 1972 assignment, window screen mesh was used to make hyperbolic paraboloids. Cutting the screen along the diagonals allows the squares to deform into rhombuses when bent, contributing to the curvature needed to form the hyperbolic paraboloids. I recognized the structure that emerged as a diamond lattice using the newly acquired knowledge of lattices. In 2015, students rediscovered the 1972 matrix of paraboloids by exploring the design potential of paper hyperbolic parabolas that came out of Alber's 1927 studio. Students allowed material, structure, and symmetry to create form and arrived at the same structure created with wire mesh half a century earlier.

The two Bauhaus exercises came together in 1974

(a) b) (c) (d) (e) (f) (g) (h)

Fig 9 Creating a paper diamond: (a) square paper creased on the diagonal; (b) & (c) square creased along both diagonals; (d) 3 fragments of an octahedron woven; (e) a "*paper diamond*"; (f) octahedral diamond as found in nature; (g) Robert Neal was first in publishing this structure in 1968. His publisher named it *six-fold ornament;* (h) David Mitchel published it as *skeletal octahedron* in 1997.

Continued explorations of material, structure, and symmetry led to a 1974 experiment, creasing a diagonal on a square piece of paper, Fig 9(a). It created a surprisingly stable triangulated tetrahedral shape. A second diagonal was inserted to see what would happen, Fig 9(b), which increased both stability and strength. Since endless hours had been spent creating octahedra with applicator sticks, the pyramid with a square base was instantly recognized as a fragment of an octahedron. This raised the question of whether several such pyramids could form a whole octahedron. In a few minutes, a structure was woven with six pyramids and named "paper diamond." Diamond because of its surprising hardness. Also, the six squares needed to weave it were reminiscent of carbon's atomic number. Paper had said, "I like a diamond!"

David Mitchell, the author of Mathematical Origami, independently discovered the 6-square octahedral weaving in the 1980s. He named it "skeletal octahedron."[11] While researching his book, he identified artists who had independently created this paper structure and given it various names. Naming it *paper diamond* created the problem that crystals are not unitary structures but grow through self-assembly. This launched a multi-year exploration to find how to seed the growth of paper crystals into A-B and A-B-C lattices. The paper crystal was an opportunity to continue the first assignment using a faster method of assembling octahedral/ tetrahedral lattices than slow-drying Elmer's glue on applicator sticks.

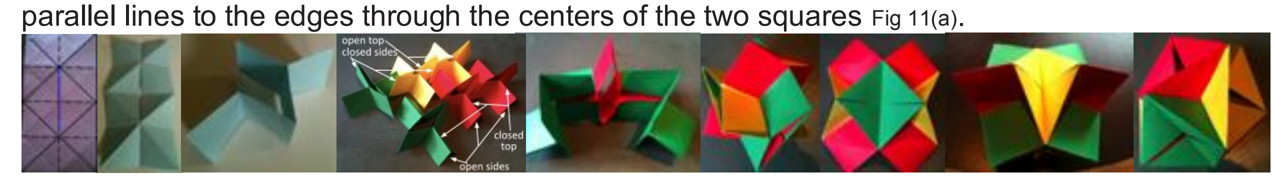
(a) (b) (c) (d) (e) (f) (g) (h) (I)

Fig 10 Two ways for growing paper crystals. I: (a) Square paper with cross slits; (b) Square paper folds into an octahedron; (c) and weaves into four cube corners outlining a tetrahedron; (d) little cubes are inserted into each other creating strong face-centered bonds; II: (e) A rectangle with a rotated square in its center flanged by two small rectangles; (f) the rotated square is folded so that the rectangles on either side are folded into squares; (g) the folded small squares are flipped into wings that can be inserted linking two octahedral units; (g) two units linked with inserted wings; (I) an octahedron with wings to grow the paper crystal in 6 directions.

Two ways to grow paper crystals were invented. The original paper octahedron was modified by cutting and folding the six vertices into 4 cube corners, which led to a square paper with cross slits weaving into a unit with four cube corners outlining a tetrahedron, Fig 10(a). The four-cube units are joined by inserting the little cubes into each other, forming strong face-centered bonds of both A-B and A-B-C paper crystals Fig 10(d). The second way found for growing paper crystals is along edges. A two-square rectangle, Fig 10(e), is folded into a pyramid with right triangle wings that can be inserted into each other, Fig 10(f) & (g), forming a malleable body-centered bond Fig 10(i).

A failed experiment led to the creation of the Polymorphic Elastegrity

Dactylognostic explorations attempted to improve growing paper crystals in terms of ease of assembly and stability. As we saw, one of these efforts described above required creating octahedra with wings Fig 10(g), so the wings could be inserted into each other to grow the crystal Fig 10(h). This required a two-square rectangle creased along the diagonals of each square and parallel lines to the edges through the centers of the two squares Fig 11(a).



(a) (b) (c) (d) (e) (f) (g) (h) (I)

Fig 11 A failed experiment to create a steadier paper crystal: (a) A 2-square rectangle with creases required to fold a pyramid with "wings"; (b) slit is torn from center to center of the two squares; (c) the rectangle with slit folded along long axis pushing the two ends together creates a cross of squares; (d) weave six crosses of little squares by placing one leg open on top over the leg with a closed ridge on top; (e) & (f) place legs open on top, over legs with a ridge on top; (f) result is three intersecting squares woven with a gaping open slit on top; (g) pushing the corner vertices towards the center creates a square closing the opening; (h) pushing the corner vertices further towards the center bisects the square and creates an elastic hinge; (I) When all 12 vertices have been flipped inside out, an icosahedron appears with 8 asymmetrical tetrahedra levitating on 12 elastic hinges. When the hinges expand to dihedral 180º, the structure becomes a cuboctahedron, and when they close to 0º, the structure contracts into an octahedron, at 90º the vertices outline a regular icosahedron.

In an experiment trying to improve paper crystal-growing, a slit was torn between the two centers of the squares, Fig 11(a) & (b). This experiment was also reported in G4G14; with more detail here, we aim to instruct how to create the PE at home. **Visit the video for the 6 yellow laser scored**

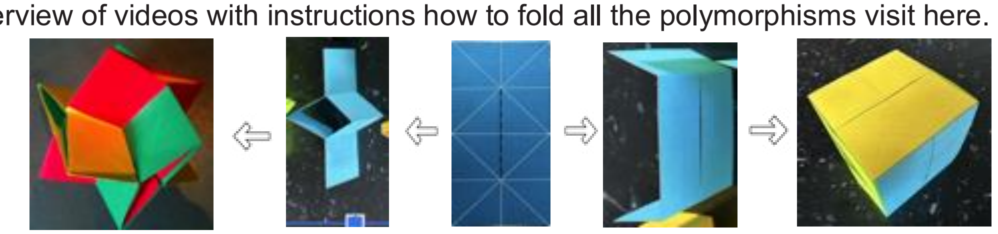
acetate in the gift exchange bag. If all else fails email epavlides@gmail.com for a zoom tutorial.

By folding the rectangle along its long axis into two and squeezing the two ends together, the slit opens and closes again perpendicular to the long axis forming a cross Fig 11(c). One axis of the cross is made with 4 little squares closed with a ridge on top and open on the sides; the other axis has 4 little squares open on top and closed on the sides Fig 11(c). Placing the squares open on top over the squares with a ridge on top, Fig 11(d) & (e), results in 3 intersecting squares Fig 11(l). However, this structure of 3 intersecting squares is not firm. It does not stay tight together as the slits gape ajar. With disappointment, this flaccid structure was discarded as a failed experiment, deemed too unstable to be of interest.

After forgetting the failed experiment, an identical structure was woven a few months later. Realizing this had been tried earlier and had failed, in an attempt to salvage wasted time, the slits were pried open Fig 11(g) & (h). Flipping the 12 outer vertices towards the center of the structure folds each of the 12 little squares into two elastically hinged triangles. 12 springs are formed supporting the 8 tetrahedra Fig 11(l). It stabilizes the entire structure into a chiral icosahedron. The 12 springs "float" the 8 irregular tetrahedra into a resilient structure that maintains its form in an elastic equilibrium. It deforms when a force is applied on one of the tetrahedral axes, but springs back pirouetting when released, returning to the original form.

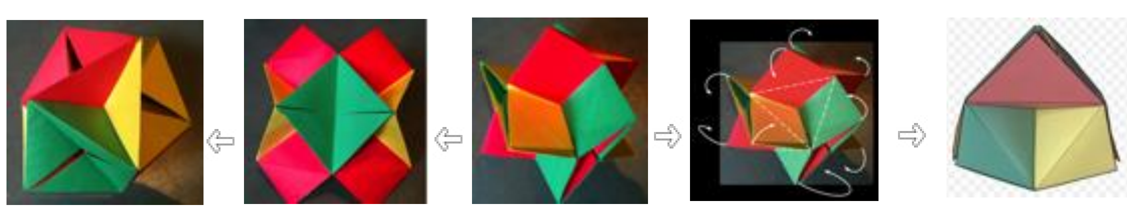
Fifteen Polymorphisms followed the discovery of the lateral icosahedron

In past G4G meetings, various versions of the tree of shapes were reported. Below, we focus on the processes that led to unexpectedly discovering 3D objects, some known in mathematics, such as the five Platonic polyhedral, some novel. The multiplicity of forms this structure assumes through folding is surprising and the same form is created at various scales as you see below. For an overview of videos with instructions how to fold all the polymorphisms visit here.



(a) (b) (c) (d) (e)
Fig 12 (a) three intersecting squares; (b) cross of little squares; (c) the 2-square rectangle with the diagonals (d) C shape; (e) cube

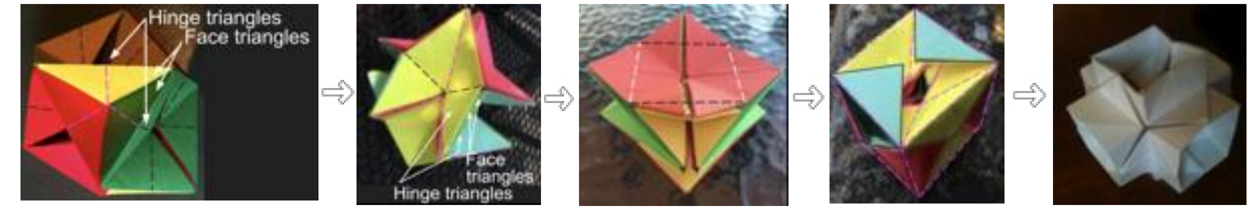
The journey of discovery starts with six 2-square rectangles with slits cut from center to center of the two squares Fig 12(c). These rectangles can be folded into crosses of little squares Fig 12(b) that can then be woven into 3 intersecting squares with gapping openings Fig 12(a) as we saw above or can also be folded into C shapes Fig 12(d) and can be woven into a cube with slits Fig 12(e).



(a) (b) (c) (d) (e)
Fig 13 (a) the Polymorphic Elasticity resulting from opening the slits of the 3 intersecting squares; (b) slit of 3 intersecting squares opened and closed forming the Polymorphic Elastegrity hinge; (c) 3 intersecting squares with gapping slits; (d) squares are folded along their diagonal in groups of 3 with tetrahedral symmetry; (e) tetrahedron.

In addition to bisecting the squares of the 3-intersecting squares 13(c) by opening the slits Fig 13(b) and pushing them down into an icosahedron Fig 13(a), they can be bisected by folding down in groups of three towards the same asymmetrical tetrahedron Fig 13(d) turning the entire structure into a regular tetrahedron Fig 13(a).

Icosahedral elastegrity=>to 8 triradiational groups=>to square=>to cube=>to hypercube



(a) (b) (c) (d) (e)
Fig 14 (a) Icosahedral Elastegrity, red dash line bisects tetrahedral faces, black dash line bisects hinge triangles; (b) 24 face right triangles bisected and squeezed between 24 bisected hinge right triangles form spokes of 8 triradiational groups of triangles. The structure is nicknamed "tumbleweed". Black dash lines are 24 newly created hinges; (c) two opposite spokes opened and flattened create a square; (d) folding corners with black dashes up 180º, corners with white dashes fold down 90º, creates a cube; (e) Unfolding triangles with pink creases under the flaps out and straightening the corners creates a hypercube representation in 3D.

Form-experimenting continued after creating the icosahedral elastegrity. Shortly after its discovery the right triangle faces were bisected along the red dashed lines while at the same time the three hinge triangles attached to the equilateral faces were bisected along the black dashed lines Fig 12(a). The simultaneous bisection was accomplished by crushing the rigid asymmetrical tetrahedra and flattening them into squares in three directions. Adjacent right triangle faces, and their attached hinge triangles were bisected and merged into four layers Fig 14(b). The new structure is nicknamed "tumbleweed".

Folding and merging the tetrahedral faces with their attached hinge triangles creates 24 new hinges, three for each crushed tetrahedron Fig 12(b black dashed lines). 24 former hypotenuse hinges were merged and bisected becoming top edges of the triradiational triangles Fig 12(b). The 12 hinges between pairs of hinge triangles now are hinges between the 8 triradiational groupings.

Further folding of the tumbleweed gives rise to numerous new objects, including a square, a cube, a hypercube, and an object symmetrical to a twelve-strut tensegrity.

Square Opening and flattening two opposite spokes of the tumbleweed creates a square Fig 12(c). The discovery that the structure can flatten into a square came before the "tumbleweed" was discovered. During the initial visit with Professor Banchoff, who had been recommended as someone who may be interested in examining the Polymorphic Elastegrity, he skillfully squeezed the icosahedron into a square. This was before he became involved with the project in a later year when he discovered a path of monododecahedra presented at G4G12.

At that meeting, Professor Banchoff's exquisite line quality when sketching demonstrated he possessed high artistic competence. After retiring, Professor Banchoff took a drawing class at the Providence Art Club, and as a member, he was allowed to take advanced art classes. Initially uncertain if he qualified for a class open only to artists, he was pleasantly surprised that he possessed advanced artistic skills, confirming the evaluation based on the line quality of his sketch. The discovery that the Polymorphic Elastegrity flattened into a square, just like his 3D hypercube model, was dactylognostically, not computationally derived. Some STEM researchers with no formal art training have acquired advanced artistic skills through alternative means.

Cube Folding 180º the corners of the square along the black dashed lines into the center of the square and 90º the white dashed lines, the tumbleweed turns into a cube Fig 14(d). The 12 edges are hinges allowing opposite faces to rotate in either direction collapsing into a flat square. Moving symmetrically in two directions is referred to as jitterbugging.

Hypercube Folding the triangular flaps up 90º from the faces of the cube and unfolding the triangles under the flaps and straightening them creates a hypercube Fig 14(e).

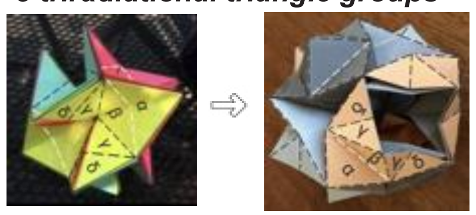
8 Triradiational groups of triangles=>to monododecahedron=>path of monododedahedra

(a) (b) (c)

Fig 16 (a) tumbleweed with 12 spokes, each spoke made of two right triangles; (b) flattening the two triangles of each spoke into a square and folding down 51.83…º along the dashed lines from center to triradiational center of the tumbleweed creates a monododecahedron when the dihedral angle of the flaps to the cube face θ = ~38.17… as presented in G4G12; (c) Further folding creates a monododecahedral path that includes the vertices of the regular dodecahedron shown above.

As reported in G4G12, with Professor Banchoff, we discovered a monododecahedron path.

8 triradiational triangle groups=> to object with 12-bar tensegrity symmetry

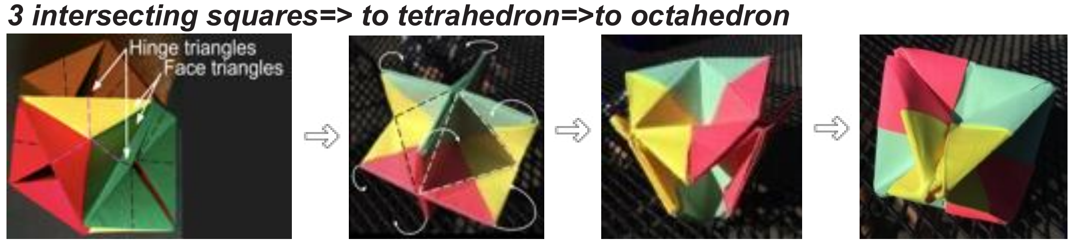


(a) (b)

Fig 17 (a) tumbleweed with 12 spokes, each spoke made of two right triangles bisected with white lines, black lines link the center of the structure to the center of the triradiation; (b) flattening the spokes and pulling triangles α, β, γ, δ up opens the slit into four vertices outlining a tetrahedron; (b) object symmetrical to a 12-bar tensegrity.

The 8 triradiational groups of triangles can turn into an object with the symmetry of a twelve-bar tensegrity by pulling up the merged triangles as shown Fig 17(a) & (b).

3 intersecting squares=> to tetrahedron=>to octahedron



(a) (b) (c) (d)

Fig 18 (a) & (b) 3-Intersecting Squares are created by bisecting the four hinge triangles around each slit and squeezing them together between four bisected face triangles; (b) &(c) squares are folded with dihedral angle 54.83…º along their diagonal of the squares in groups of 3 with tetrahedral symmetry; (c) tetrahedron; (d) by pushing in the 4 corners of the square 180º gives rise to an octahedron.

We saw above crashing the rigid asymmetrical tetrahedra by bisecting the 48 triangles of the structure with elastic hinges to create 8 triradiational groups of triangles. The same creases may be used to push four bisected triangles back-to-back around each slit to create 3 intersecting squares Fig 18(b). The faces of the bisected tetrahedral right triangles squeeze in between the bisected hinge triangles. They create the 12 little squares that compose the 3 Intersecting Squares Fig 18(b). Further folding of the 3-interesting squares gives rise to a tetrahedron and an octahedron.

Tetrahedron: Fold the 12 squares along their diagonals in 4 groups of three, with white dashed diagonal indicating folding together and black dashed diagonal indicating folding apart. At a dihedral angle of 54.83…º, the folded triangles form a tetrahedron Fig 18(c).

Folding the triangles of the tetrahedron at 180º from their original place in the 3-Intersecting Squares creates an octahedron Fig 18(d).

3 intersecting squares=>to tetrahedron=>to octahedron

(a) (b) (c)

Fig 19 (a) 3-Intersecting Squares; (b) tetrahedron; (c) octahedron

3-Intersecting Squares created when we first wove the 6 crosses of little squares; (b) folding along the diagonals of the square as we did in Fig18(b) creates a tetrahedron; (c) folding the flaps down 180º from the 3 intersecting squares as we did in Fig18(b) creates an octahedron.

Cube=>to dodecahedron=>to rhombicosahedron

(a) (b) (c) (d)

Fig 20 (a) Cube with slits on the faces; (b) the vertices outline a regular dodecahedron; (c) rhombicosidodecahedron, (d) rhombicosidodecahedron adorning the office of John Conway hanging from the ceiling on the extreme left of the photo.[12]

Pushing in the corners of the original cube with slits woven with six C-shaped elements, Fig 12(e) lifts the midpoints as the 6 slits open creating 12 vertices. The 20 vertices consisting of the 12 midpoint vertices when the dihedral angles of the hinge triangles are lifted to 90º combined with the 8 corners of the cube outline a regular dodecahedron Fig 20(b). The prescored diagonals make it possible to assemble the cube with the slits open. The resulting dodecahedron was delicate and required glue to stabilize. Additional folds were added to increase friction to firm it up, Fig 20(c). Inspecting the vertices of the resulting spherical structure, we discovered they outlined 12 pentagons, 20 triangles, and 30 squares. Later, we found out that the structure we had created by weaving 6 folded rectangles with slits as an art project was first described by Archimedes and called it ρομβοεικοσιδωδεκάεδρο. Still interesting to mathematicians, we see a rhombicosahedron hanging adorning John Conway's office.

Conclusion

We left the mathematical proofs out because of space constraints, but also as a puzzle-opportunity for the mathematically inclined. Given that all triangles are right triangles, they are derived through bisection and have well-defined dihedral angles, one can provide geometric and computational proofs of the mathematical objects suggested by the physical objects.

This article provides an example of how art skill sets could facilitate inquiry in STEM areas. Discoveries resulted from curiosity about how materials would behave when certain physical actions were attempted. The processes of discovery were the ones used by sculptors to create form. Carefully examining the resulting structures revealed interesting mathematics, suggested useful applications in engineering,[13] and provided explanatory properties for biology.[14]

How important are artistic skills in the education of a scientist or an engineer? Were Einstein's violin playing and Feynman's bongo drumming a coincidence or materially connected to their scientific contributions? It is possible that the work of artists can contribute to advances in engineering and science?[15] At least one Nobel Prize depended on expertise in water coloring: Fleming's discovery of penicillin. An art experiment that went awry requiring visual interpretation.[16]

[TREE OF FORMS VIDEOS](#)

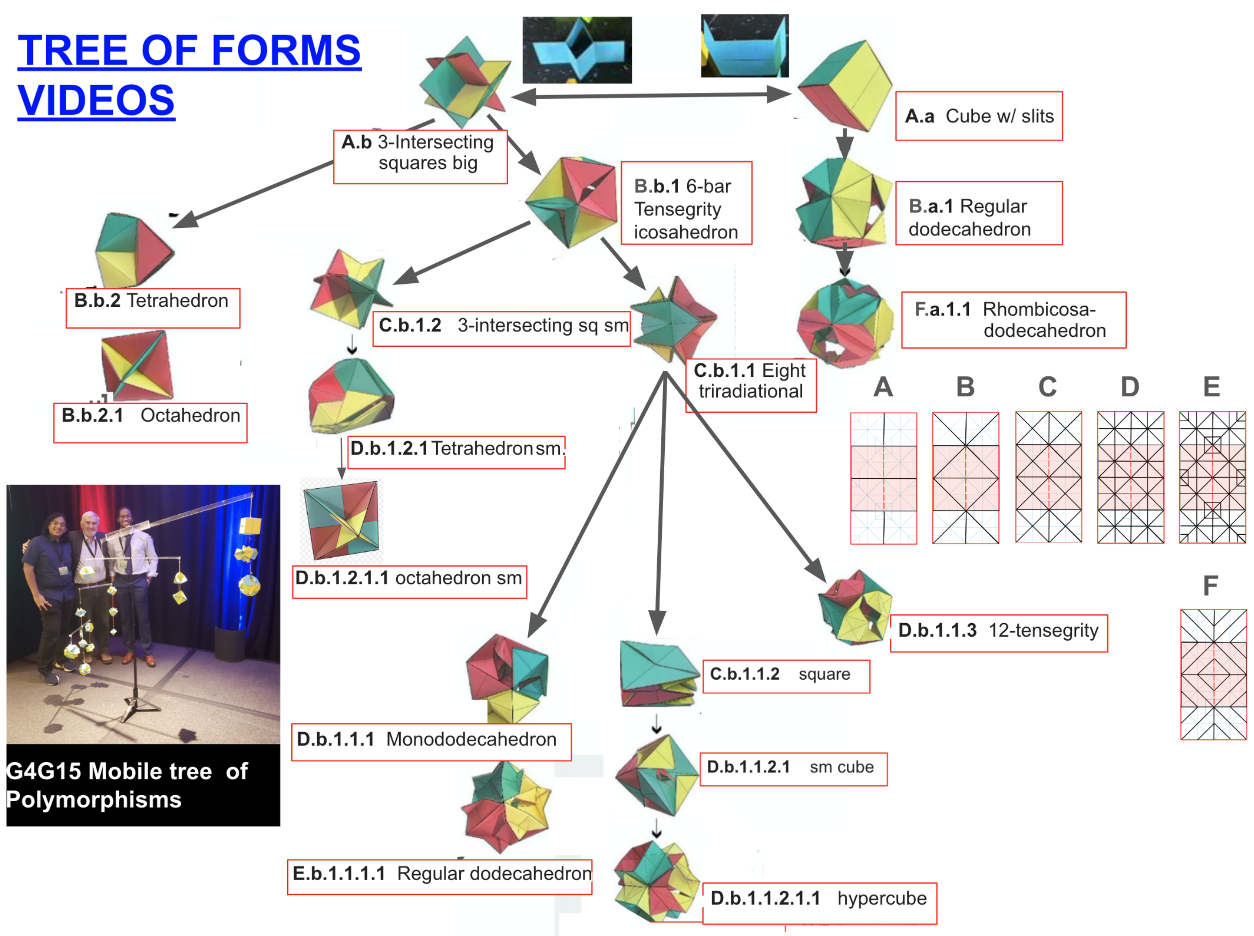
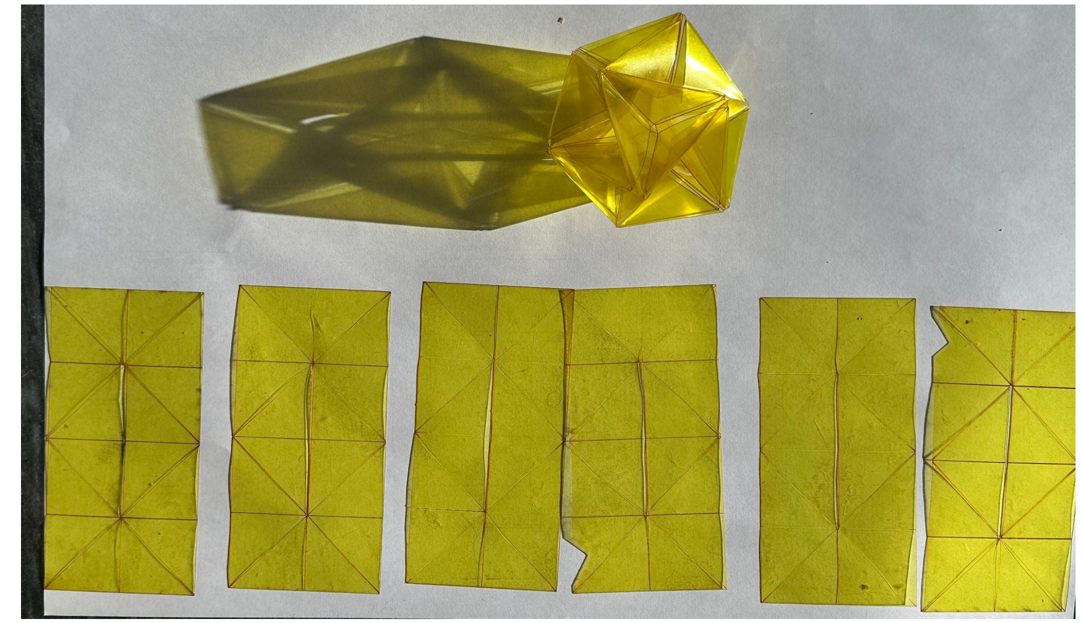


Fig 21 Tree showing derivation of forms through further folding. Instructions on how to fold them can be found here.

[1] The chiral icosahedron has the same symmetry as a 6-bar tensegrity and initially referred to as tensegrity. The term elastegrity was suggested when inventor of tensegrities Kenneth Snelson pointed out the term had been misapplied to structures with moment connections such as geodesic domes, which did not maintain integrity of shape due to tension. Elastegrity more accurately describes structures that maintains integrity of shape due to elasticity.
[2] E. Pavlides, T. Banchoff *Chiral Icosahedral Hinge Elastegrity's Shape-shifting* G4G12, 2016.
[3] E. Pavlides, P. Fauci *Chiral Icosahedral Hinge Elastegrity's Geometry of Motion*, G4G13, 2018.
[4] Eleftherios Pavlides, Thomas Banchoff, Alba Malaga, Chelsy Luis, Kenneth Mendez, Ryan Kim, Daanish Qureshi, *Tetradecahedron as Palimpsest of the Monododecahedral 1-Parameter Family of the Polymorphic Elastegrity*, G4G14.
[5] At G4G12, it was termed "chiral icosahedral hinge elastegrity." In 2020 an editor simplified it to "Pavlides Elastegrity", by analogy to the Hoberman Sphere and the Rubik's Cube. We concluded polymorphic is more descriptive.
[6] Rachel Hooper, *Picasso Black and White*, March 11, 2013.
[7] The exercises were assigned by sculptor and beloved Yale Professor Kent Bloomer.
[8] F. Horstman, Preliminary Course, and the Matière *Josef Albers Papers*, Josef & Anni Albers Foundation, Bethan, CT.
[9] W. Lesser. *You Say to Brick: The Life of Louis Kahn*, Farrar, Straus and Giroux, 2017.
[10] Frank Lloyd Wright: From Within Outward Audioguide (NY: Antenna Audio, Inc. & the S. R. Guggenheim Found, 2009)
[11] Mitchell, D. *Mathematical Origami*, Tarquin, 1997. Mitchell named it "Skeletal Octahedron". In correspondence, he identified Bob Neale as the earliest independent creator, who published it in 1968 as "Sixfold Ornament."
[12] John Horton Conway in his office at Princeton University in 1993. Photo credit Dith Pran/The New York Times
[13] RI NASA funded students with summer stipends to work on the Polymorphic Elastegrity 2019, 2020, 2021, & 2022. The PA has -1 Negative Poisson's Ratio w/ 36 hinges contracting when pressured. It has very large specific energy absorption. Made with 48 right triangles, has tiny mass and enormous energy absorption and dissipation as 36 elastic hinges contract and compress air as the volume of each cell contracts exponentially.
[14] There is an extensive literature on tensegrity as the basis of all biological structure at all size scales in numerous species including wood, for example D. Ingber Tensegrity as the architecture of life IASS, Boston, MA, 2018, pp. 1-4. and L. Brownell The "architecture of life," described by computer modeling, Wyss Institute, 2018. The Polymorphic Elastegrity augments the tensegrity theory at NSF Mechanobiology Symposia 2019 and 2021.
[15] J. L. Aragón et al. Turbulent Luminance in Impassioned van Gogh Paintings. J Math Imaging Vis 30, 275–283 (2008).
[16] Rob Dunn, Painting With Penicillin: Alexander Fleming's Germ Art, Smithsonian Magazine, July 11, 2010.

Creating Polymorphic Elastegrity (PE) Science Puzzle click for animations



Video on how to assemble the 6 element into jumpy polymorphic elastegrity click here
Or you may prefer this video
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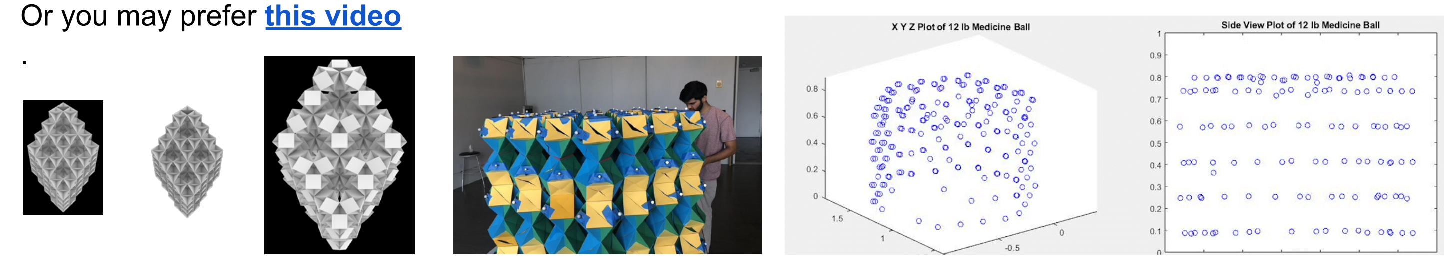
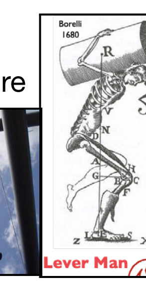
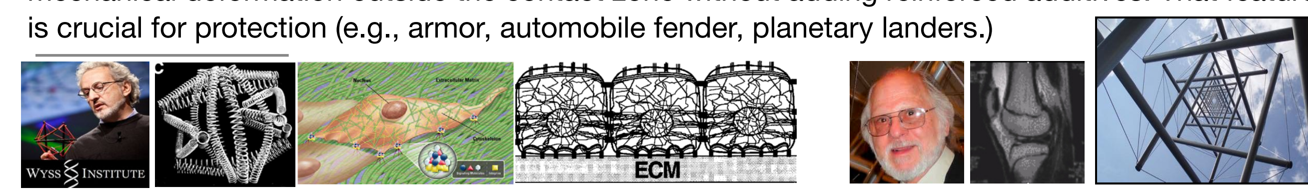


a. Matrix contracted; b. pulsating; c. expanded; e. drop sphere on185 units + motion sensors; f. 6660 hinges contracting simultaneously.

Engineering NASA-funded the PE in hopes of developing impact mitigation technology that is highly auxetic, with Poisson's ratio -1 along 4 axes, enabling it to resist impact by migrating mechanical deformation outside the contact zone without adding reinforced additives. That feature is crucial for protection (e.g., armor, automobile fender, planetary landers.)





a. Ingber; b. tensegrity w/springs; c. cytoskeleton; d.ExraCelularMatrix. e. Levin; f. bones do not touch; g. tensegrity tower; h. Borelli's Lever Man

Biology: addressing objections to the tensegrity conjecture and augmenting it:

Ingber, Wyss Institute director conjectured that the cytoskeleton is a tensegrity, not balloon w/molasses.

Levin, an orthopedic surgeon, conjectured the musculoskeletal is a tensegrity, not levers as per Borelli.

The PE addresses persisting objections to the tensegrity theory that persist even after 40 years:

(1) The 36 hinges per unit allow stability and pulsing in sync; tensegrity exhibits asynchronous pulsing;
(2) Explains pumping biofluids by exponential expansion and contraction of the PE's interior volume;
(3) Has simple assembly through folding a membrane by contrast to tensegrity that requires two materials & scaffolding;
(4) Conformation by folding a single membrane accommodates shape-shifting vital to biological function;
(5) Can be created easily at all size scales as smaller unit matrices can create larger units;
(6) Explains extreme auxeticity in biology found experimentally in certain directions;

Patterns in a Botanical Garden

Ivars Peterson, Santa Fe, New Mexico

Abstract: *Mathematics can help illuminate and enrich our understanding of the many patterns found in a botanical garden. Instances of tilings, spirals, fractals, Fibonacci numbers, spherical geometry, and many types of symmetry, all among Martin Gardner's favorite topics, appear among the plants and structures at the Santa Fe Botanical Garden.*

When we look at the world around us, we don't usually think about mathematics, or even notice math that may be right in front of our eyes. Yet an eye for math can greatly enrich our appreciation and understanding of what we see.

The Santa Fe Botanical Garden is a wonderful place for exploration with mathematics in mind, from the bilateral symmetry of leaves to branching fractal forms and Fibonacci numbers embedded in spiral patterns.

Counting and Measuring

Most people associate the term mathematics with numbers and, indeed, numbers do play a role in mathematics. At the same time, we encounter numbers in all sorts of ways in everyday life.

Let's start by characterizing the Santa Fe Botanical Garden, noting how we use numbers as key parts of these descriptions.

The Garden sits about 7,200 feet above sea level, near the southern end of the Rocky Mountains, which were formed 80 million to 55 million years ago.

Left: The Sangre de Cristo Mountains near Santa Fe represent the southernmost subrange of the Rocky Mountains.

The Garden gets about 9 to 13 inches of precipitation (rain and snow) annually, putting the area in the climate category of semi-arid steppe.

Now covering nearly 9 acres, this botanical garden is relatively new; its oldest section opened to the public in 2013.

Note how numbers help us describe, measure, and understand what we experience or encounter.

But there's much more to mathematics than just numbers and counting (and arithmetic). More broadly, we can think of mathematics as the study (or science) of patterns, though the patterns may themselves involve numbers. At the same time, patterns play an important role in botany, especially for identifying and characterizing plants.

Spheres and Symmetry

The leaves of a beaked yucca (Yucca rostrata) grow in a distinctive spherical shape. In effect, the plant looks the same from any direction, displaying spherical symmetry. For a sphere, the distance from its center to any point on the surface is the same.

*Left: Beaked yucca (*Yucca rostrata*) has a roughly spherical shape).*

Here's an interesting botanical question: How does this species of yucca achieve its spherical shape? What "rules" do its cells follow so that each leaf ends up roughly the same length?

In studying patterns, one key concept is that of symmetry. Take a look at the individual leaves of an agave plant.

*Left: Havard's agave (*Agave havardiana*).*

You'll notice that the left side of each leaf is just about identical to the right side. These leaves have mirror (or bilateral) symmetry: one side is a reflection of the other.

The leaves of many plants, large or small, have the same left-right symmetry.

Reflection is arguably the simplest type of symmetry. More generally, an object has some form of symmetry when, after a flip, slide, or turn, the object looks the same as it did originally.

Four Edges

The Garden's Rose and Lavender Walk features a wide variety of roses and several types of lavender (Lavandula).

Feel the stem of a lavender plant. You'll notice that the stem is not rounded but has edges. Indeed, the stem has (roughly) a square cross section.

The square stem is a characteristic of plants in the mint family (Lamiaceae). This family includes not only mint and lavender but also basil, rosemary, sage, thyme, salvia, and others.

*Left: Garden sage (*Salvia officinalis*) is a member of the mint family and has a square stem.*

Five Petals

The number 5 comes up repeatedly when you examine members of the rose family of plants (Rosaceae). The flowers of these plants typically have five sepals and five petals.

Wild roses have just five petals, as do a few varieties of cultivated roses such as 'Golden Wings.' The sweetbriar rose (*Rosa eglanteria*) is another example of a rose with five petals found in the Garden. However, most cultivated roses, which are bred for their appearance, have many more petals, generally multiples of five (though they still have just five sepals).

The fruit trees in the Orchard Garden are all members of the Rosaceae family. In springtime, the apple, apricot, cherry, plum, peach, and pear trees all produce blossoms with five petals.

The number 5 can also come up in surprising ways. Cut across an apple to reveal its core, and you'll find a five-pointed star shape in the center.

Cactus Spirals

If you look closely at a cactus, you can often detect distinctive patterns (though the spines may sometimes hide the underlying pattern), particularly spirals and helices. Note, for example, the way in which the spines and ridges on a cane cholla (Cylindropuntia spinosior) create a helical (spiral) pattern.

*Left: Cane cholla (*Cylindropuntia spinosior*) helix.*

The helical pattern is even more evident in the woody skeleton that serves as the framework for a cholla cactus.

Left: The woody skeleton of a tree cholla shows a helical pattern, as seen in the offset slits of the limb.

Similarly, observe how the leaves of an agave appear to grow in a spiral fashion. The leaves are not lined up like the spokes of a wheel.

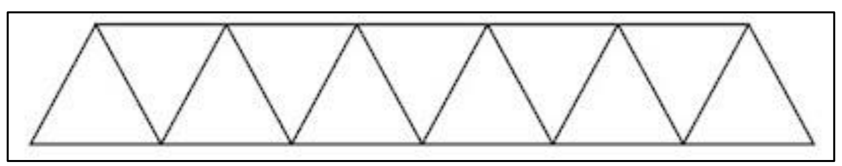
An agave's spiral growth pattern is also evident when a stalk (inset) forms at the end of the plant's life.

You may also notice a resemblance between an agave stalk and the young shoot of an asparagus plant. It turns out asparagus, agave, and yucca are genetically related and all belong to the Asparagaceae family.

Triangles, Squares, and Symmetry

Kearny's Gap Bridge is a recycled structure, originally built in 1913 for a highway near Las Vegas, New Mexico, and installed at the Garden in 2011 to connect the two sides of the Arroyo de los Pinos.

The most important geometric element is the use of equilateral triangles, characteristic of what is called a Warren truss, named for British engineer James Warren, who patented the weight-saving design in 1846.



A truss is a framework supporting a structure. A Warren truss consists of a pair of longitudinal (horizontal) girders joined only by angled cross-members (struts), forming alternately inverted equilateral triangle-shaped spaces along its length.

It’s a particularly efficient design in which the individual pieces are subject only to tension or compression forces. There is no bending or twisting. This configuration combines strength with economy of materials and can therefore be relatively light.

Look at the pattern of struts along the “railing.” This is an example of translational symmetry. Shifting the pattern to the left or right leaves the pattern the same.

Left: Behind the railing is another geometric feature: a protective fence in the form of a square grid.

In general, the repeated patterns of a symmetrical design make it easier for engineers to calculate and predict how a structure will behave under various conditions. They are characteristic of a wide range of human-built structures.

Counting Petals

During seasons when flowers are in bloom, it can be rewarding to examine the blossoms of individual plants, paying close attention to the number of petals characteristic of a given type of blossom.

*A chocolate daisy (*Berlandiera lyrata*) blossom appears to have eight "petals."*

Certain numbers come up over and over again: 3, 5, 8, 13, 21, 34. We don’t often find flowers with four, seven, or nine petals, though they do exist. For example, sundrop (*Oenothera hartwegii*) blossoms have four petals.

The larger numbers are generally characteristic of daisies, asters, and sunflowers, all belonging to the Asteraceae family. However, in this case, each "petal" is actually an individual flower, known as a ray floret. And these florets are associated with the spirals of seeds in the plant's central disk.

The 'Arizona Sun' blanket flower (Gaillardia x grandiflora 'Arizona Sun') belongs to the Asteraceae family of plants. This particular example has 34 petal-like ray florets.

The numbers 3, 5, 8, 13, 21, and 34 all belong to a numerical sequence named for the 13th-century Italian mathematician Leonardo of Pisa (also known as Fibonacci). Each consecutive number is the sum of the two numbers that precede it. Thus, 1 + 1 = 2, 1 + 2 = 3, 2 + 3 = 5, 3 + 5 = 8, 5 + 8 = 13, 8 + 13 = 21, 13 + 21 = 34, and so on.

Is it just a coincidence that the number of flower petals is more often than not a Fibonacci number, or does it point to something deeper—a pattern—about the way plants grow? That's a question that's been pondered for centuries.

Perhaps the statistics are skewed. For example, the number of flower petals can be characteristic of large families of plants. The flowers of plants in the rose family (Rosaceae), which includes many fruit trees such as apple, peach, and cherry and shrubs such as fernbush, serviceberry, and mountain mahogany, typically have five petals. So, we are likely to find the number 5 come up again and again when counting petals in the Garden.

Fibonacci numbers also come up in other ways. Take a look at the bottom of a pine cone. Pine cones have rows of diamond-shaped markings, or scales, which spiral around both clockwise and counterclockwise. If you count the number of these spirals, you are likely to find 5, 8, 13, or 21.

Left: The overlapping scales of a pine cone produce intriguing spiral patterns.

You find similar spirals among the seeds at the center of sunflowers and in the helical patterns that many cacti and succulents such as agave feature.

Left: The number of ray florets (above) displayed by a sunflower is often a Fibonacci number, as is the number of clockwise and counterclockwise spirals of seeds at a sunflower's center.

The patterns are intriguing (though sometimes difficult to discern and count), and mathematicians, physicists, and other scientists have, over the years, proposed various sets of "rules" that might govern how plants grow and produce the patterns observed in nature. One such set of rules, for example, leads to an efficient three-dimensional packing of "cells." It's a growth pattern that results in the optimal spacing of scales or seeds to reduce crowding, and applies to the helixes of cacti, the spiraling leaves of an agave, the scales of a pine cone, or the seeds and ray florets of members of the aster family.

Branches and Patches

Examine the leaf of a bigtooth maple (*Acer saccharum*).

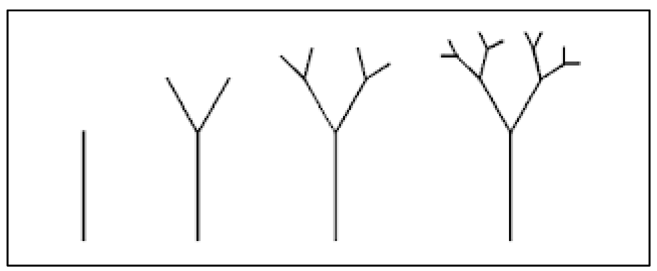
You'll notice that the left side of the leaf is just about identical to the right side. These maple leaves have mirror (or bilateral) symmetry: one side is a reflection of the other.

But there's another pattern on display. If you look closely, you will also see a network of veins: a main vein that branches into smaller veins, and these veins in turn branch into smaller veins, and so on.

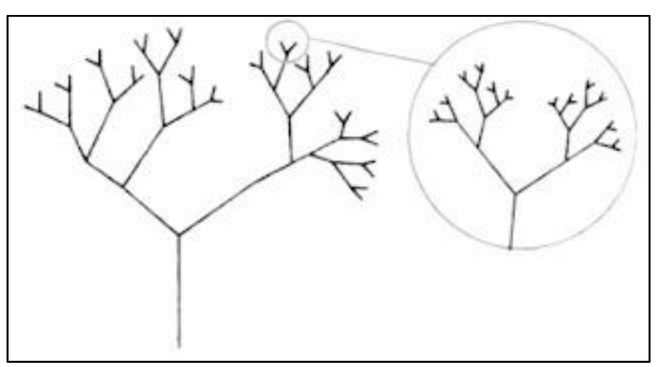
Such branching structures are characteristic of many natural forms. Cypress and juniper trees, for example, have fronds that show this type of pattern.

Left: The fronds of an Arizona cypress (Cupressus arizonica) have a distinctive branching structure.

In many cases, the branches look (at least roughly) like miniature versions of the overall structure. Such patterns are said to be self-similar. Mathematicians can create self-similar forms simply by repeating the same geometric structure on smaller and smaller scales to create an object known as a fractal. Each part is made up of scaled-down versions of the whole shape.



This example illustrates the first few steps in creating a simple geometric branching structure that has a self-similar, or fractal, pattern.



A magnified portion of a fractal looks like the overall structure.

The notion of self-similarity can also apply in other ways to natural forms. Just as a tree's limbs and twigs often have the same branching pattern seen near its trunk, clouds keep their distinctive wispiness whether viewed distantly from the ground or close up from an airplane window.

Left: The edge of a cloud may have many indentations, and those indentations when examined closely reveal smaller indentations, and so on.

Take a look at a raw stone surface. Do you see any straight lines, circles, triangles?

Instead, you may see some large hollows and ridges, and when you look closely, you see smaller hollows and ridges within these features, and so on. So there is a kind of pattern, even if the features are irregular.

Left: The patchiness of lichen growth on a stone surface has a fractal quality.

In general, in nature, you often see patterns in which shapes repeat themselves on different scales within the same object. So clouds, mountains (rocks), and trees wear their irregularity in an unexpectedly orderly fashion. In all these examples, zooming in for a closer view doesn't smooth out the irregularities. Objects tend to show the same degree of roughness at different levels of magnification or scale.

The characteristic furrows and ridges of Ponderosa pine (Pinus ponderosa) bark have a self-similar, or fractal, quality.

Where else might you find fractal patterns? Try a grocery-store produce department, where you'll see striking fractal patterns in such vegetables as cauliflower and Romanesco broccoli.

Left: This image looks like a fern, but the self-similar, or fractal, form on display was actually generated point by point by a computer following a simple set of rules.

Although the Garden doesn't have any ferns, it does have fernbush (Chamaebatiaria millefollum). Its leaves have roughly the same branching pattern displayed by fern fronds.

Left: Fernbush (Chamaebatiaria millefollum) leaves display a branching structure similar to that of a fern.

Patterns

There are many other patterns to observe in the Garden. For example, you could study and catalog the arrangements of leaves on plant stems (phyllotaxis).

Studying patterns is an opportunity to observe, hypothesize, experiment, discover, and create. By understanding regularities based on the data we gather, we can predict what comes next, estimate if the same pattern will occur when variables are altered, and begin to extend the pattern.

In the broadest sense, mathematics is the study of patterns—numerical, geometric, abstract. We see patterns all around us, in a botanical garden and just about anywhere else, and math is a wonderful tool for helping us to describe, understand, and appreciate what we are seeing.

See also "DC Math Trek" and "Where's the Math?"

ROOT(Math Success) = Childhood + Concrete Analogs + Challenges

Debbie Denise Reese, PhD
Zone Proxima LLC
ceo@zoneproxima.com

The current USA landscape for mathematics achievement is bleak. Nationwide assessment in 2022 reported 73% grade 8 and 64% of grade 4 students lacked mathematics proficiency (as in structure mapping U.S. Department of Education et al., 2023). Proficiency deficiency persists across grade levels. Recent data are only the latest in a national trend over decades (see also, National Center for Educational Statistics (NCES), 2022): the majority of our young people from early years through high school graduating class lack mathematics proficiency. Our students lack the skills to succeed in "typical modern economy jobs" (Siegler et al., 2012, p. 691). Effective concrete, early childhood mathematics preparation is a formative step toward mathematics and lifelong achievement (Mullis et al., 2020b; National Association for the Education of Young Children (NAEYC) & National Council of Teachers of Mathematics (NCTM), 2002; National Council of Teachers of Mathematics, 2022). Large scale research studies have found that early childhood math readiness

- Predicts high school math achievement (Watts et al., 2014)
- Predicts adult socioeconomic status (i.e., adult age 42, Ritchie & Bates, 2013).

The Zone Proxima Math initiative for early childhood (Reese, 2022) was designed to address the need for effective early childhood mathematics.

Origins of Zone Proxima Math

Zone Proxima Math derived from a Metaphorics analysis of the issues in USA mathematics education. Metaphorics is applied analogical reasoning theory. Analogical reasoning is a fundamental, ubiquitous cognitive process (Holyoak & Thagard, 1995; Hummel & Holyoak, 1997). People learn by matching relational structure from a concrete, relatively familiar domain to an unfamiliar one (Gentner, 1983), leading to inferences about the abstract or unfamiliar domain. The process is automatic, with low cognitive overhead. Immediate goals and context guide selection of analogs and mapping (Gentner & Holyoak, 1997). Analogical reasoning is recognized as a key to scientific and mathematical learning and discovery (e.g., Gentner, 1980; Kuhn, 1993; Polya, 1954). Metaphorics applies analogical reasoning structure mapping and pragmatics constraints to the design (especially domain specification), development, assessment, and evaluation of instruction (Reese, 2003, 2009, 2015a). Fundamentally, novice learners should manipulate effectively designed and sequenced concrete experiences. These concrete analogs should share profound relational structures (a deep system composed of layered and branched relational connections) with the to-be-learned domain. The learning environment should produce challenges as goals that motivate and orient the learner to discover viable connections within the domain of the concrete analogs. This will produce learner construction of viable mental models of foundational concepts in a domain such as mathematics (see also, Reese, 2015c). Related new learning, such as transfer from the concrete analog to more abstract symbols, becomes intuitive. When learning is intuitive, learners have a greater chance of success.

Low-risk (Reese, 2010, 2015a, 2015b), well-designed, embodied learning challenges can effectively guide young children to discover and apply foundational mathematics concepts. During guided discovery learning, children can experience and learn to recognize joy in mathematics thinking and activity. Given that early childhood learning in the home and within other child care environments produces associated gains in mathematics achievement (Mullis et al., 2020a, 2020b), Zone Proxima teaches **adults how to effectively mentor, recognize, and reward guided mathematics discovery and application by guiding their young children**:

- to manipulate concrete learning objects designed to embody mathematical concepts,
- to recognize and guide children to self-regulate (includes tenacity),
- to engage in metacognition while engaged in mathematical problem-solving, thinking and activity,
- to enjoy the rewards of effortful cognitive engagement (hard work can be good fun).

Zone Proxima children construct the integrated, coherent, well-formed mental models of foundational math required for mathematical understanding and success. The Zone Proxima Math vision and intervention for academic and personal flourishing derives from (a) over 25 years of learning science research and development, (b) fieldwork identifying the USA mathematics proficiency need, and (c) fieldwork applying cognitive science to mathematics instruction. Zone Proxima Math instruction works the way the mind does: leveraging cognitive recognition between the abstract and the concrete.

What's Metaphoric About Zone Proxima Math?

Zone Proxima Math employs pedagogical tools like 5-frames, ten-frames, counters, subitizing, tangrams, and pattern blocks as expertly described in the editions of the Van de Walle mathematics pedagogy textbook (e.g., Van de Walle et al., 2016). Zone Proxima Math employs other sources as well: for additional number sense, e.g., Jo Bohler's youcubed program (Boaler & youcubed Team, 2024), learning trajectories (e.g., Clements & Sarama, 2021), growth mindset (e.g., Dweck, 1989; Dweck et al., 2011), metacognition (e.g., Flavell, 1979; National Research Council Committee on the Foundations of Assessment et al., 2001, see p. 78; Schoenfeld, 2004), and flow (Csikszentmihalyi & Csikszentmihalyi, 1988; Reese, 2010, 2015b; as well as the related construct of flourishing, see Zhang, 2022). But the core Metaphorics component of Zone Proxima Math is the concrete analog of Cuisenaire Rods (invented by Emile-Georges Cuisenaire) coupled with the early childhood mathematics curriculum for using the Rods with guided discovery as developed by Caleb Gattegno (1970), and Zone Proxima evaluation, approach, and enhancements see Figure 1.

Although theorists and educators have extolled the virtues of Cuisenaire Rods, many lack the solid grounding of why and how the Rods are such a powerful learning tool. This might explain why many Cuisenaire Rods inhabit educators' classroom closets with but rare expeditions into the light for discrete "fun" or "hands-on" activities. Rather, the Rods should be core components of on-going, coherent infusion within the school year's curriculum over many grades (at least preschool through middle school). Before John Holt's crusade promoting homeschooling and unschooling, he extolled the virtues of Cuisenaire Rods:

> The beauty of the Cuisenaire Rods is not only that they enable the child to discover, by himself, how to carry out certain operations, but also that they enable him to satisfy himself that these operations really work, really describe what happens. (1964, p. 78)

Figure 1. Children ages 4 to 5.5 during Zone Proxima Math sessions (mother, child, and a virtual—via Zoom—Zone Proxima mentor: mother and mentor not pictured) using Cuisenaire Rods, a modified Gattegno Math curriculum, and Zone Proxima learning aids. Copyright 2024 by Zone Proxima LLC.

Yet, Holt's published writings present strong evidence that he did not understand—or at least apply—the true power of the Cuisenaire Rod analog: the deep relational mapping from the Rods' physical characteristics to mathematical relationships. For example, consider one section of the same book in which Holt described his fifth-grade students and lessons using the Rods as **counters** rather than relationally rich analogs. He directed the student to divide quantities of Rods into cups. Placing the Rods into "containers" should always privilege the learner's concrete interaction with length x width x depth. Cups obscure any physical properties of the individual and/or combined rods. The mathematical relational properties are infused into the physical characteristics of the Rods. Holt's implementation removed the power of the analog. The implementation removed the relational correspondence between the analog and multiples, factors, and equivalence.

Like many educators —be they parents, caregivers, or teachers—Holt seems unaware of the shortcomings of his implementation of the Cuisenaire Rod learning technology.[i] Holt wrote

> This work has changed most of my ideas about the way to use Cuisenaire rods [*sic*], and other materials. It seemed to me at first that we could use them as devices for packing in recipes[ii] much faster than before, and many teachers seem to be using them this way. But this is a great mistake. What we ought to do is use these materials to enable children to make for themselves, out of their own experience and discoveries, a solid and growing understanding of the ways in which numbers and the operations of arithmetic work. Our aim must be to build soundly, and if this means that we must build more slowly, so be it. Some things we will be able to do much earlier than we used to—fractions, for example. Others, like long division, may have to be put off until later. The work of the children themselves will tell us. (p. 120)

I agree with Holt that guided discovery learning is powerful instructional approach. So, too, are apt concrete instructional learning objects.

An apt instructional analog is a relationally dense and aligned in one-to-one correspondence with a targeted learning domain. Primary concepts and terminology are

- Mapping (as defined within structure mapping theory, Gentner, 1983): the cognitive process of placing two domains into correspondence according to their shared relational structure. Based upon the source (concrete or familiar) domain, the analogizer (learner) infers the existence of to-be-learned relations in the target (to-be-learned) domain. Metaphorics designers of instruction work in reverse, specifying the targeted to-be-learned domain and then designing a relationally isomorphic concrete analog (source domain), goal structures, and learning context designing the source domain (e.g., here the Cuisenaire Rods and early childhood lessons). This "backward analogizing" is the essence of Metaphorics (Reese, 2015a, 2015c).
- Systematicity: the degree of relational density within a domain and within the mapping from the source (concrete or familiar) domain to the to-be-learned domain. Apt instructional analogs embody deep systematicity with the targeted learning domain (Gentner, 1983; Reese, 2009).

- Isomorphic: a one-to-one relational correspondence between the concrete (or familiar) domain and the to-be-learned domain (Gentner, 1983; Reese, 2009).
- Embodied: (Reese, 2015c): When humans use their senses and perceptual processing to interact with the physical (real or virtual), those experiences are embodied. Seymore Papert (1980) coined the term body syntonicity for this condition: human learning when it is related to "individuals' sense and knowledge about their bodies" (p. 63). The more that interaction corresponds to the individual's sense of the individual's body and its cause and effect with the physical world, the greater its body syntonicity, the greater the syntonicity, the more natural the learning experience and analogical mapping from the source domain to the targeted learning domain.[iii]
- Pragmatic constraints: An analogizer's situation-specific goal structures dictate (constrain (i.e., constrain, see Spellman & Holyoak, 1992; Spellman & Holyoak, 1996) selection of analog domains and the mappings between the domains (Holyoak et al., 2001). During game-based learning, a game's goal structure can provide the necessary constraint to guide analog selection and cross-domain mapping (e.g., Reese, 2012). During Zone Proxima Math, "challenges" —which are the session-level (lesson-level) problems posed to learners within guided discovery learning—serve as the pragmatic constraint.

Memorization and rote application of algorithms or formulas leads to discrete, disconnected, isolated, and inert knowledge. Today's math education quandary has produced a significant proportion of youth with inert, disconnect mathematics knowledge. When these individuals do attempt mathematics, they often do not know what algorithm to use, when/why to use it, if the memory of the algorithm is correct, and if calculated results are viable. Meaningful knowledge is connected knowledge. A human can hold 7 ∓ 2 chunks of knowledge in working memory (cognitive load theory further posits a limit of only 2 or 3 active nodes within working memory, see Paas et al., 2003; Reese et al., 2016; Sweller et al., 1998). However, each chunk may contain an infinitely large and expandable knowledge network. Expert knowledge contains chunks of large, integrated knowledge networks. This is why experts can bring so much accrued knowledge to bear on a problem. Knowledge networks lessen cognitive load. The goal of sound education—and the Zone Proxima Math enterprise—is to guide children to discover, apply, and build dense knowledge networks of viable (see von Glasersfeld, 1995 for discussion of human knowledge and viability) domain knowledge. Instruction should guide learners to construct "robust, cohesive, and normative views" (Linn et al., 2004, p. 37).

Powerful instructional and learning technologies must be implemented appropriately. It is incumbent that the educator implementing an apt concrete learning object (i.e., an apt, manipulative concrete analog) appropriately apply the pedagogical content knowledge (Shulman, 1986) that supports its instructional application. Without the pedagogical content knowledge that informs instructional delivery, inadequately executed the lessons are, at best, a waste of instructional/learning time. At their worst implementation, such lessons may produce intractable misconceptions—doing more harm than good.

Metaphorics to the Rescue

Zone Proxima Math serves a mission: to rectify early childhood math instruction and ready youth for mathematics success that will follow them through academic preparation and through professional and personal lives. Zone Proxima provides instruction to the adults who care for and about young children (see Figure 2). Zone Proxima mentors train adults, during sessions for adults and sessions for adult-child dyads, to effectively and knowledgably guide young children to discover and apply foundational mathematics concepts. Typically, training involves one adult session per week and an adult-child dyad per week. Adults and their dyads may train individually or as part of a group class. Zone Proxima also provides professional development. Ideally, Zone Proxima Math and Cuisenaire Rods should integrate from early childhood through middle school education, with high school courses using the manipulatives to reactivate prior Cuisenaire Rod-related knowledge as necessary. Zone Proxima Math trains adults to provide effective mathematics education by leveraging what we know about the human mind. Zone Proxima Math is designed to work. . . the way the mind does.

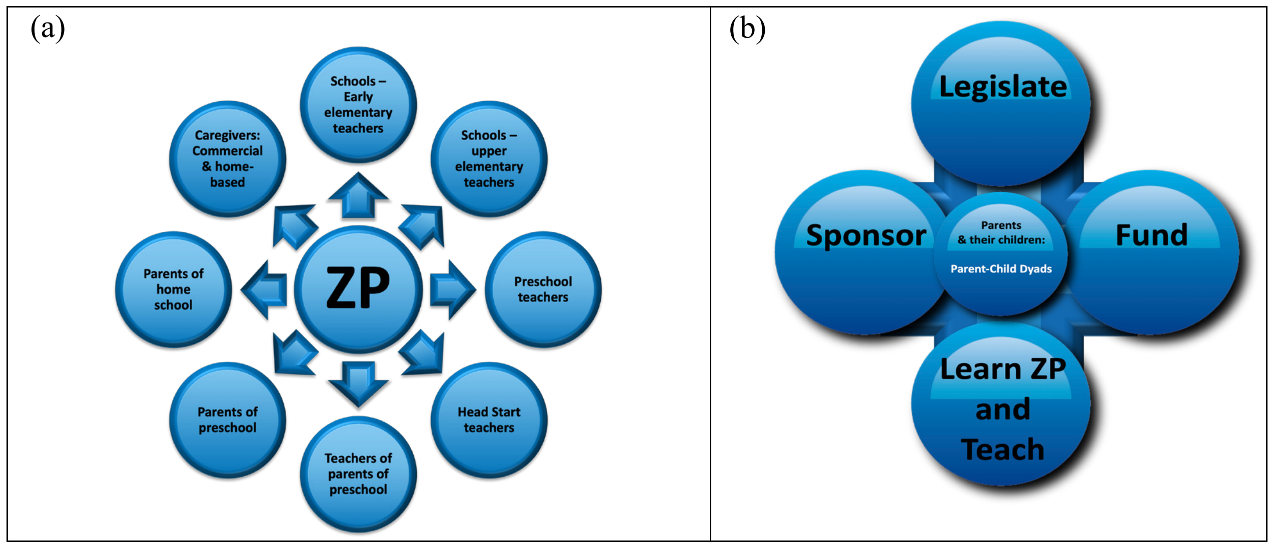


Figure 2. (a) Zone Proxima flexibly provides and designs training for educators working with early childhood in diverse environments. (b) This call to action (a) recruits adults who train to provide effective early childhood mathematics education, (b) those who might sponsor/fund trainings in populations lacking financial resources to pay for training, and (c) those who might propose, advise, and legislate Zone Proxima Math solutions for early childhood mathematics achievement.

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[i] Indeed, John Caldwell Holt's position as a leader in educational reform and the home schooling movement began with this publication.

[ii] Here, I assume Holt uses "recipes" to refer to the standard algorithms, as delineated in current Common Core standards (National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010).

[iii] It is well established: a primary obstacle for fundamental, introductory knowledge acquisition in scientific disciplines is the misalignment with the physical phenomena such as the scale at which it occurs, human perception of their environment, and human interpretations of their experiences of cause and effect. (Hestenes, Wells, & Swackhamer, 1992).

Commonsense belief about how the physical world works derived from years of personal experiences. Many can be incompatible with complex concepts - such as those in introductory Newtonian physics or genetics. Specifically, it has been established that (1) commonsense beliefs about motion and force are incompatible with Newtonian concepts in most respects, (2) ROOT(Math Success) =

conventional physics instruction produces little change in these beliefs, and (3) this result is independent of the instructor and the mode of instruction. The implications could not be more serious. Since the students have evidently not learned the most basic Newtonian concepts, they must have failed to comprehend most of the material in the course. They have been forced to cope with the subject by rote memorization of isolated fragments and by carrying out meaningless tasks. No wonder so many are repelled! The few who are successful have become so by their own devices, the course and the teacher having supplied only the opportunity and perhaps inspiration. (p. 141)

A Beware the Ides of Math

Dana Richards

The first number to be triangular and hexagonal etc. Who cares!

It is a lucky number. But it is also a deficient number (sum of the factors < 2n)!

In the Hebrew numbering system 15 is not referred to as "ten and five" unlike 13, 14, 16, 17, etc. It is called "nine and six". Ten and five is pronounced "yodh heh" which sound suspiciously like a name for God. And let's not get started on Gematria.

Knuth recommended doing a lateral Bible study, looking and 16th verse of the third chapter of each book of the bible. So, let's look at Genesis 3:15

> And I will put enmity
> between you and the woman,
> and between your offspring and hers;
> he will crush your head,
> and you will strike his heel.

Let's just stop there!

Who would like to be world-famous but only for 15 minutes?

Phosphorus, atomic number 15, is so darn reactive that it does not occur on earth by itself. It is known as Lucifer.

Ah! the good old days when the tip was just 15%.

Sam Loyd's infamous 15 puzzle, with the 14 and 15 initially reversed, wasted vast amounts of time and made the public question the trustworthiness of puzzlers.

An average employer spends 15 seconds on each job application.

James Buchanan was the 15th United states president. He is often ranked the worst president but does compete with Donald Trump and Andrew Jackson for that honor.
He fought the Mormons, appointed political hacks, and bent over backwards to appease the South. Most of his cabinet joined the Confederacy.

Rugby Union teams have the rare number of 15 members; remember Rugby is a blood sport: "Give blood - play rugby"

Kentucky is the fifteenth state of the Union and the flag with 15 stars, which had 15 stripes, was considered so cluttered that there was no flag with 16, 17, 18 or 19 stars.

The 15th Amendment granted blacks the right to vote but was so narrow that it left open huge loopholes to disenfranchise by other means, such as bogus literacy tests.

Tipper Gores "Filthy Fifteen": Prince, Sheena Easton, Judas Priest, Vanity, Motley Crue, AC/DC, Twisted Sister, Madonna, W.A.S.P., Def Leppard, Mercyful Fate, Black Sabbath, Mary Jane Girls, Venom, Cindi Lauper.

A popular meme: Maybe you weren't a terrible person maybe you were just fifteen.

> And you just might find who you're supposed to be
> I didn't know who I was supposed to be
> At fifteen. [Taylor Swift]

CDC: 15 year olds feel a lot of sadness or depression, leading to poor grades at school, alcohol or drug use, unsafe sex, etc. And in 32 states you can drive at the age of fifteen (including Georgia!)

On average women sleep 15 minutes longer than men.

"Fifteen Men on the Dead Man's Chest" is a fictional sea shanty by Robert Louis Stevenson. It turns out that Blackbeard marooned 15 men on one of the British Virgin Islands, called Dead Man's Chest or Dead Chest. He left them for a month with a cutlass and a bottle of rum, hoping they would kill each other; they all survived the month.

The Tarot card for 15 is called the Devil. The devil is portrayed with breasts, a face on the belly, eyes on the knees, lion feet, male genitalia, bat-like wings, and antlers.

It took 15 blows to decapitate Mary Queen of Scots.

The iPhone 15: easily scratched metal, overheating, fragile, slow transfers, OLED burn-in, darker images, poor battery life.

A cockroach can survive 15 minutes under water.

"Beware the Ides of March"
Recall the Ides was only on the 15th in March, July, October and May.
Two years earlier Julius Caesar had moved New Year's Day from the Ides of March.
On March 15th:
 Louis XV declared war on England.
 (Louis XV (15) died of smallpox as a defeated and unpopular king.)
 Hitler invades Czechoslovakia, 1939.
 Riots erupt in Watts, 1966.
But there never has been any superstition about the date.

Finally: Compared to fifteen love is nothing.

[At G4Gx there is a talk disparaging x; I got the job for G4G15.]

The Epolenep Principle (Restacked)

David Rutter

February 13, 2024

1 An unadorned self-working impromptu effect

While the performer's back is turned, a participant takes a small packet of cards and begins dealing them into two piles, stopping whenever they like. Tossing one of the piles aside, they remember the top card of the other pile before burying it under the cards remaining in their hand. They pick up this pile and deal it out completely into two piles. The performer divines which piles their card landed in. The participant deals through the cards in this pile until the performer says stop, upon which time the performer divines the selected card. The card the participant was stopped on is revealed to be this card.

2 Background on the principle

Over the years since Alex Elmsley published it, dozens of magicians and mathematicians interested in mathematical card tricks have constructed routines based on Penelope's Principle. If you're not familiar, this is the principle that says that if a selected card begins $\lfloor n/2 \rfloor$ cards from the top of a packet consisting of n cards, and x cards are removed from the top of this packet, then after a "nearly perfect" bottom out-faro of the remaining $n - x$ cards, the selection will be x cards from the top of this packet. See `https://www.vanishingincmagic.com/blog/location-location-location` for a proof of this principle.

The fact that this calls for a perfect faro has scared away some of the more recreational card tinkerers, along with the magicians who like to allow their participants to be a bit more hands-on. And while a partial Klondike (or Milk) shuffle can be substituted for the faro, and a participant can be taught how to do this themselves, they may not find it particularly easy or intuitive.

Here, instead, I will be discussing the *reverse* Penelope's Principle (heretofore called the Epolenep Principle). Ostensibly, this is just Penelope's Principle with the implication in the other direction: If a selection begins x cards from the top of an $n - x$ card packet, then after an out-anti-faro, the selection will be $\lfloor n/2 \rfloor$ cards from the top of the packet. This direction of the principle is best known from the work of John Born (*Seeking the Bridge*, 2012) but has not been given nearly as much attention, which is a shame.

Why? *Because an anti-faro is just dealing into two piles* which means any participant can do it themselves. Of course, dealing reverses the order of the cards, but we can deal with that. So here's the "participant-workable" Epolenep Principle stated:

> EPOLENEP PRINCIPLE: If a card begins x cards from the bottom of a packet of $n - x$ cards, after dealing into two piles and placing the pile that does not contain the card on top of the one that does, the card will be $\lceil n/2 \rceil$ cards from the top of the packet.

3 A proof

Of course, this statement leads to a natural question which we will need to answer before this theorem can be proved: which dealt pile contains the selected card? This is simple enough to answer. As there are $n - 2x$ cards above the selection at the beginning, the card is at position $n-2x+1$, which has opposite parity to n: If n is even, the card is at an odd position from the top and will be in the first pile dealt to. If n is odd, the card is at an even position and will be in the second pile dealt to.

So now the proof, which we will handle separately for the even and odd cases. First, we note that after dealing the $n - x$ cards into two piles, the first pile dealt to will contain $\lceil \frac{n-x}{2} \rceil$ cards and the second pile will contain $\lfloor \frac{n-x}{2} \rfloor$.

For n even:

Of the last x cards of the original pile, of which the selection was the first, $\lceil \frac{x}{2} \rceil$ will land on the first pile. Therefore, the selection will be in the first pile, exactly $\lceil \frac{x}{2} \rceil$ cards from the top. If the other pile is placed on top of this pile, the selection will be at position $\lfloor \frac{n-x}{2} \rfloor + \lceil \frac{x}{2} \rceil = \frac{n}{2}$. (Noting that x and $n - x$ have the same parity, so either both are divisible by 2 or neither is, and in the latter case the rounding up of the first term is canceled by the rounding down of the second.)

For n odd:

Of the last x cards of the original pile, of which the selection was the first, $\lceil \frac{x}{2} \rceil$ will land on the second pile. Therefore, the selection will be in the second pile, exactly $\lfloor \frac{x}{2} \rfloor$ cards from the top. If the other pile is placed on top of this pile, the selection will be at position $\lceil \frac{n-x}{2} \rceil + \lceil \frac{x}{2} \rceil = \lceil \frac{n}{2} \rceil$. (Noting that x and $n - x$ have opposite parity, so one of them will always be divisible by 2, so exactly one rounding up always occurs.)

4 Working the principle into an effect

It is tempting to apply the principle in the fashion of Marlo's Automatic Placement: The participant cuts off and secretly counts a small pile of cards. The performer deals off another pile of cards while counting, having the participant remember the card at their secret number. After counting $\lfloor \frac{n}{2} \rfloor$ cards in this fashion, the performer drops the remaining cards on top. Now the selected card is x cards from the bottom of the packet as desired. This is functional and justifiable, but it is time-consuming and draws attention to the mathematical nature of the method by directly involving numbers. The simple effect described at the beginning of this article points to another way:

> I'm going to look away while you start dealing these cards one to your left then one to your right, back and forth. Go ahead and start, but don't deal out the whole pile. Make sure you leave some cards in your hand. And for the last two cards you deal, turn one face up onto the left pile and then the other one face up onto the right pile. Study these two cards and decide which you like better. The one you don't like, get rid of it and all the cards under it. Throw them on the floor. Sit on them. Put them in the box. Whatever. We don't need them. The one you do like, memorize it, turn it card face down on top of its pile, and drop all the cards remaining in your hand on top of it to bury it somewhere in the middle of the pile.

Given a pile of about twenty cards, this procedure achieves the same result, takes about a minute, and gives a lot of opportunities for byplay: for example, consider having them brutally reject the card they card they don't want with a declaration of hatred before confessing their undying love to the one they do. You never touch or look at the cards the whole time, so you can drive home that the card and its position was selected by a combination of their deliberate choice and randomness such that you had no say in the outcome. With this done, you can turn around and watch again. Next, still without touching the cards, have the participant perform the anti-faro:

> It's not true that I have no idea whatsoever where your card is. I know it's not close to the top because those are the cards you kept in your hand. So, just to make this as fair as possible, go ahead and deal the whole pile out into two separate piles. Since the two piles will be about equal, there's a roughly 50-50 chance your card will land in either pile.

Of course, that's a bald-faced lie. You know exactly which pile the card will land in as described above. But you have a job to do during this dealing. You're going to count how many cards are dealt into the pile that does not contain the selection, which you can then use to infer where in the other pile the selection is. For example, let's say the participant started with 20 cards. (Much more than this and the whole effect becomes tedious.) You know the card will be in the first pile, and you count five cards dealt to the second pile. The Epolenep Principle says that if the second pile were placed on the first, the selection would be $\frac{20}{2} = 10$ cards from the top. Thus, you know the selection is now $10 - 5 = 5$ cards from the top of the first pile. You can now use whatever method you want to "divine" which pile the card ended up in. (Or use equivoque to have the participant do it.)

> I'm sure now. I'm getting nothing from that pile and a strong feeling from this one. Go ahead and turn it over. Spread them out. I'm right, aren't I? Your card's here isn't it?

Of course, you already know what position the card is in, so having the cards spread face up is just so that you can see which card is in that position. With the pile face-down again, have the participant deal the cards one at a time while you "divine" which one is the selection, and then, without them even turning it over yet, which card it is. Or skip this last dealing down and just "divine" the selection directly from the face-up pile.

But... what if you don't want to *ever* look at the faces of the cards and still be able to divine the selection?

5 Combining the principle with a stack

Well, you have to sacrifice something, so we'll have to give up letting the participant shuffle so we can have the deck stacked. For the sake of simplicity, let's assume the deck is set up in Si Stebbins stack, but the facts discussed herein are equally applicable to other stack systems. If you've got Mnemonica down cold, use that instead. We'll assume the participant-driven selection procedure described in the previous section is used and the participant put the rejected pile of cards into the tuck case.

FACTS:

- The card on top of the selection in its final pile is the one that was originally four cards above it in the deck. In the case of Si Stebbins, this card will be the card of the same suit that is greater in value by one (where K+1=A and A+1=2).
- The bottom cards of the two final piles will be the two cards following the selected and rejected cards in the original stack. That is, they will be the first, second, or third card after the selection in the stack. For example, in Si Stebbins, if the selection was the 3♠, the bottom card of either of the final piles will be from among the 6♦, 9♣, or Q♥.

5.1 A richer effect

Let's build on the basic effect using these principles.

To begin, allow the participant to cut to a random part of the deck and cut off "maybe around half of the cards." Restacking the remaining cards to set them aside, you glimpse the top and bottom cards and calculate exactly how many cards were cut. Si Stebbins describes how to do this for his stack in *Card Tricks And The Way They Are Performed.* Direct the selection procedure described above followed by the dealing into two piles. You can now add "I don't even know how many cards you started with" to your list of false claims when recapping the procedure, but of course, you still know which pile their card will land in and where it will be in that pile. Have them deal the cards out face-up one at a time from that pile. Stop them before they deal their card. Using the card they just dealt, go four cards past it in stack order (in Si Stebbins, just subtract one from its value) to calculate what card they are now holding. Reveal it.

And now for the kicker: you can now turn both face-down piles face-up and spread them out–no apparent order is visible even to those who are aware of Stebbins stack. And yet you have one more revelation up your sleeve:

> At the beginning, you put another pile of cards inside the box didn't you? Yep, I hear them rattling around in there. In fact, it sounds like...five cards...and I think the card on top, the one you decided you didn't want...is the four of clubs.

How do you get this information?

The number of cards in the box you can easily determine from the selected card and the card you glimpsed on the bottom of the deck earlier—it is simply half (rounded up) the number of cards to get from the latter to the former in stack order. For Si Stebbins, you can use the same calculus you used to find how many cards were cut off.

The identity of the rejected card you will determine using the second fact above. You've just flipped over the two piles, so you know the two cards on the bottoms of the two piles. You also know the selection. The rejected card is simply the card that completes the group of four cards in order. In more detail, if the two cards on the bottoms of the piles are the two that come directly after the selection in stack order, the rejected card came immediately before it. If the two cards on the bottoms of the piles come second and third after the selection in stack order, the rejected card is the one that fills that gap by coming immediately after it.

If you really want to go above and beyond, you can also name all the other cards in the box. They are simply every other card that came before the rejected card in stack order—work your way backwards through the stack and name every other card until you've named enough to match the number you determined.

5.2 Or just do it all verbally

You could also use the second fact above to determine the identity of the selection before you've even seen any of the cards in the pile the selection is in at the end. Simply flip over the pile that doesn't contain the selection and spread it out to prove it isn't there. You know that the selection is one of the three cards that come before (in stack order) the card on the face of this pile, so you only need two fishing questions to narrow down which one it is. For example, let's continue with the example from the second fact. You were in Stebbins stack and you see a Q♥ on the bottom of the "wrong" pile. You know the selection must be 3♠, a 6♦, or a 9♣.

> Go back in time mentally to when you finally made that decision between those face up cards. I'm sensing it was a black card you were looking at...

If you get a head shake here, you know the selected card was the 6♦ and that in deciding on it, the participant rejected either the 3 or the 9:

> ...when you put its pile in the box because you had finally decided on the six of diamonds.

A nod on the other hand narrows you down to the 3 or 9.

> ...it wasn't a low-valued card, was it?

For a no:

> But it wasn't particularly high either... the nine of clubs, right?

For a yes:

> Yeah, I thought so. Like... maybe the three of spades? Wasn't that it?

At this point, you still have not found the card despite knowing exactly where it is. You have a lot of options. For example, you could deal the other pile out and have the participant stop *you* on a card using a timing force. Or eliminate some of the remaining pile and force the selection with equivoque. Or note that their name or yours spells in the number of letters equal to the position. It's a great time to improvise because it's basically a bonus effect.

(You could also arrange to glimpse the bottom card of *both* piles and, in so doing, narrow the selection down to just two cards so that you need only one question.)

6 Other presentational ideas

6.1 Video calls

The original impromptu unstacked version of the effect is great for performance over the phone or in a Zoom call because 100% of the card handling can be done by the participant. The only consideration is that you will need to know how many cards the participant begins with *somehow*. You don't want to use a full deck due to the tedium of dealing the whole thing out, and you can be honest about this fact:

> This experiment would take too long if we use the whole deck, so cut off a pile, half the deck or less. Good. Can you quickly count those for me? 23 cards? Yeah, I think we can work with that. Now, move aside from the camera or turn it so I can't see the cards while you do this next part. You have a table or desk nearby, I hope?

For a Zoom call, another possibility is to have the selected pile of cards spread face up on a table in view of the camera to take a screenshot. Then you not only know the number but you have a full stack to work with too. This can be arranged in advance of the actual performance if possible with the justification of making sure the cards show up on camera.

6.2 Lie Detector

One possible method to use to "divine" the thought-of card is along these lines:

> I will ask you a series of yes or no questions. No matter what the correct answer is, say "no," but as you do, think about whether you're telling the truth or lying. Try to feel bad about lying when you do. Is this your card? That seems true. Is this your card? Yep, truth again. Is your card somewhere in this pile? Yep, I believe you. Let's try the other pile just to be sure. Is your card anywhere in this pile? Yep, that was definitely a lie. Okay, here I don't even need to look at the cards. Just take them one at a time off the top. Just look at it and tell me "This is not my card." I'll stop you when I hear a lie. Okay, that was true. That was true again. Oh, you were definitely lying that time. Wait, just tell me "yes" every time now, but again, be conscious of whether you're lying or not. Is your card red? That was true. Is it a heart? Hmmm... sounded like a lie. Is your card high? True. Is it a court card? True again. Is it female? That was a lie. Is it a king? Nope, you lied. Show me the jack of diamonds!

7 And now for something completely different! (But we'll circle back, I promise)

7.1 Another lie detector effect

A participant cuts off a pile of cards from a deck in which each card which has a word written on it and looks at the word written on the face card of that pile. The performer turns the other way.

> I'm going to ask you a series of questions, and to avoid asking things I could possibly have known about in advance, I'm only going to ask about the word you've selected—nothing personal, don't worry. But my goal here is not to guess what word you picked, but merely to try and determine from the sound of your voice when you're lying. I can't tell anything without a good control sample, so for seven of the eight questions I'm going to ask, I want you to tell the truth, but for one, I want to to tell a cold, premeditated lie. It can be the first question if you want to catch me unawares. Or you could try to trick me early by telling the truth like it's a lie and tell

> the real lie at the end. Or you could really try to throw me for a loop by not telling a lie at all. To make it as hard as possible for me, I'm only going to ask whether the word contains one of the 8 most common English letters—in other words, only yes or no questions where the answer is equally likely to go either way. And I can only use the sound of your voice. I'll face away the whole time so I can't look for you to give away a lie with your facial expression or body movement. All I can do is take notes on the sound of your voice and hope I can pick out the one that sounds off.

After asking about and noting down whether the selected word contains eight specific letters, the performer reveals whether the participant lied, how many times, and about which letters.

7.2 The method

You'll need a deck of cards with words on them, with the 16 in the middle being a very carefully chosen set. In a deck of 36 word cards (which I've simply written with a felt-tip pen on blank tarot-size cards), the 10th through 26nd cards should be precisely these words in this order:

1. murmur
2. softball
3. therapy
4. gumshoe
5. uncouth
6. husband
7. lemonade
8. nests
9. biography
10. physicist
11. exploit
12. aimless
13. dainty
14. fusion
15. children
16. relationship

So, a participant cuts into the middle of this deck of words and as long as they are in the middle third of the pile, they will get one of the above words. What's so special about these words? Well, they just so happen to form a Hamming(8,4) code on the letters INESTAOH. This means that we can correct any one incorrect bit (lie) and at least detect if there was more than one. But don't worry, we can figure out where the lie was without having to try to match up the set of yeses and nos with a word directly. We can just calculate it without worrying about the words.

With the answers to those 8 questions in hand, first we need to determine how many lies there were. If the number of "yes" answers is odd, the number of lies is odd, so we assume there was only one. If the number of "yes" answers is even, the number of lies is even, so we assume there were zero or two. (In general, the above script of insisting on a "yes" answer to "H" should only be followed if you're confident that the participant will not make a mistake because there is no way of recovering from three incorrect answers.)

Next, we calculate where any lies were. The key to this lies in the three words "exploit", "lemonade", and "softball". Or rather, to the subsets of letters in these words that are on the question list, namely, EOIT, EONA, and SOTA. An odd number of "yes" answers in the first set, EOIT, is worth 1 point. An odd number of "yes" answers in the second set, EONA, is worth two points. An odd number of "yes" answers in the third set, SOTA, is worth 4 points. A "yes" to H is worth 8 points. Even numbers of "yes" answers in any set are worth nothing. Add up all the points to get your index.

If you know there was only one lie, this index is precisely an index into the list {I,N,E,S,T,A,O,H} telling you which letter was the lie. For example, if you calculated 5 as the index, you know the lie was regarding T.

If you get zero for the sum and the original "yes" answer total parity was even, you know there were no lies.

If you get a nonzero value and the original parity was even, you know there were two lies/mistakes. The index you calculated is the bitwise exclusive-or of the indices of the two lies. If you're lucky, you get a number from 9 to 15, and you know that the lies were H and whatever letter is at the index 8 below your calculated index. Otherwise, you'll have to ask what one of the lies was and xor its index with the calculated one to find the other lie.

7.3 And now we circle back to the Epolenep Principle

> Now that I know which letters you were lying about, I know enough about which letters are in your word that I could figure out which of those words you actually selected. In other words, that word is no longer a useful piece of secret information. So, I'm going to have you select another, but this time we'll use a process that gives you a little bit more deliberate choice. Since words are often connected to emotion, I want to use your emotional connection to a different word to actually figure out which word you've selected in spite of *any* lies you will try to tell me about it, and the baseline I've just established should make that even easier. So I want you to take that pile of cards in your hand and deal it out onto the table into two piles, one card to the left, then one to the right, back and forth...

In order to use the Epolenep Principle, you do need to know how many cards the participant started with. Luckily, as described, this deck is stacked to make that easy to calculate. Simply construct a binary number from the correct answers to the first four letter questions (I, N, E, S) in that order, where yes=1 and no=0. Add 10 to this number. This is the number of cards the participant is holding. For example, if we know the word has an I and an E but no N or S, we make the binary number 1010=10, add 10 to it to get the 20th position, which is where their chosen word "exploit" will be. We can now lead right into the lie detector routine described in section 6.2 above but for words.

7.4 And we can stack it too.

If you have the deck of words memorized, you can use the stack principles described above to identify which word was selected without ever seeing it. I recommend coming up with 10 words starting with the letters B,P,E,A,D,F,C,R,I,O to start the deck of words, and another 10 that start with I,O,M,S,T,G,U,H,L,N to end the deck. That way the deck consists of two repeats of the 18 initial letters B,P,E,A,D,F,C,R,I,O,M,S,T,G,U,H,L,N, and you only have to remember that letter sequence, the two words for each letter, and which one comes first. Perhaps just make it so the shorter of the two always comes first. It's much easier to go forward four cards in stack order when that only entails going forward four letters in your 18 letter sequence and recalling the first word that started with that letter (since you're never going to have to deal with the words on the bottom of the deck anyway).

8 Conclusion

While nothing you've read here is completely original, the Epolenep Principle has spent a lot of time buried in books with too few readers. And not that many either. Dozens of effects have been released based on Penelope's Principle, while only a handful have exploited its inverse operation. And while I've suggested a few possible applications for it here, my true goal is just to get more people thinking about it by understanding what it does: a card in a position selected by a participant ends up in a known pile in a known position in that pile *without the performer ever having to touch the cards*. And every step of that process is easy for even the most inexperienced card handler to pull off. In short, I want to see more people coming up with ways to apply it. Is there a way to use it with objects other than cards? Are there alternate selection procedures that exploit it while rearranging a stack in even more favorable ways? Does it combine nicely with any other self-working effects?

For a video explanation of the Epolenep Principle, along with a performance of a basic effect, see `https://youtu.be/gpxtqmdhg_c`

For a performance of the Lie Detector trick with the Hamming Code, see `https://youtu.be/jNOUiyHIMlE`

For a new use of the more popular Penelope's Principle, see `https://youtu.be/hzFyyfrnC-c`

Combinatorial Jenga

G4G15 Gift Exchange

Tali, Lila, and Aaron Siegel

February 21, 2024

Introduction

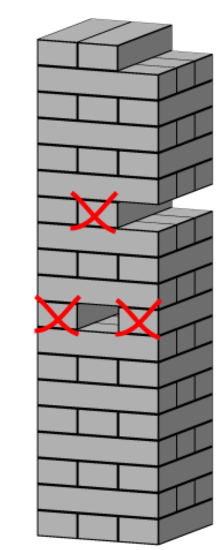
Jenga hardly needs an introduction. One of the most popular and best known tabletop games in the world, it was published by Leslie Scott in 1983 and has since sold more than 80 million copies.

Jenga is usually thought of as a dexterity game, in which coordination is the primary component of skill. While this is no doubt true, *Jenga* also has some interesting mathematical properties, which become apparent if we simply disregard the dexterity elements.

In particular, let's suppose that there are just two players, and they have become so skilled that they can execute any physically feasible move with perfect accuracy.[1] (We'll shortly give a precise definition of "feasible".) Then at some point during the game, one of the players will encounter a position in which no remaining move is feasible. At that point, any move at all will cause the tower to collapse.

We can simplify a bit by assuming the players will *only* make physically feasible moves, and that as soon as a position arises where no such move remains, they agree to end the game immediately and simply declare the player with the move to be the loser. With these assumptions, the winner is the player who makes the last move! This is known in game theory as a *normal-play combinatorial game*.

We'll call this idealized variant *Combinatorial Jenga*, and in this paper we'll give a perfect winning strategy. The strategy has essentially no application to actual *Jenga* (since even in games between players of great skill, the tower invariably collapses well before reaching its theoretical limit), but we have created a specially constructed, 3D printed set that makes it possible to play *Combinatorial Jenga* as a separate game.



Move Structure

Certain moves are prohibited by gravity: they will inevitably cause the tower to collapse, no matter how skilled the players are. The various kinds of prohibited moves can be seen in the figure on the right. If any layer is missing its center brick, then the two other bricks on the layer can never be removed. Likewise, if an end brick is missing, then the center brick on the same layer may never be removed, although it is still feasible to remove the brick on the other end of the layer.

By a **feasible move**, we mean any move that is not prohibited by one of these two constraints. With this definition, it is easily seen

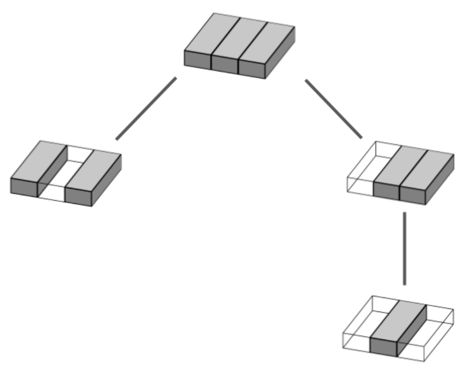
[1] We also have to assume that the blocks are competely smooth, perfectly arranged, free of any irregularities in size, and so forth.

that each layer is independent of all the others. Moreover, no player will ever move on a layer that is missing its center block, nor one that contains only a center block. We can therefore disregard such “stopped” layers from consideration, and an entire *Combinatorial Jenga* position reduces to three integer parameters:

- *m* = the number of playable complete layers (with a full complement of three bricks)
- *n* = the number of playable incomplete layers (with a center brick and one end brick)
- *k* = the number of bricks on the partial layer on top (always 0, 1, or 2)

Following standard *Jenga* rules, it is never permissible to play on the topmost complete layer, nor on the partial layer above it, and so they are *not* counted in the calculation of *m* and *n*.

Also note that *k* will always be 0, 1, or 2. As soon as a third brick is placed on top, a new playable complete layer is introduced (so that *m* increases by 1 and *k* resets to 0).



For brevity, we’ll write C for a playable complete layer and I for a playable incomplete layer, and we’ll denote a typical position by [C*m*,I*n*,+*k*]. For example, [C3,I4,+1] means three playable complete layers, four playable incomplete layers, and one brick on top. A standard *Jenga* set has 18 layers, but the topmost layer is initially unplayable, so the initial position is [C17,I0,+0].

Now every layer has the game tree shown at left; a C can be terminated immediately or replaced by an I, while from an I the only move is terminal. Thus the possible moves can be described as follows, where it is understood that no parameter may ever be decreased below 0:

- If *k* = 0 or 1, either:
  - Decrement *n* and increment *k*; or (take from an I)
  - Decrement *m* and increment *k*; or (take from the middle of a C)
  - Decrement *m*, increment *k*, and increment *n*. (take from the end of a C)
- If *k* = 2, either:
  - Decrement *n*, set *k* = 0, and increment *m*; or (take from an I)
  - Set *k* = 0 (only allowed if *m* > 1); or (take from the middle of a C)
  - Increment *n* and set *k* = 0 (only allowed if *m* > 1). (take from the end of a C)

For example, from [C3,I4,+1], one can move to [C3,I3,+2], [C2,I4,+2], or [C2,I5,+2]. Note that the moves for *k* = 2 are similar to the others, but with an extra increment given to *m*, corresponding to the introduction of a new playable complete layer.

The *P*-Positions

Following standard practice in combinatorial game theory, we call a position an *N*-position (*N*ext player wins) if a winning move exists; otherwise it’s a *P*-position (*P*revious player wins). Clearly every position is one or the other, and a perfect winning strategy may be found by classifying the *P*-positions. This we have done with the help of the *cgsuite* software.

[C1,I1,+0]	[C0,I0,+1]	[C0,I0,+2]
[C2,I0,+0]	[C2,I0,+1]	[C0,I1,+2]
[C2,I2,+0]		[C1,I2,+2]

The eight types of *P*-positions in *Combinatorial Jenga* are shown at left. All of the numbers are to be interpreted modulo 3: for example, [C*m*,I*n*,+*k*] is a *P*-position whenever $m \equiv n \equiv 1 \pmod 3$ and *k* = 0.

There is one exception to this classification: when *m* and *k* are both exactly 0. However, this exceptional situation can never arise during actual play, since any move that sets *k* to 0 necessarily also leaves *m* ≥ 1. We leave it as an exercise to the reader to determine the *P*-positions when *m* = *k* = 0.

The Strategy

As with any combinatorial game, the winning strategy is:

- Always move to a *P*-position.
- If you can’t, resign (or hope your opponent makes a mistake)!

The strategy is easier to remember if stated in a slightly different form:

- If there are no bricks on top (*k* = 0), play to leave C0,I0 or C2,I0;
- If there is one brick on top (*k* = 1), play to leave C0,I0 or C0,I1 or C1,I2;
- If there are two bricks on top (*k* = 2), play to leave C1,I1 or C2,I0 or C2,I2.

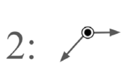
Memorize this strategy, and you can play perfect *Combinatorial Jenga*. Then all you have to do is develop flawless coordination, and you can master *Jenga* as well!

A Proof

Here’s a simple visual proof that the strategy works. To prove that the classification of *P*-positions is correct, we must show that:

- From a given *N*-position, there is at least one move to a *P*-position;
- From a given *P*-position, *every* move is to an *N*-position.

C \ I	0	1	2
0	12	2	
1		0	2
2	01		0

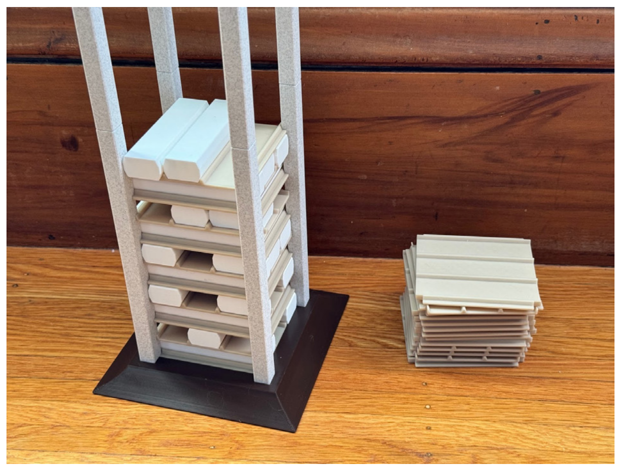
k = 0 or 1:  *k* = 2: 

The diagram on the right gives the proof. The box at row *m*, column *n* contains all the values of *k* for which [C*m*,I*n*,+*k*] is a *P*-position. From a position with *k* = 0 or 1, one can “move” west, north, or northeast on the picture; from a position with $k = 2$, one can move east or southwest, or simply stay put. Moves that stray from the diagram “wrap around” to the other side.

The *P* and *N* recurrences are easily verified from the diagram. The base cases $m = 0$ and/or $n = 0$ must be checked separately, since in those cases some of the options are prohibited. (And as mentioned above, the specific case $m = 0$, $k = 0$ is not incorporated into the diagram, since it can never arise during actual play.)

A Combinatorial Jenga Set

In an actual game of *Jenga*, our strategy is only helpful if the players are able to execute every feasible move flawlessly. In the real world, this is of course never the case. Insofar as *Jenga* is concerned, the strategy is more of a mathematical curiosity than anything that might be useful in competition.



In order to more easily explore the mathematics of *Jenga*, we’ve created a specially constructed, 3D printed *Combinatorial Jenga* set. It uses a system of stabilizing plates to reduce the dexterity component, so that it is easy to execute any *feasible* move, whereas *infeasible* ones still cause the plates to collapse.

If you have a 3D printer and want to print your own copy, scan the QR code at the bottom of this page for a link to the print files (STLs).

That’s All!

Scan this QR code for more *Combinatorial Jenga* resources, including a rules sheet and a link to the 3D print files.

Exercises:

1. Is the starting position [C17,I0,+0] a *P*-position or an *N*-position?
2. If it’s an *N*-position, what is the winning move? If it’s a *P*-position, what is the winning response to each possible opening move?

Acknowledgement

We wish to thank the Lost Iguana Resort in Arenal, Costa Rica for the giant-sized Jenga set in their lobby, which provided the inspiration for this work.

A FIBONACCI ARRAY

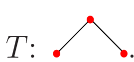
RICHARD P. STANLEY

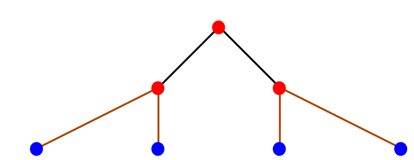
1. Introduction

We will define a certain numerical array, which we call the *Fibonacci array* $\mathfrak{F}$, and will state some properties of this array related to Fibonacci numbers and the golden mean. Proofs are omitted; for further details see the reference at the end of this article.

Define a diagram as follows. At the top there is a single vertex (or point or node), denoted T (for “top”). Now continue recursively using the following rules:

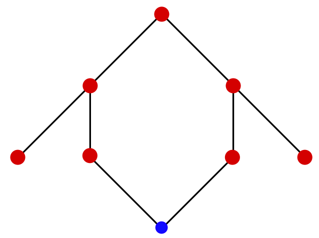
(P1) Each vertex is connected to exactly two vertices in the row below.
(P2) The diagram is planar, i.e., edges cannot cross.
(P3) Given a vertex t and the two adjacent vertices u, v to t in the row below, complete this figure to a hexagon by adding a vertex u' below and adjacent to u, a vertex v' below and adjacent to v, and a vertex w below and adjacent to both u' and v'.

Thus the first step is to add two vertices below T: . We cannot add a vertex below both of the two bottom vertices, because we must complete to a hexagon, not a quadrilateral. Since the two bottom vertices must each be adjacent to two vertices below, at the next step we get

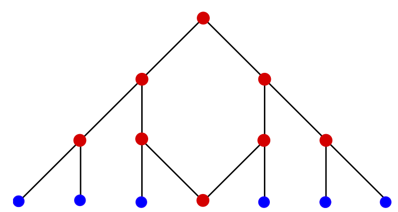


Now we add a vertex adjacent to the two middle vertices on the bottom row in order to complete to a hexagon:

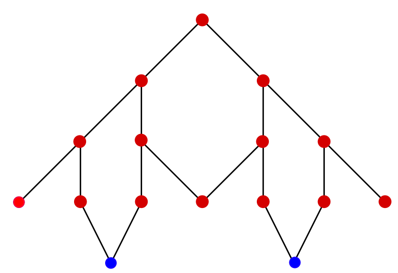
Date: February 11, 2024.



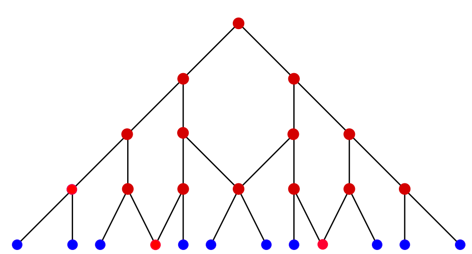
Add remaining vertices on bottom row so that rule (P1) is satisfied:



Complete the two hexagons:



Add remaining vertices on bottom row:



Continuing in this manner produces a diagram consisting of infinitely many levels. We denote this diagram by $\mathcal{D}$. The top element T is defined to be at level 0. The two vertices immediately below T are at level one, etc. The number of vertices at the levels $0, 1, 2, \dots$ is $1, 2, 4, 7, 12, 20, 33, 54, \dots$. In fact, the number of vertices at level n is $F_{n+3}-1$, where F_i denotes a Fibonacci number (defined by $F_1 = F_2 = 1$ and $F_{i+1} = F_i + F_{i-1}$ for $i \geq 2$). This gives the first glimpse of the connection of our diagram with Fibonacci numbers.

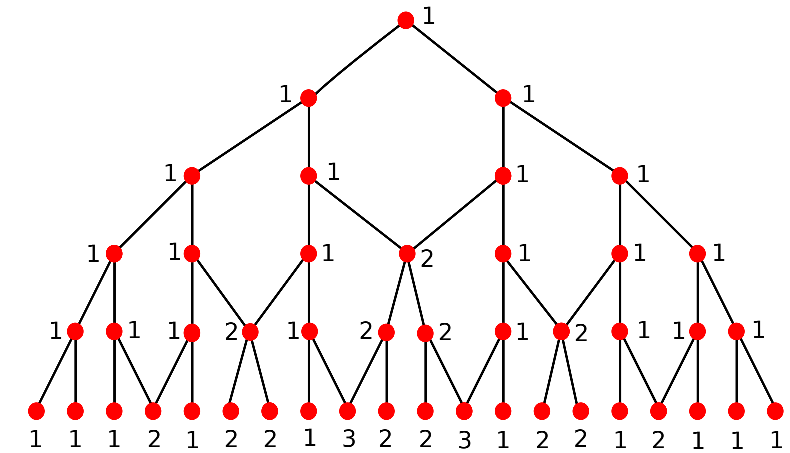


Figure 1. The Fibonacci array $\mathfrak{F}$

The next step is to attach a positive integer (a label) to each vertex of $\mathcal{D}$ by the following recursive procedure. The top element T is labelled 1. Once we have labelled all the vertices at level n, label a vertex v at level $n+1$ by the sum of the labels of the elements on level n that are adjacent to v. This procedure is analogous to the usual recursive definition of Pascal's triangle[1]. A nonrecursive description of the label of a vertex v is that the label is equal to the number of paths from T to v (along the edges of the diagram $\mathcal{D}$). We denote the resulting labelled diagram by $\mathfrak{F}$, called the *Fibonacci array*. Figure 1 shows the levels 0 to 5 of $\mathfrak{F}$.

2. The numbers $\left\langle {n \atop k} \right\rangle$

What are the numbers appearing in $\mathfrak{F}$? Let $\left\langle {n \atop k} \right\rangle$ denote the kth number on level n of $\mathfrak{F}$, beginning with $k = 0$. Thus for instance from Figure 1 we see that

$$\left\langle {5 \atop 0} \right\rangle = \left\langle {5 \atop 1} \right\rangle = \left\langle {5 \atop 2} \right\rangle = 1, \ \left\langle {5 \atop 3} \right\rangle = 2, \left\langle {5 \atop 4} \right\rangle = 1, \dots.$$

The numbers $\left\langle {n \atop k} \right\rangle$ may be regarded as "Fibonacci analogues" of the binomial coefficients $\binom{n}{k}$. The binomial coefficients satisfy the binomial theorem

$$\binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n = (1+x)^n. \tag{2.1}$$

[1] In fact, if we modify the rule (P3) by saying that we complete a vertex and the two adjacent vertices u, v to a quadrilateral rather than a hexagon and use the same labeling rule, then we obtain Pascal's triangle.

The numbers $\left\langle{n \atop k}\right\rangle$ satisfy

$$\left\langle{n \atop 0}\right\rangle + \left\langle{n \atop 1}\right\rangle x + \left\langle{n \atop 2}\right\rangle x^2 + \cdots + \left\langle{n \atop F_{n+3}-2}\right\rangle x^{F_{n+3}-2}$$

$$= (1+x^{F_2})(1+x^{F_3})\cdots(1+x^{F_{n+1}}), \tag{2.2}$$

a "Fibonacci analogue" of the binomial theorem. For instance,

$$(1+x)(1+x^2)(1+x^3)(1+x^5)$$

$$= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11},$$

so the labels on the fourth level of $\mathfrak{F}$ are $(1,1,1,2,1,2,2,1,2,1,1,1)$.

3. Sums of powers of $\left\langle{n \atop k}\right\rangle$

In Pascal's triangle the sum of the numbers on level n is 2^n. In symbols,

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

This formula follows from the fact that every number in Pascal's triangle is used twice in forming the next row. Alternatively, we can set $x = 1$ in the binomial theorem (2.1). Exactly the same reasoning applies to the Fibonacci array. Each number on some row is used twice in forming the next row, essentially a restatement of property (P1). Alternatively, we can set $x = 1$ in equation (2.2), so we get

$$\left\langle{n \atop 0}\right\rangle + \left\langle{n \atop 1}\right\rangle + \cdots + \left\langle{n \atop F_{n+3}-2}\right\rangle = 2^n. \tag{3.1}$$

The situation becomes more interesting when we consider powers $\left\langle{n \atop k}\right\rangle^r$ of the entries. The main result is the following. Let r be a positive integer, and set

$$v_r(n) = \left\langle{n \atop 0}\right\rangle^r + \left\langle{n \atop 1}\right\rangle^r + \cdots + \left\langle{n \atop n}\right\rangle^r.$$

Thus $v_1(n) = 2^n$, a restatement of equation (3.1). In general, $v_r(n)$ satisfies a linear recurrence with constant coefficients, i.e., there are integers $c_1, ..., c_k$ (which depend on r, as does k) such that

$$v_r(n) = c_1 v_r(n-1) + c_2 v_r(n-2) + \cdots + c_k v_r(n-k)$$

for all $n \geq k$. For instance,

$$\begin{aligned}
v_2(n) &= 2v_2(n-1) + 2v_2(n-2) - 2v_2(n-3)\\
v_3(n) &= 2v_3(n-1) + 4v_3(n-2) - 2v_3(n-3)\\
v_4(n) &= 2v_4(n-1) + 7v_4(n-2) + 2v_4(n-4) - 2v_4(n-5)\\
v_5(n) &= 2v_5(n-1) + 11v_5(n-2) + 8v_5(n-3)\\
&\quad +20v_5(n-4) - 10v_5(n-5).
\end{aligned}$$

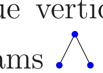
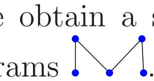
Nothing like this is true for the ordinary binomial coefficients $\binom{n}{k}$.

Note (for readers with sufficient mathematical background). Define the power series $V_r(x) = \sum_{n\geq 0} v_r(n)x^n$. Since $v_r(n)$ satisfies a linear recurrence with constant coefficients, $V_r(x)$ is a rational function. For $1 \leq r \leq 6$ it is given by

$$\begin{aligned}
V_1(x) &= \frac{1}{1-2x}\\
V_2(x) &= \frac{1-2x^2}{1-2x-2x^2+2x^3}\\
V_3(x) &= \frac{1-4x^2}{1-2x-4x^2+2x^3}\\
V_4(x) &= \frac{1-7x^2-2x^4}{1-2x-7x^2-2x^4+2x^5}\\
V_5(x) &= \frac{1-11x^2-20x^4}{1-2x-11x^2-8x^3-20x^4+10x^5}\\
V_6(x) &= \frac{1-17x^2-88x^4-4x^6}{1-2x-17x^2-28x^3-88x^4+26x^5-4x^6+4x^7}.
\end{aligned}$$

Note that the numerator of $V_r(x)$ is the "even part" of the denominator. It was proved by Ilya Bogdanov that this fact continues to hold for any r (MathOverflow 457900).

4. Two consecutive levels

We now turn to a completely different aspect of $\mathfrak{F}$: the structure of two consecutive levels. Consider for instance levels four and five, shown as blue vertices in Figure 2. We obtain a sequence of three-vertex diagrams  and five-vertex diagrams . Thus we can represent the structure of two consecutive levels as a sequence of 3's and 5's. For instance, rows 4 and 5 correspond to the sequence $(3,5,3,5,5,3,5,3)$. In general, the number of terms in the sequence corresponding to rows n and $n+1$ is F_{n+2}.

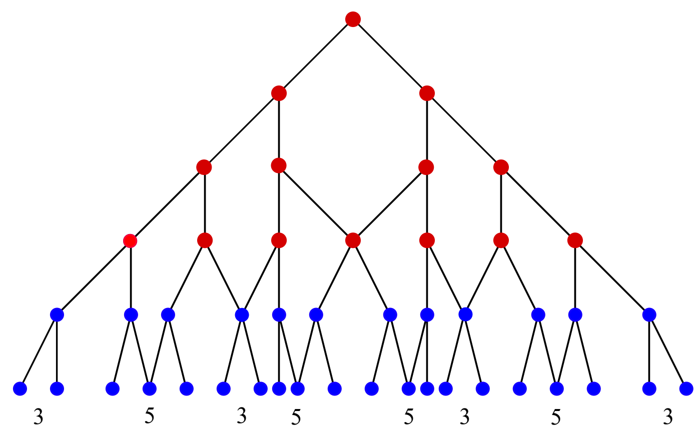


FIGURE 2. Levels four and five of $\mathfrak{F}$

How can we describe the sequence corresponding to levels n and $n+1$? It is palindromic (reads the same backwards as forwards), so we only have to describe the first half. The result is that the kth term (beginning with $k=1$) is given by

$$1+2\lfloor k\phi\rfloor-2\lfloor (k-1)\phi\rfloor, \tag{4.1}$$

where $\phi=(1+\sqrt{5})/2$, the *golden mean*. As usual, $\lfloor x\rfloor$ denotes the greatest integer $m\le x$.

The numbers in equation (4.1), beginning with $k=1$, are

$$\gamma=(3,5,3,5,5,3,5,3,5,5,3,5,5,3,5,3,5,5,\dots). \tag{4.2}$$

The first four terms are $3,5,3,5$, agreeing with the description of the first half of levels 4 and 5.

The sequence (4.2) has several other descriptions.

- If we remove the first term, then the remaining sequence (5,3,5,5,3,5,...) is characterized by invariance under $3\to 5$ and $5\to 53$ (the *Fibonacci word* in the letters 3,5).
- We have $\gamma=3z_1z_2z_3\cdots$ (concatenation of words), where $z_1=5$, $z_2=35$, and $z_k=z_{k-2}z_{k-1}$ for $k\ge 3$:

$$(3)\ 5\cdot 35\cdot 535\cdot 35535\cdot 53535535\cdots.$$

- If we replace 3 by 1 and 5 by 2 in γ, then we obtain the sequence that records the number of 5's between consecutive 3's in γ:

$$3\underbrace{5}_{1}3\underbrace{55}_{2}3\underbrace{5}_{1}3\underbrace{55}_{2}3\underbrace{55}_{2}3\underbrace{5}_{1}3\underbrace{55}_{2}3\cdots.$$

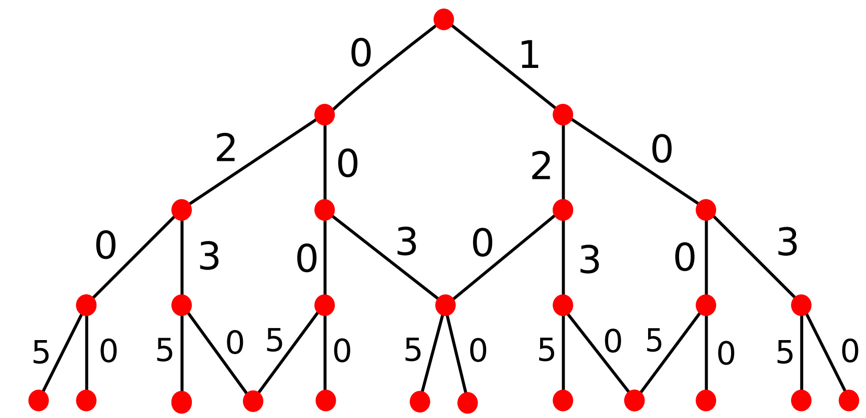


FIGURE 3. An edge labeling of $\mathfrak{D}$

5. AN EDGE LABELLING

Label the edges of $\mathfrak{D}$ as follows. The edges between levels $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2},\dots$ from left to right. The edges between levels $2k-1$ and $2k$ are labelled alternately $F_{2k+1}, 0, F_{2k+1}, 0,\dots$ from left to right. Figure 3 shows the first four levels of this labeling.

If t is a vertex in $\mathfrak{D}$, then the sum $\sigma(t)$ of the edge labels on any path from t to the top depends only on t, not on the choice of path. Figure 4 shows these sums for the points at level four. At level n we obtain the integers from 0 to $F_{n+2}-2$ once each. As we go down a path from the top to level n, there are two choices for each step. These choices correspond exactly to expanding the product (2.2). For each of the n factors there are two choices: choose the constant term 1 or the monomial $x^{F_{i+1}}$.

Moreover, if i appears to the left of j at level n, then i appears to the left of j at all subsequent levels. Thus we can define a linear ordering, denoted $\prec$, on the nonnegative integers by letting $i\prec j$ if i appears to the left of j at some level n (and thus at all subsequent levels). Figure 4 shows that

$$7\prec 2\prec 10\prec 5\prec 0\prec 8\prec 3\prec 11\prec 6\prec 1\prec 9\prec 4.$$

The order $\prec$ on the nonnegative integers is *dense*, meaning that whenever $i\prec k$, there is some (hence infinitely many) j satisfying $i\prec j\prec k$. The description of this order is based on *Zeckendorf's theorem*, which says that every nonnegative integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, where a summand equal to 1 is always taken to be F_2. The description of the order $\prec$ is a little

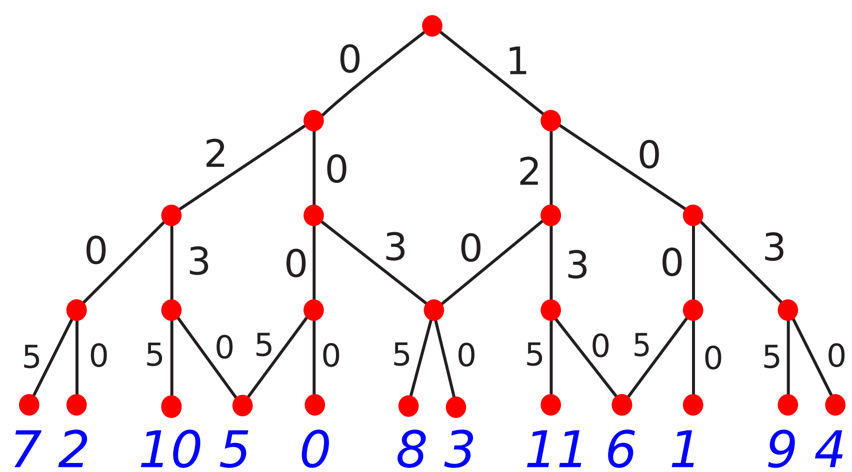


FIGURE 4. An ordering of the integers from 0 to 11

too complicated to describe here, but to give the flavor we give the condition for $n \succ 0$. Namely, let $n = F_{j_1} + \cdots + F_{j_s}$ be the Zeckendorf representation of $n > 0$, where $j_1 < \cdots < j_s$. Then $n \prec 0$ if j_1 is odd, while $n \succ 0$ if j_1 is even. For instance, $45 = 3 + 8 + 34 = F_4 + F_6 + F_9$. Since the first index (subscript) 4 is even, we have $45 \succ 0$.

REFERENCE. R. Stanley, Theorems and conjectures on some rational generating functions, *Europ. J. Math.*, to appear; arXiv:2101.02131.

Email address: rstan@math.mit.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MIAMI, CORAL GABLES, FL 33124

Database of Common Nets of Polyhedra

Daniel Valente-Matias*, Nuno Araújo*

February 21, 2024

Abstract

Polyhedra can be unfolded to form nets. Some nets, named common nets, can be folded into multiple polyhedra. This has been extensively studied for some types of polyhedra (usually with regular faces) but barely touched for more complex shapes. In this paper we introduce a database with examples of common nets.

1 Introduction

While researching the most efficient nets to allow self-folding of micro structures we stumbled upon some nets that were able to fold to more than one polyhedra. This revealed to be a very interesting problem. In fact, Demaine et al. proved that every convex polyhedron can be unfolded and refolded to a different convex polyhedron [1].

Different researchers have named this type of phenomenon as "common unfolding", "common development" or "ambiguous unfolding", here we use the denomination of "common net" for a net that can be folded onto several polyhedra. One of the simplest examples is given in Figure 1.

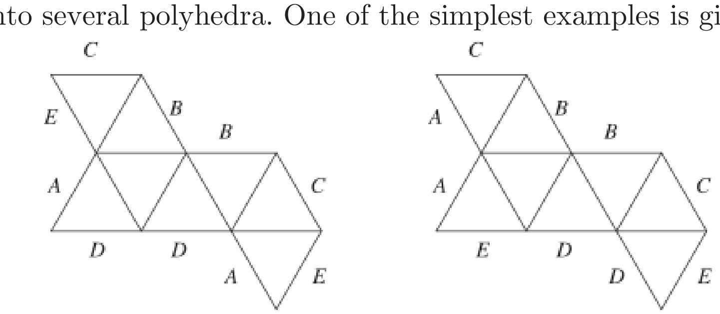


Figure 1. Common net of both an octahedron and a Tritetrahedron [2].

In this context we can have two types of common nets, strict edge unfoldings and free unfoldings. Strict edge unfoldings refers to common nets where the different polyhedra that can be folded use the same folds, that is, to fold one polyhedra from the net of another there is no need to make new folds. Free unfoldings refer to the opposite case, when we can create as many folds as needed to enable the folding of different polyhedra. We also add the concept of multiplicity, which refers to the number of common nets for the same set of polyhedra.

*Centro de Física Teórica e Computacional, Faculdade de Ciências da Universidade de Lisboa

In this paper we present a comprehensive database of nets that can be folded to more than one polyhedron. This problem has already been studied extensively for some kinds of polyhedron but still has a lot of open questions. To complement this paper a Wikipedia page has been created where new cases can be added as they are found [3].

2 Classes

Regular Polyhedra

Finding a common net between two regular polyhedra has been one of the most sought after problems. Open problem 25.31 in Geometric Folding Algorithm by Rourke and Demaine reads:

> *"Can any Platonic solid be cut open and unfolded to a polygon that may be refolded to a different Platonic solid? For example, may a cube be so dissected to a tetrahedron?"* [4]

This problem has been partially solved by Shirakawa et al. with a net that is conjectured to fold to a tetrahedron and a cube with infinite iterations.

Table 1 lists nets of regular polyhedra that can also be folded onto other polyhedra. All cases represent free unfolding common nets.

Table 1. Common nets of regular polyhedra. *Fractal net. $^{+}$20 polyhedra foldable from the Latin cross (seven tetrahedra, three pentahedra, four hexahedra, and six octahedra).

Multiplicity	Polyhedra 1	Polyhedra 2	Reference
1	Tetrahedron*	Cube*	[5]
1	Tetrahedron	Cuboid ($1 \times 1 \times 1.232$)	[4], [6]
87	Tetrahedron	Jonhson Solid J17	[7]
37	Tetrahedron	Jonhson Solid J84	[7]
2	Cube	Tetramonohedron	[8]
9	Cube	$1 \times 1 \times 7$ and $1 \times 3 \times 3$ Cuboids	[9]
1	Cube	Octahedron (non-regular)	[5]
1	Cube	20 Polyhedra^{+}	[4]
1	Octahedron	Tetramonohedron	[4]
1	Octahedron	Tetramonohedron	[8]
1	Octahedron	Tritetrahedron	[2]
1	Icosahedron	Tetramonohedron	[8]

Cuboids

Common nets of cuboids have been deeply researched, mainly by Uehara and coworkers. To the moment, common nets of up to three cuboids have been found. It has, however, been proven that there exist infinitely many examples of nets that can be folded into more than one polyhedra [10]. Table 2 shows the different common nets of cuboids found to the moment. With the exception of the marked ones, all the nets present an strict orthogonal folding despite still being considered free unfoldings.

Table 2. Common nets of Cuboids. *Non-orthogonal foldings

Area	Multiplicity	Cuboid 1	Cuboid 2	Cuboid 3	Reference
22	6495	$1 \times 1 \times 5$	$1 \times 2 \times 3$		[11]
22	3	$1 \times 1 \times 5$	$1 \times 2 \times 3$	$0 \times 1 \times 11$	[12]
28	1	$1 \times 2 \times 4$	$\sqrt{2} \times \sqrt{2} \times 3\sqrt{2}$*		[13]
30	30	$1 \times 1 \times 7$	$1 \times 3 \times 3$	$\sqrt{5} \times \sqrt{5} \times \sqrt{5}$*	[9]
30	1080	$1 \times 1 \times 7$	$1 \times 3 \times 3$		[9]
34	11291	$1 \times 1 \times 8$	$1 \times 2 \times 5$		[11]
38	2334	$1 \times 1 \times 9$	$1 \times 3 \times 4$		[11]
46	568	$1 \times 1 \times 11$	$1 \times 3 \times 5$		[11]
46	92	$1 \times 2 \times 7$	$1 \times 3 \times 5$		[11]
54	1735	$1 \times 1 \times 13$	$3 \times 3 \times 3$		[11]
54	1806	$1 \times 1 \times 13$	$1 \times 3 \times 6$		[11]
54	387	$1 \times 3 \times 6$	$3 \times 3 \times 3$		[11]
58	37	$1 \times 1 \times 14$	$1 \times 4 \times 5$		[11]
62	5	$1 \times 3 \times 7$	$2 \times 3 \times 5$		[11]
64	50	$2 \times 2 \times 7$	$1 \times 2 \times 10$		[11]
64	6	$2 \times 2 \times 7$	$2 \times 4 \times 4$		[11]
70	3	$1 \times 1 \times 17$	$1 \times 5 \times 5$		[11]
70	11	$1 \times 2 \times 11$	$1 \times 3 \times 8$		[11]
88	218	$2 \times 2 \times 10$	$1 \times 4 \times 8$		[11]
88	86	$2 \times 2 \times 10$	$2 \times 4 \times 6$		[11]
160	1	$4 \times 4 \times 8$	$\sqrt{10} \times 2\sqrt{10} \times 2\sqrt{10}$*		[12]
532		$7 \times 8 \times 14$	$2 \times 4 \times 43$	$2 \times 13 \times 16$	[10]
1792		$7 \times 8 \times 56$	$7 \times 14 \times 38$	$2 \times 13 \times 58$	[10]

Polycubes

Maybe the first cases of common nets of polycubes found was the work by George Miller, with a later contribution of Donald Knuth, that culminated in the Cubigami puzzle [14]. It's composed of a net that can fold to all 7 tree-like tetracubes. All possible common nets up to pentacubes were found. Table 3 lists the known common nets of polycubes. All the nets follow strict orthogonal folding despite still being considered free unfoldings.

Table 3. Common nets of Polycubes.

Area	Multiplicity	Polyhedra	Reference
14	29026	All tricubes	[15]
14		All tricubes	[11]
18	6495	All tree-like tetracubes	[14]
22		23 pentacubes	[16]
22	3	22 tree-like pentacubes	[16]
22	1	Non-planar pentacubes	[16]

Deltahedra

Deltahedra pose a class with many developments to be made. To our knowledge, there has been no extensive search for common nets of deltahedra.

Table 4. Common nets of deltahedra. *Not all faces are equilateral triangles.

Area	Multiplicity	Polyhedra	Reference
8	1	Both 8 face deltahedra	[2]
10	4	7-vertex deltahedra	[17]
8*	1	Both 8 face deltahedra (non regular)*	[18]

3 Conclusions

This paper presents a comprehensive list of common nets found to the moment. A Wikipedia page [3] has been created as a repository for new cases found in the future, we invite everyone to contribute to this collaborative page. The knowledge of a wider range of common nets can help us answer some open problems but also raise new ones.

4 References

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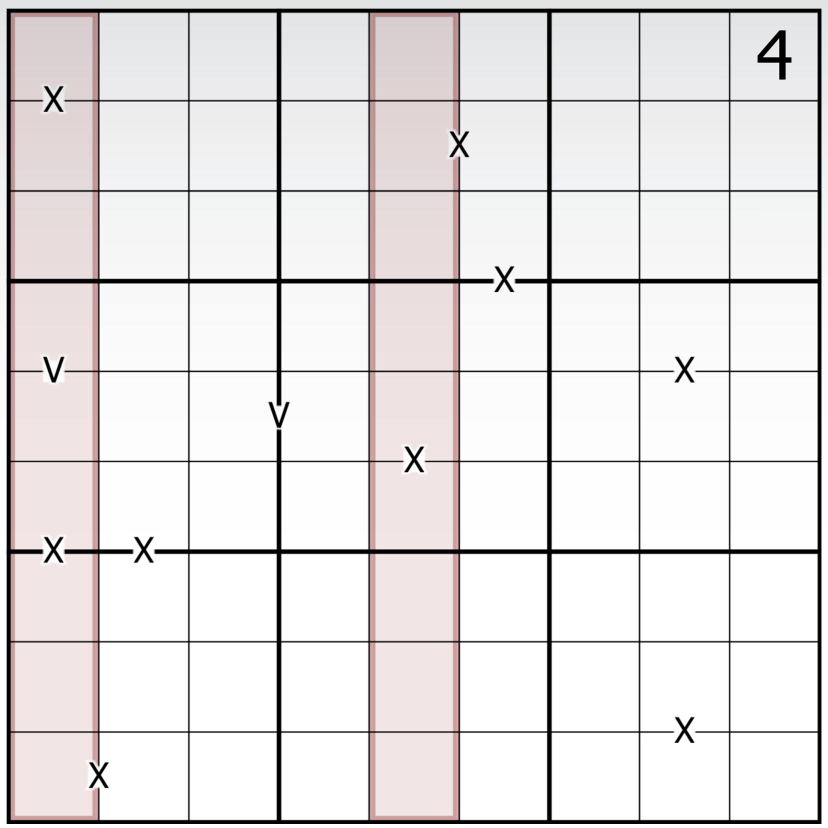
PUZZLES



Father & Son Sudoku | Graham & Gabriel Kanarek | Page 292

The Crossover Spiral

Original puzzle by Ian Andrew

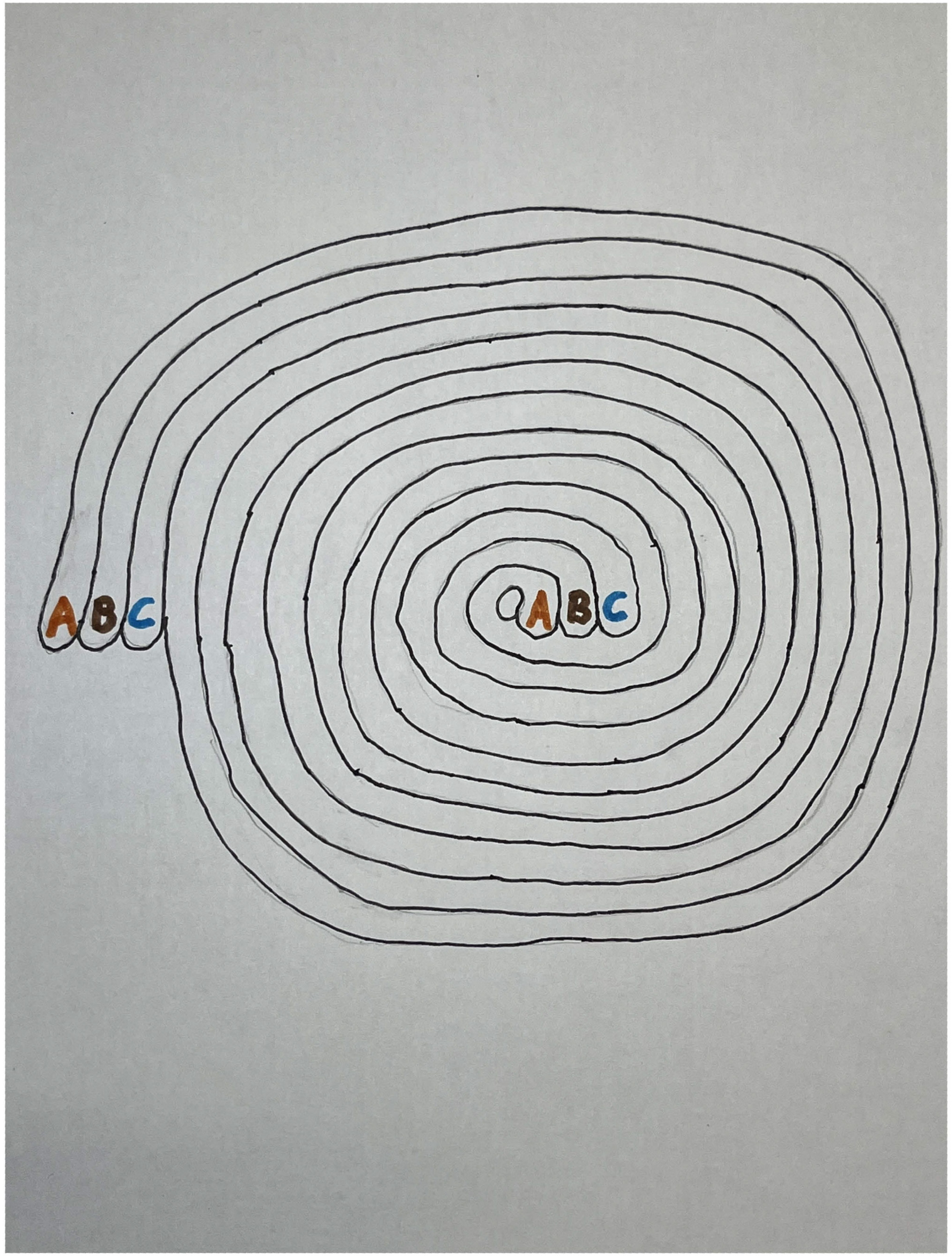
The 3 routes ABC lead to ABC down a spiral.

Use Amber pen to trace A to A
Use Brown pen to trace B to B
Use Cyan pen to trace C to C

First follow the A route to join A and A. Success.

However…
Routes B and C crossover without any visible means of doing so.
B now leads to C and C now leads to B.

How come?



Sliding Block Puzzle Page – Update Feb 2024

In 1999 I built a web site that brought to life many poplar and rare sliding block puzzles. But about 10 years ago, all common web browsers discontinued support for Java applets, killing the play features of my site.

https://www.puzzleworld.org/SlidingBlockPuzzles/

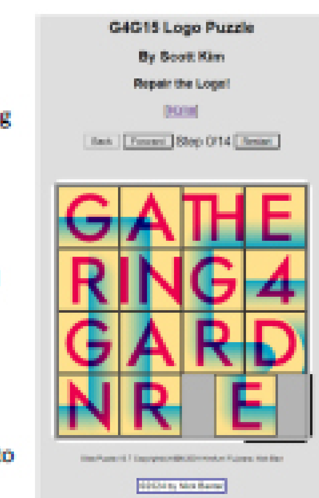
Recently, Andrea Gilbert helped me take advantage of a browser plug-in that emulates Java in Javascript, and thus brings my web site back to life!

Additionally, I have completed some significant reorganization, with the following groups of playable puzzles:

- **The 15 Puzzle –** problems presented in Jerry Slocum's *The 15 Puzzle*
- **Classic Designs –** Here are the best known puzzles from the early 1900's, including Get My Goat, Dad's Puzzler, and L'Âne Rouge.
- **Modern Designs –** including designs by Nob Yoshigahara, Serhiy Grabarchuk, Rodolfo Kurchan, Junichi Yananose, and Neil Bickford.
- **Minoru Abe –** Minoru's puzzles were the original inspiration for this web site. Now I present a playable selection my favorites.

Other resources:

- **Minoru Abe Gallery –** This is a near complete photo gallery of all Minoru Abe sliding block puzzles, including scans of the Japanese instruction sheets describing rules, goals, and additional challenges.
- **Panex –** Various papers about the Panex puzzle, including my G4G5 exchange paper, the original unpublished 1982 research paper asserting an optimal solution algorithm, and a recent report corroborating those results.



To celebrate, and in the spirit of G4G15, I have added the **G4G15 Logo Puzzle** to the web site! Designer Scott Kim asks about the implicit challenge…here I give you the opportunity to try it out for yourself!

Designing the Trapdoor Octahedron puzzle

George Bell 12 / 31 / 23

My G4G15 exchange is the **Trapdoor Octahedron** puzzle (Figure 1), a 3-piece **coordinate motion** puzzle. Coordinate motion refers to the fact that to come apart, all three pieces must move at the same time.

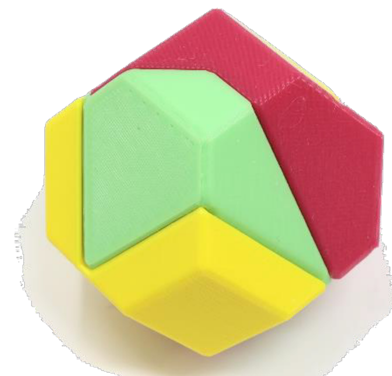


Figure 1. Trapdoor Octahedron puzzle, 3D printed.

The puzzle is not difficult to take apart, but most will struggle to reassemble it. In cryptography, a trapdoor function is one that is easy to compute in one direction yet difficult in the opposite direction (finding its inverse). Thus, the **Trapdoor Octahedron** puzzle is a mechanical analog of a trapdoor function.

Coordinate motion puzzles are difficult to design, as the pieces must work together to constrain their movement. Any change will generally ruin the puzzle. This document outlines the design process for this puzzle—the last page gives the solution.

Step 1: The basic piece

Coordinate motion puzzles often have identical pieces. This is not a requirement—in fact the **Trapdoor Octahedron** uses three different pieces! However, it is a good starting point in the design of a coordinate motion puzzle.

I wrote a program to search for dissections of the rhombic dodecahedron into three identical pieces. The program also requires that the dissection be symmetric about the 3-fold axis of symmetry. See [1] for details. The piece shown in Figure 3 is one out of 231 found by my program.

An easy way to build the basic piece uses a special building block, the green pyramid in Figure 2. The green pyramid can be made from a rhombic dodecahedron by cutting along every triangle connecting an edge to the center of the polyhedron. This dissects the rhombic dodecahedron into 12 identical pyramids, one for each face. In BurrTools [2], this green pyramid is four voxels in the “Rhombic Tetrahedra” geometry; Stewart Coffin calls it a “Rhombic Pyramid Block” [3]. We will also use this pyramid cut in half lengthwise (orange in Figure 2).

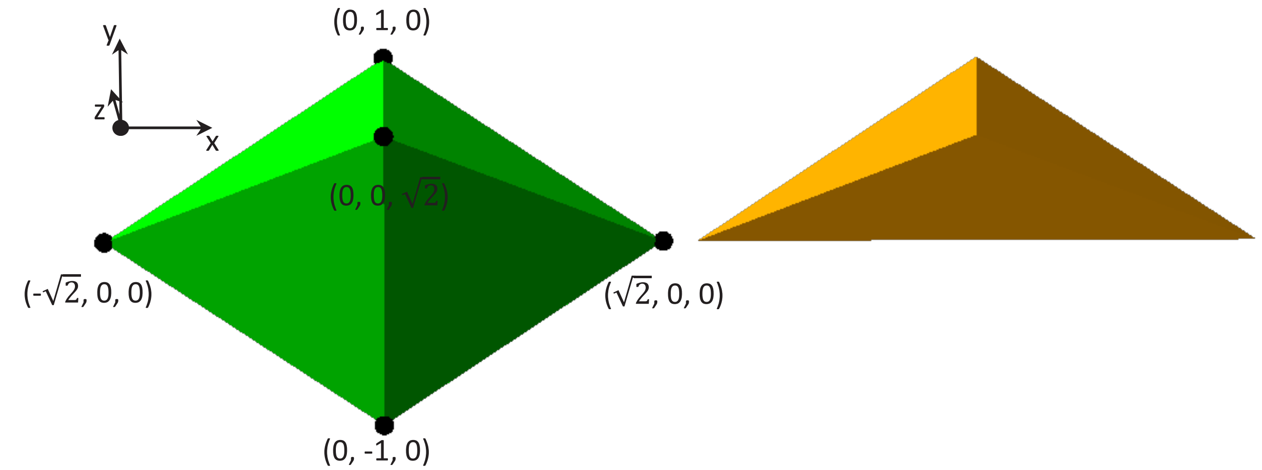


Figure 2. Left: the building block (green); right: cut in half lengthwise (orange).

We now combine three green pyramids and two orange halves to make the basic puzzle piece (Figure 3). Each basic puzzle piece has the volume of four pyramids, and three copies form a rhombic dodecahedron. This basic piece has 180° rotational symmetry about the z-axis (coming out of the page).

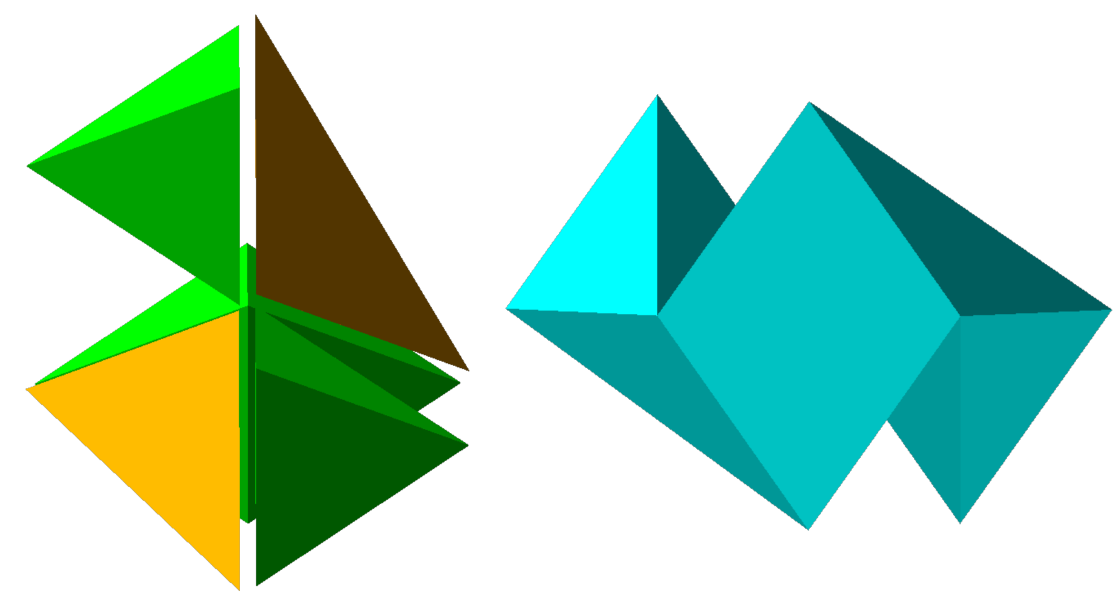


Figure 3. Three pyramids (green) and two half-pyramids (orange) form the basic piece. Left: pyramids slightly separated; right: completed piece (bottom view).

Step 2: Partial stellation into an octahedron

Three copies of the basic piece assemble via coordinate motion to form a rhombic dodecahedron. Coordinate motion means that all three pieces must move at the same time. This process is shown in Figure 4 (left, looking down the 3-fold symmetry axis). The assembly feels loose, because there is more than one way for the pieces to go together (the puzzle is “CM+” as defined in [4]).

One way to fix this looseness is to stellate the rhombic dodecahedron, which adds twelve green pyramids to the outer faces. To remain a three-piece puzzle, we need to connect the added pyramids to the existing pieces, and there are many ways to do this. I have chosen only a partial stellation, adding four green pyramids. This results in a non-regular octahedron.

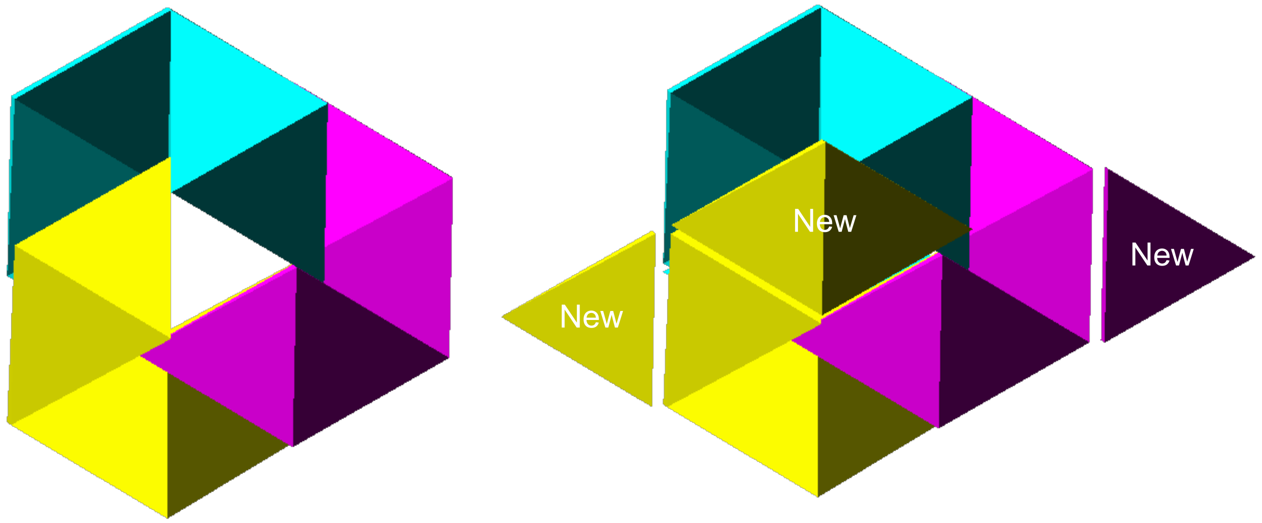


Figure 4. Left: coordinate motion assembly; right: same view showing the added pyramids (marked “New”). The 4^{th} (aqua) pyramid is hidden behind.

This partial stellation is the most important step in the design because the added pyramids restrict the piece motion so that the puzzle comes apart in only one way. In addition, after partial stellation the three pieces are all different. The assembled shape is a non-regular octahedron (Figure 5, right). Each face is not an equilateral triangle but an isosceles triangle with odd angle from the rhombic dodecahedron face, 70.53°.

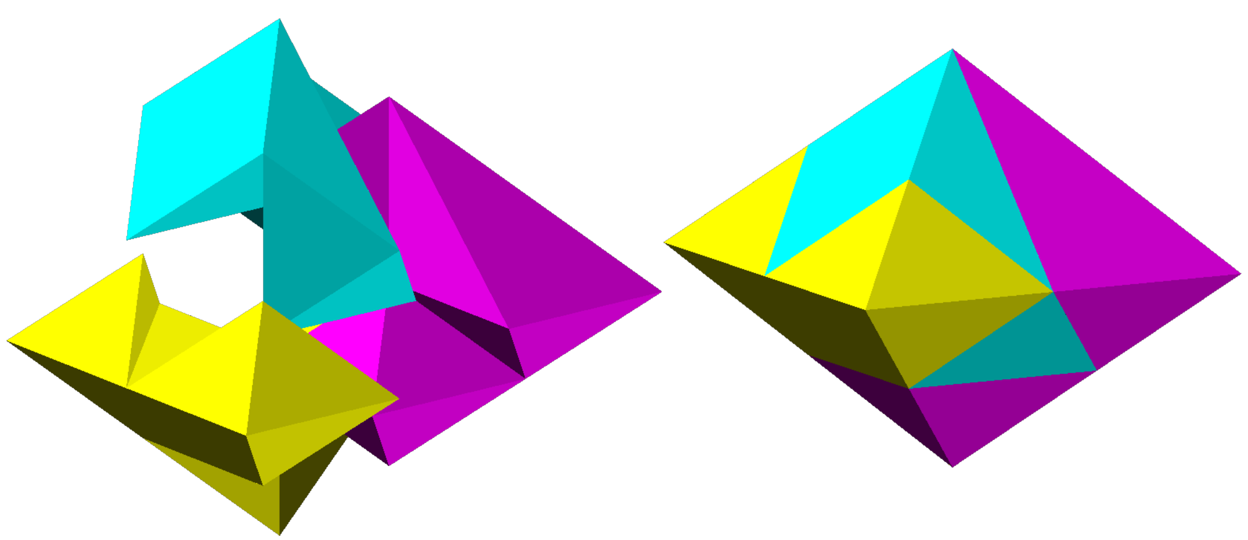


Figure 5. Top view of the puzzle being assembled, final assembly into a (non-regular) octahedron.

Step 3: Vertical stretch into a regular octahedron

If we stretch the octahedron along its short axis by a factor of $\sqrt{2}$ it becomes a regular octahedron. Remarkably, the coordinate motion is unchanged by this transformation. The reason is that the pieces have planar faces and move linearly without rotation (see [1]).

Step 4: Truncation

The final step is to truncate the octahedron. The truncation makes it easier to 3D print, plus I find it aesthetically pleasing. Figure 6 shows the three pieces 3D printed.

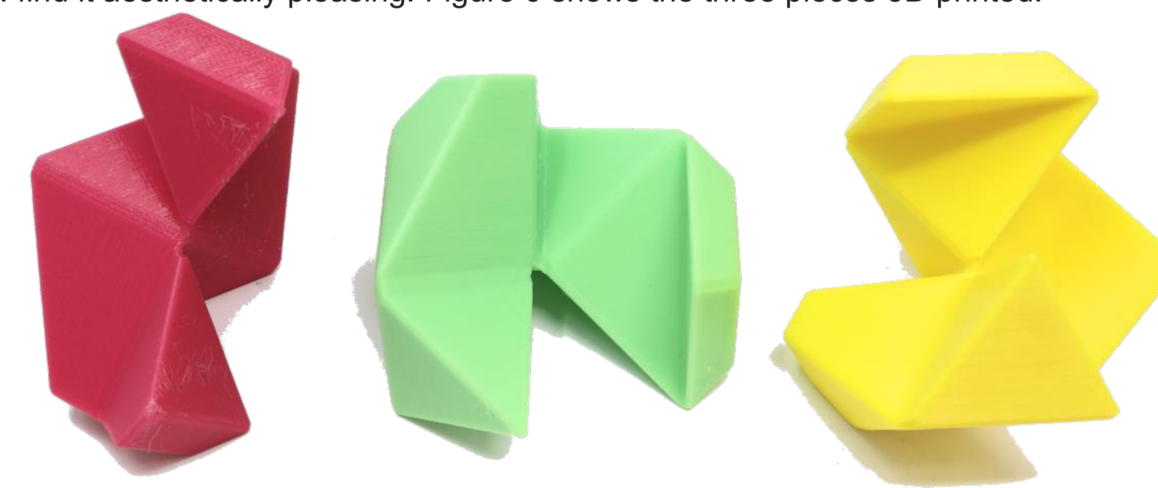


Figure 6. The final **Trapdoor Octahedron** pieces 3D printed (normal version). All pieces are now different, and all have different volumes.

The dimensionless units in Figure 2 are multiplied by a scale factor to get the final puzzle size. The scale factors used by the two versions are given in Table 1. The exchange version is the small size. The puzzle was designed using BurrTools [2] and OpenSCAD [5]. You can 3D print your own copy at Printables [6].

		BurrTools [2]			edge length of		
version	scale factor	Unit Size	Bevel	Offset	rhombic dodecahedron	regular octahedron	truncated octahedron
normal	15.59	13.5	0.500	0.05	27	54	18
small	11.67	10.1	0.375	0.05	20.2	40.4	13.47

Table 1. Sizing numbers for the final puzzle (all dimensions in mm).

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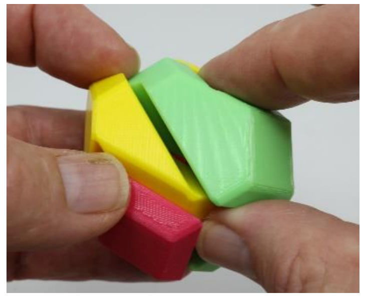
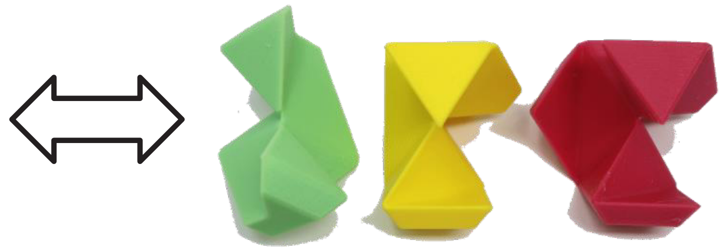
Trapdoor Octahedron Solution



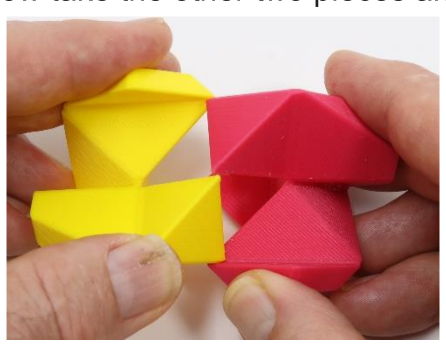


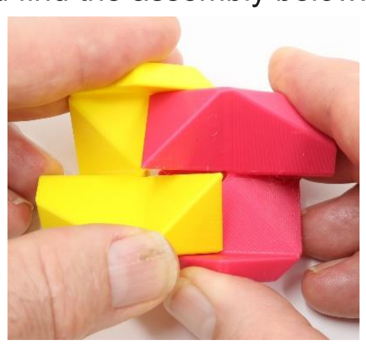
Designer: George Bell
Goal: Take it apart and put it back together
Solution video: https://www.youtube.com/watch?v=6pLIXKmy0CU

Disassembly:
Note that two of the hexagonal faces contain a triangular face of a different color. To make the puzzle expand, press in these two faces with your thumb and forefinger (as shown above). Use your other hand to stop the motion. If you expand it far enough, it will fall apart.

Assembly: This is much harder. First, identify the smallest piece. This piece is also the only one with 180° rotational symmetry. In the above photos, it is the green piece—find this smallest piece in your puzzle (it likely has a different color).

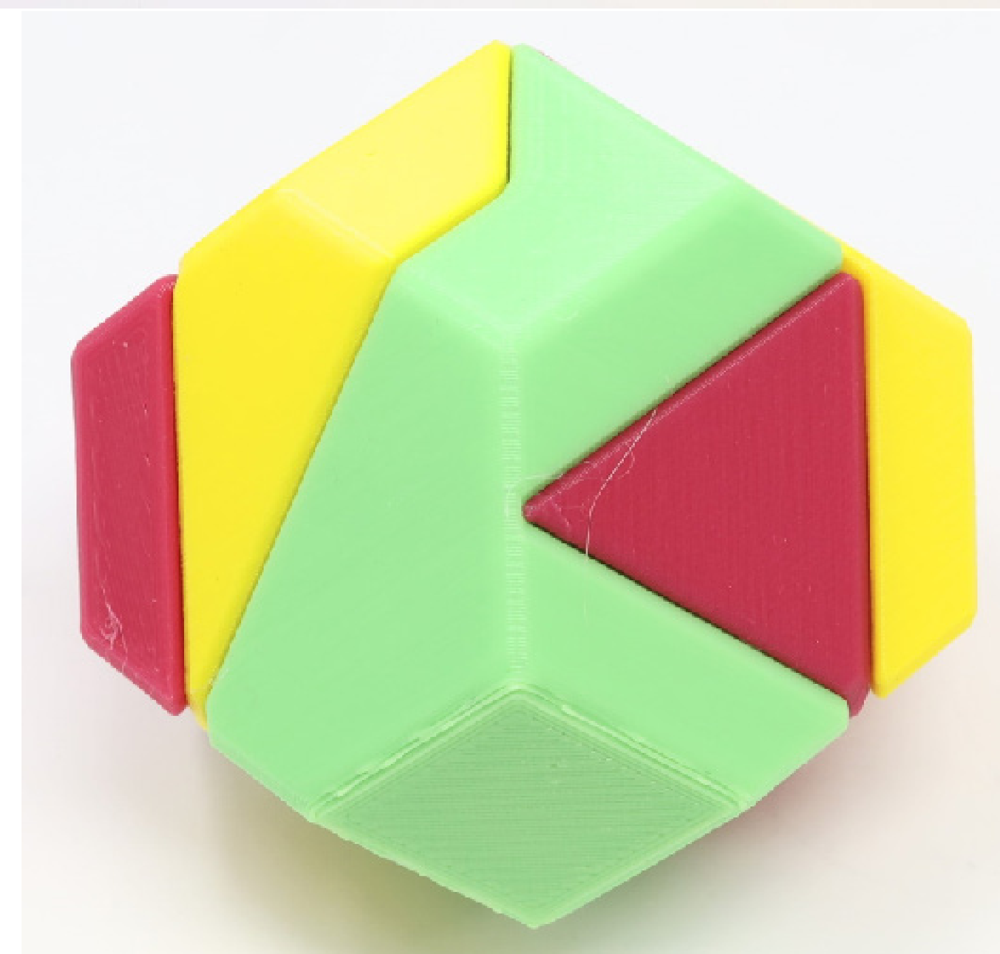
Now take the other two pieces and find the assembly below.





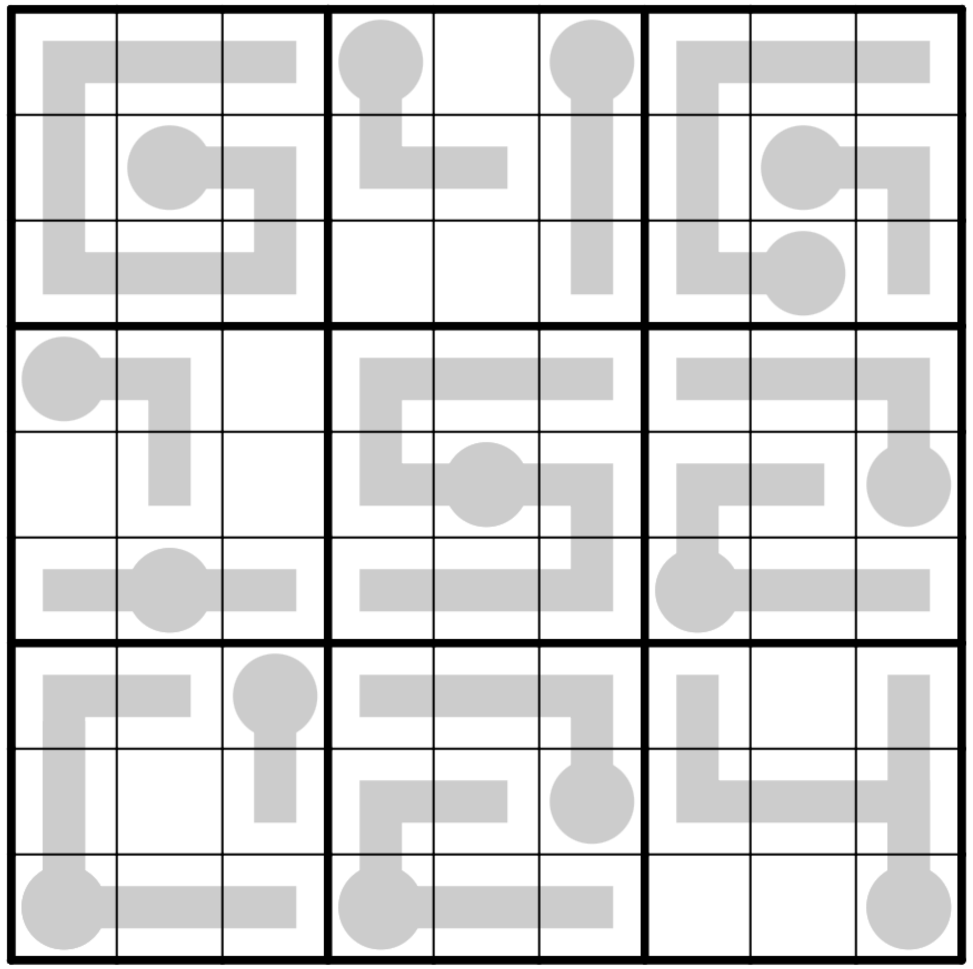
The yellow piece slides in from the left, the red piece from the right. In the right photo, these two pieces are in their assembled configuration. We now need to somehow get the green piece between them.

To get the green piece in there, back the red and yellow pieces up until you are in the position of the left photo. Note that the two central points, as well as those above and below, should be just touching. With the red and yellow pieces in this position, add the green piece from above. There are four points which must engage into grooves. If you can engage all 4 simultaneously, the three pieces will slide together.



G4G15 2024 Thermometer Sudoku

Tantan Dai | tdai44@gatech.edu



Online solving with answer check: http://tinyurl.com/2c69ezlx

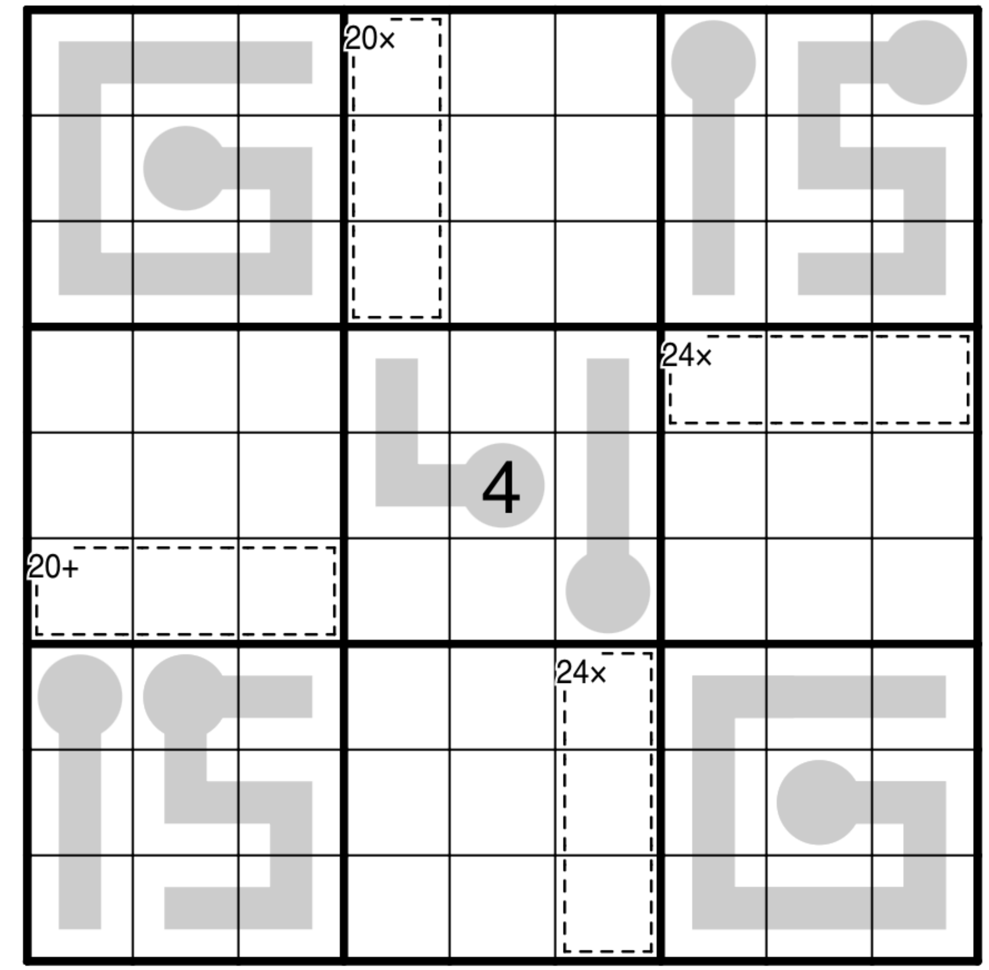
Rules:

- Place a digit from 1 to 9 into each empty cell in the grid so that no number repeats in any row, column, or outlined 3×3 region.
- Some thermometer shapers are in the grid; numbers must be strictly increasing from the round bulb to the flat end.

Solving path by Tantan: http://tinyurl.com/2xpfbjmk#Replay

G4G15 2024 Thermo Killer (+,×) Sudoku

Tantan Dai | tdai44@gatech.edu



Online solving with answer check: http://tinyurl.com/297wrw23

Rules:

- Place a digit from 1 to 9 into each empty cell in the grid so that no number repeats in any row, column, or outlined 3×3 region.
- Some thermometer shapes are in the grid; numbers must be strictly increasing from the round bulb to the flat end.
- The sum or the product (indicated by + or ×, respectively) of the numbers in each cage must equal the value given in the upper-left corner of that cage.

Solving path by Tantan: http://tinyurl.com/24bdb2fo#Replay

For extra challenge: You can remove the given digit 4 in the middle of the grid and solve.

Four prophets and three sages

Riddle and solution by Ivo Fagundes David de Oliveira and Yogev Shpilman, 14th of December 2023

A long time ago four prophets spoke. Ever since, no more prophets have ever nor will ever come to be under the sun. Each prophet came to the land after the other had passed away, and each knew and agreed with their predecessors: The second knew and agreed with everything the first said, the third knew and agreed with everything the first two said, and finally, the fourth knew and agreed with everything the first three said.

The sacred writings testify that:
One prophet is known to have said "there will be, or has been, *at least one false prophecy uttered by a false prophet*", another prophesied that "*there will be, or has been, at least one true prophecy uttered by a true prophet*", yet another prophesied "*there will be, or has been, at least one true prophecy uttered by a false prophet*", and finally one that prophesied that "*there will be, or has been, at least one false prophecy uttered by a true prophet*".

Each prophecy was prophesied by a different prophet. However, **the order in which these prophecies were uttered remains unknown to us**.

It is known that a true prophet only says true things and never agrees with false prophecies. It is also known that a false prophet may or may not prophesy something true; a false prophet is false either because he prophesied a false prophecy, or because he agreed with a false prophecy, or both.

Now it came to pass that three wise men, **who knew the order in which the prophets came**, strongly disagreed on which prophecies were fulfilled and which weren't. Thankfully, in the first great council, chaired by the three wise men, after heated debates and long meditative moments of prayer, they finally decided to resolve the conflict by simple majority voting: they placed votes for each of the four prophecies simultaneously, deeming each one to be either true or false. To their surprise, the majority outcome of all of the votes completely aligned with the opinion of the eldest of the wise men, and from that moment onward he was considered the greatest among equals.

Question: In which order did these prophecies come? Which prophecies were true and which were false?

SOLUTION:

We label the prophecies as follows:

Prophecy FF	There will be or has been, at least one **false prophecy** uttered by a **false prophet**.
Prophecy FT	There will be or has been, at least one **false prophecy** uttered by a **true prophet**.
Prophecy TF	There will be or has been, at least one **true prophecy** uttered by a **false prophet**.
Prophecy TT	There will be or has been, at least one **true prophecy** uttered by a **true prophet**.

It is possible to immediately notice that prophecy **FT must be false**, as no true prophet would prophesy a false prophecy. This means that prophecy **FF must be true**, as prophecy FT is a false prophecy uttered by a false prophet. Because the three sages strongly disagreed, **the truthfulness of prophecies TT and TF must remain ambiguous**. We also notice the following:

A. If prophecy FF came after FT, that makes the prophet of FF a false prophet (because he agreed with a false prophecy that was uttered before him), which means prophecy TF must be true. That wouldn't leave room for three sages to disagree, and we thus conclude **prophecy FT came after prophecy** FF (but maybe not right after).
B. If FF was the first prophecy, its prophet must have been a true prophet, which means prophecy TT must be true. That wouldn't leave room for three sages to disagree, and thus **prophecy FF wasn't first**.
C. If TF was the first prophecy, then prophecies TF and TT are either both true or both false. That wouldn't leave room for three sages to disagree, and thus **prophecy TF wasn't first**.

Thus, from A, B and C we conclude that **prophecy TT was the first** under the sun. Now, prophecy TF could be 2nd, 3rd, or 4th. Let's explore these options:

D. If TF came 2nd under the sun, then its truthfulness is opposite to that of TT, which wouldn't leave room for three sages to disagree. (We consider the order: TT, TF, FF, FT. If TT is false, then TF must be true as FF is a true prophecy by a false prophet. If TT and TF are true, then the first three prophecies were uttered by true prophets. Prophecy FT is as always false, which means there was no true prophecy by a false prophet, in contradiction to the assumption that TF is true. Thus if TT is true in that considered order, TF must be false).
E. If TF came 3rd and TT was true, there would be a paradox between prophecy TF and itself (We consider the order: TT, FF, TF, FT. If TT is a true prophet, then so is FF. That means, for TF to be false it must be uttered by a true prophet, which is a contradiction, and for it to be true it must be uttered by a false prophet - but since there was no false prophet before it, it also cannot be true), which means TT must be false and the sages would agree on it unanimously.

These facts uniquely determine the position of Prophecy TF to be the fourth and thus the full order is: TT → FF → FT → TF

Prophecies TT/TF could be true/true, true/false, or false/true (but not false/false as if TT is a false prophet, so do the other prophets, which makes FF a true prophecy which makes TF a true prophecy). After the sages vote for these options, we conclude that prophecies TT, TF, and FF were fulfilled, while prophecy FT wasn't.

The Hell, Michigan Round at the 2024 MIT Mystery Hunt

Exchange Gift for Gathering for Gardner 15 from Joe DeVincentis

I am a member of an MIT Mystery Hunt team. I have been on five different teams over the 25 years I have been participating, though to some degree every year's team is different since people come and go. In 2023 I was writing for the hunt in January 2024, my fourth time to do so, and I really feel like I did something special this year. Something that generated reactions like "That was amazing!" and "How could you possibly make that work?" and had a lot of people saying it was their favorite part of the hunt. I'm going to explain how I did it in this paper. Spoilers occur later on, with a warning and link if you want to try the puzzles yourself first.

About Puzzle Hunts

Puzzle Hunts are different from simple puzzles in that there are a group of puzzles meant to work together. The form of that interaction varies. A common style is the *metapuzzle*: Several individual puzzles lead to final answers that are usually words or short phrases. The metapuzzle is an incomplete puzzle that needs the answers from the other puzzles to complete it. Solving enough individual puzzles (called *feeder puzzles* because they *feed into* the metapuzzle) lets you solve the metapuzzle. It's usually possible to solve the metapuzzle missing some of the feeder answers; typically you need 60%-80% of the answers.

Sometimes metapuzzles use the answer words or their letters alone, with only some hints in the metapuzzle to help you figure out what to do with the answers; puzzlehunters call this a *pure meta*. Other times you may be given additional information in the metapuzzle itself, such as a grid, or you may get information along with confirming each feeder answer, which is used instead of or in addition to the feeder answers to solve the meta. These are called *shell metas*.

The Mystery Hunt has seen many variations on the general metapuzzle structure. Sometimes the feeder answers aren't words or short phrases. One round (a *round* refers to an entire group of puzzles and their metapuzzle(s); because of differing structures, some rounds include more than one metapuzzle) had every answer be a picture solvers had to find and verify with HQ (*headquarters*, the team running the hunt, also called *GC* or *Game Control* in some puzzle hunts). In another round, each answer was a physical object. In another, each answer was an emoji character. In yet another, each answer was a grid of letters and spaces arranged in a specific way. Sometimes individual feeder puzzles have multiple word-or-phrase answers you can find and confirm separately.

Other times it's the metapuzzles that work differently, especially when there are multiple metas. Sometimes you have to figure out which feeder answers belong to each meta. Possibly some feeder answers belong to multiple metas. Sometimes rounds build up in stages, with one metapuzzle using an initial set of feeder answers, and a second metapuzzle using those answers again plus the answers of new feeder puzzles you are given after solving the first meta, potentially iterating more times.

About Our Mystery Hunt

There are no spoilers for the puzzles in the Hell, Michigan round in this section. The theme for the overall hunt is spoiled, but only to the degree that someone attending the opening skit would have seen (you can watch that opening skit at https://youtu.be/BwnsHRNyTns). The puzzle spoilers will begin in the next section (on the next page).

Puzzle hunts usually have themes and some sort of story that gives some reason for why participants are solving puzzles, though that often involves ludicrous leaps of logic and the assumption that everybody everywhere does puzzles all the time. You have to accept this, just like the suspension of disbelief necessary to enjoy TV shows and movies based on magic or other obviously unreal scenarios.

We chose to use a theme based on Greek mythology, and to some extent Roman as well, since the Romans worshiped many of the same gods by different names. Specifically, because of the popularity of Pluto in recent years, we chose to base our hunt on the demotion of Pluto to dwarf planet by the International Astronomical Union in 2006.

Our team had connections at Caltech, where Mike Brown is a professor; Mr. Brown was the astronomer most responsible for the IAU's decision, and wrote a 2010 memoir *How I Killed Pluto and Why It Had It Coming* about the event. We were able to get Mr. Brown to record a short video ranting about how he wanted to demote Pluto even further. During the opening, we played this video, then had a group of made-up astronomers vote to demote Pluto to "unimportant space debris."

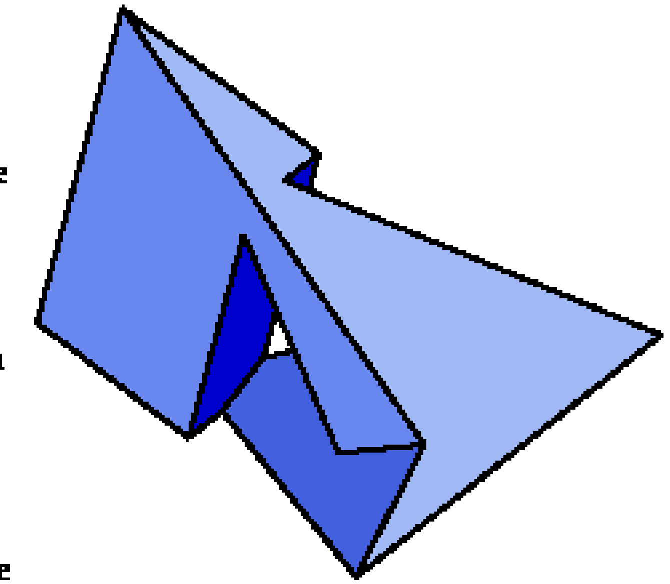
Moments after the vote concludes, the stage is invaded by Persephone (Pluto's wife) and several other gods who are incensed that these *mortals* (they emphasize the word several times) have just killed Pluto. After some exposition about how the gods are the planets and vice versa, the gods present blame the entire group of mortals present at the event (the astronomers on stage and the solving teams watching in the auditorium) for Pluto's death, and banish them all to the underworld. Solvers are already presented with multiple tasks, all of them to be achieved by solving puzzles: Escape the underworld, restore Pluto to life, and appease the angry gods.

A Call for Unorthodox Round Concepts, and My Answer

A few months into constructing our hunt, our team leadership was worried that we would be letting down solvers who had become used to the more ingenious round designs that have become so frequent that solvers might expect to see some in every Mystery Hunt, and they put out a call for unorthodox round designs. This is where I stepped up with the idea for the Hell, Michigan round.

Last chance before spoilers begin. At the time of this writing, the puzzles are available by going to https://mythstoryhunt.world. Click "Public Access", and then from the Rounds menu click Hell, MI.

At some point this year, and possibly before you get a chance to read this, it will move into the hunt archive at https://puzzles.mit.edu/huntsbyyear.html under 2004.



I've always been fascinated by the Szilassi polyhedron. Some of you probably know it; it's a polyhedron with 7 faces, most of them concave, and each with 6 edges, sharing one edge with every other face. As you can see here, it has a hole through it.

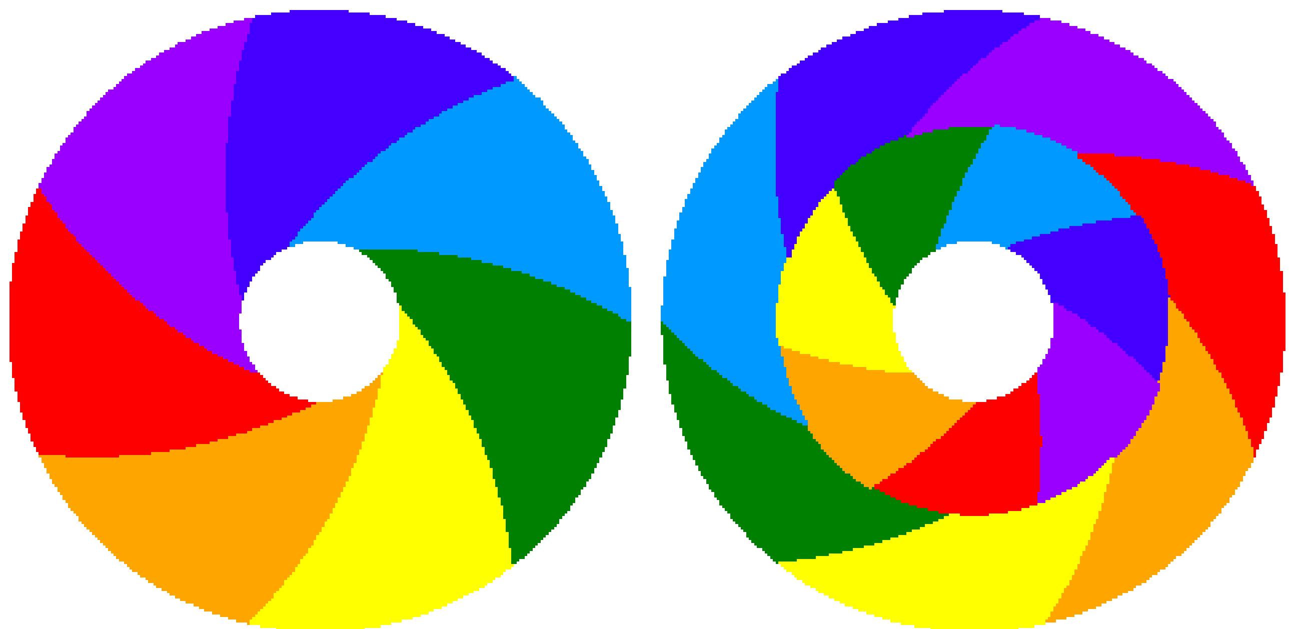
My thinking was to make a round where puzzles don't work by themselves. We had such a round back in 2009, the Reverse Dimension, where each puzzle's long title described one of the first nine Doctors from *Doctor Who* or one of their companions. You had to pair up puzzle-halves for the Doctors and their corresponding companions to get puzzles you could actually solve, though the interaction between the halves worked differently for different pairs. This proved that giving solvers half-puzzles they have to pair up to solve was feasible.

But my round was going to go further than that. Each of the seven puzzles was going to share an answer with *each* other puzzle, with some sort of interaction between the pair necessary to find the answer. That's C(7,2) = 21 answers. It seems like a lot (8 to 12 feeder answers is typical) but in rounds in the past where puzzles had multiple answers, we've had similar numbers. The *Legend of Zelda* Triforce round in the 2011 hunt featured 9 puzzles, each with 3 answers, for a total of 27 answers. (There were three metapuzzles, and each used one answer from each puzzle, but you had to figure out which.) The 2009 hunt had an Orbital Nexus round which was based on phases of the moon. This was represented within the hunt by each of the 8 puzzles in this round changing in subtle ways every 15 minutes, cycling through all four of its variations each hour. So you had to solve each puzzle four times with different data to get 32 answers. In contrast to these rounds, 21 answers wasn't too many.

Since I was putting puzzles on the faces and answers implicitly on the edges by virtue of them being shared by pairs of faces, logically the answer should extract letters from the vertices somehow. I decided the three answers meeting at each vertex would have only one letter common to all three and that letter would go there. Moreover, if possible, I'd have every two answers meeting at a vertex have at least two letters in common, to make it not too easy and give solvers an incentive to keep solving for more answers.

But then how are solvers to read the answer? A quick lookup told me there are 24 Hamiltonian cycles on the Szilassi. Combined with 14 starting points and two directions to read the answer, that's too many orders. Sure, if they have all the letters they can list all the possibilities via a program and scan the list of several hundred strings for something that looks sensible, but good metapuzzle design ensures the puzzle is solvable with a few answers missing. Turning, say, 8 of those letters into choices of two letters each (because four answers touching 8 different vertices are missing) multiplies those choices by 256. A design concept for puzzle hunts is that random anagramming is bad. To give solvers so many letters without a stronger ordering concept than "it forms a Hamiltonian" is akin to random anagramming.

This led me to the next stage in this puzzle's evolution. Instead of using the Szilassi polyhedron as the design concept, I would use the 7-colored torus. And I'd help them out a bit. By giving them the image (below left) with the 7 bands of color and referring to it as a doughnut, they'd know that images of the torus they can find online where the back side looks like the right image below are what they want. While there are still lots of Hamiltonians, there's now one obvious one where the letters read around in a circle.



And looking up 7-colored torus in Google images gives plenty of images like this, including one from somebody selling pillows made in such a shape. This was a reasonable way to clue this information to solvers, who should have figured out every pair of puzzles shares an answer from the puzzles themselves, if they didn't already get it from this image in the metapuzzle.

So I felt like I had a good metapuzzle mechanism. In most cases that's enough to start writing an actual metapuzzle with actual feeder answers that you make up leading to a metapuzzle answer that you make up but with approval from the editors of the hunt. In this case, I wanted to do a bit more. Making each puzzle interact with every other puzzle in the round is a heavy burden, much more than feeder puzzles usually carry. So, in contrast to the usual meta authoring strategy where the feeder puzzles aren't even considered until a metapuzzle is written and tested, in this case I wanted to plan out a feasible set of feeder puzzles to ensure I could do what I was proposing. I was pioneering here. I didn't want to write and test a metapuzzle and then discover that I couldn't figure out how to make the puzzles interact in all the ways necessary.

Feeder Interaction Design

In this stage I developed loose plans for what each puzzle should be and how they would interact. I wasn't actually writing the puzzles (with one exception), just considering how certain types of puzzles could work and how I could force them to interact.

The first puzzle I designed was one called Blanks. This puzzle was meant to be a clue that the puzzles needed to work together, that there wasn't possibly enough information here to solve the puzzle by itself. There would be just a set of blanks with five of them numbered like this:

```
_ _ _ _ _ _   _ _ _ _   _ _ _ _ _ _ _
    2     4       1       3       5
```

I had decided five distinct letters was the minimum any of my answers should have, just as two was the minimum any two answers meeting at a vertex should have in common. This puzzle would involve six phrases which fit on the blanks (one from each other puzzle) and would extract a five-letter answer from each. I wanted to minimize the strain on the phrases to spell good words, while minimizing the strain on the puzzles to spell out long and weird phrases (which there would be plenty of already apart from this puzzle) and this was the balancing point. The letters are well distributed among the three words and tests showed it was able to make a variety of different words with different vowel-consonant sequences.

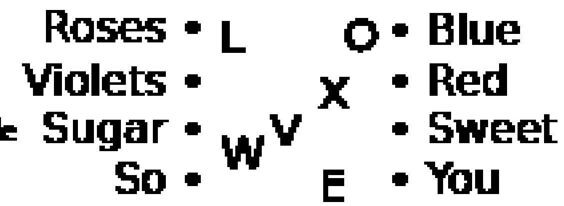
Next I decided to make three grid puzzles. A classic mechanism is to give solvers two grids of the same size to overlay. In this case, I'd write a 15x15 crossword, a 15x15 word search, and a 15x15 Akari puzzle. The last is a Japanese puzzle type sometimes called Light Up in English. You're to place lights in certain cells of a grid so as to illuminate all cells according to a number of restrictions. The light positions make such an obvious overlay that I felt a clue phrase telling solvers to overlay these puzzles was unnecessary.

But the word search and crossword should also overlay, and the mechanism there was less obvious. Perhaps the most obvious thing to do was overlay the grids and see where the same letters overlay on each other. But I didn't want to put that much more strain on the crossword and word search grids. So I decided to leave a message in unused letters of the word search to overlay it on the crossword and look through the Qs. And I'd just put enough Qs in the word search to match the answer for this pair, again to reduce the strain I was putting on the crossword puzzle. Word searches are less constrained and I felt sure I could make a word search with whatever else was needed after the other two grid puzzles were written. The Akari would be written first of the three, so that whoever was constructing the crossword knew where to put letters to spell out a message. And finally, the message among the unused word search letters explicitly referring to the crossword was another chance for solvers to figure out the puzzles had to work together, if they hadn't done that yet.

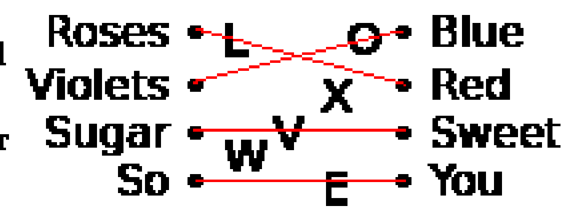
I also decided at this point that more unused letters in the word search could give the phrase to put on the blanks, the crossword could have the phrase as one of the clues, and the Akari, which hadn't been planned to have any letters in the grid at all at this point, could hide the phrase in flavor text. So this was four puzzles interacting in every combination. But I had three more to go, three which had to interact with each other *and* with each of these three.

I added one more puzzle type next because I knew it would come in handy.

This puzzle type didn't have a name until Mike Selinker called it Matchmaker in his book *Puzzlecraft*.



The basic puzzle type is simple: Match the items on one side with the items on the other, drawing lines to connect the corresponding dots.



This leads to some of the letters in the middle being crossed by lines. We might notice that the items on the right are in alphabetical order, while the ones on the left are in the order of the poem. Reading the letters crossed by the order of the left ends of the lines spelled out the answer LOVE. The letters not crossed by lines are red herrings and are not used.

But that's not the only way to read the answer from one of these. Sometimes the things in the middle are numbers, and you have to index them into the items on one side, perhaps reading them in the order of the connections on the other side. This wouldn't work for LOVE in this example since Sugar doesn't have any of those letters, and the letters on the right don't have a V. But suppose LOVE were replaced by 3, 4, 2, 1, respectively. Then, indexing the 3 into Red (and reading it first since it's on the first line from the left) gives a D. The other numbers in the same manner complete the word DEWY.

Sometimes it's the items that aren't crossed that are used. In this case there's nothing to index into, so the items generally have to be used directly. If we apply that to this puzzle, we get XW if reading the letters from top to bottom, or WX if reading from left to right. Neither of these looks useful, so we can conclude this puzzle doesn't work that way. But some Matchmaker puzzles do.

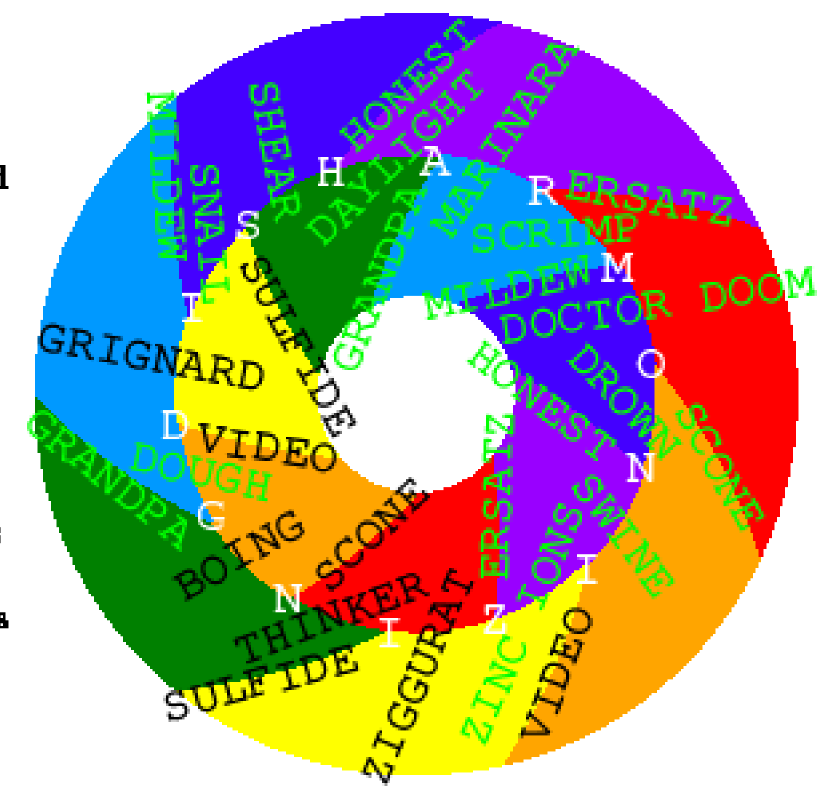
I had decided I would write the mother of all Matchmakers. There were going to be letters, numbers, and Roman numerals (in a different font to distinguish them from the letters) in the middle. Each line would cross one of each type, and some of each type would remain uncrossed, giving us 6 sets of symbols. The two sets of crossed numbers would each index into one side, read in the order of connections at the other side, using both choices of sides. The other four sets would read left to right, top to bottom, right to left, and bottom to top, a different direction for each set. The overloaded puzzle would give 6 messages, one for each other puzzle. One would be the phrase for the blanks, one would identify a crossword entry, one would identify a word search word, and I wasn't sure how to extract from the Akari.

This was when I decided the Akari would have small letters in the cells. The letters on the cells with lights could spell a message to apply to some other puzzle, and one from the Matchmaker made of unused numbers would start SHIFT BY when indexed into the alphabet, and the rest would tell how to shift the letters of the Akari message within the alphabet to give a different message. I also decided the Matchmaker would be written last. It would be able to encode any messages of reasonable length, only limited to the two extractions from the clues being the same length.

One of the editors working with me suggested a technical puzzle ("something sciencey") and an interactive puzzle as good ways to fill in the last two puzzles needed to complete the set. I chose a chemistry puzzle, because I felt I had enough knowledge to write it, and a text adventure, because it had been done before in Inform 7, so I knew it was feasible. Also, it was possible to hide easter eggs inside the game that messages from other puzzles might lead to, or give messages when completing tasks the game posed which would apply to other puzzles.

Writing and Testing the Metapuzzle

After deciding how the remaining interactions would work, including have the traditional long theme entries of the crossword identify an answer among the chemicals shown or implied in the chemistry puzzle, I started actually writing and testing the metapuzzle.



First was coming up with an answer. Some of the first answers I tried led to a problem completing feeder answer selection. Repeated letters in the answer forced so many feeder answers to contain that letter that the three answers meeting at some other vertices were forced to contain that letter. I didn't want that to happen; I wanted the letters to be unique is solvers had all three letters at the vertex.

I wrote a script to check possible answers for problems of this sort, as well as report where just two of the three answers at a vertex were forced to have the letter so I could be sure to avoid it in the third answer. With this sorted out, and a working meta answer chosen, I selected the other answers, choosing five-letter answers for the Blanks puzzle (shown here in orange) that I could make good phrases for, and chemistry-related words for some answers for the Chemistry puzzle (yellow) where I expected other puzzles to pick out aspects of the chemistry with a clue phrase.

Then we tested the metapuzzle. Testing rarely goes as smoothly as you hope, and this one was sure to be extra-tricky. Testers were spoiled on the idea that the feeder answers we gave them came from interactions from two puzzles and were each accepted as answers for both those puzzles. In one test, we gave them just the colors for each answer; the plan was that the puzzles would be labeled with their ROY G BIV colors and linked from those sections of the torus image we gave them. They got too distracted by there being multiple chemistry words in the yellow puzzle to actually focus on what they were supposed to be doing.

After a couple other configurations, we finally settled on this plan: Puzzles would not initially be associated with colors, but when solvers got all the answers for a particular color, the round page would reveal to them which color it was associated with. And we tested this with 16 answers given (all the answers from three puzzles and one other answer) which seemed barely enough for them to get it, and in some other combinations, and it seemed to work.

But one editor pointed out that none of these testers tried to brute-force the solution. The meta answer DISHARMONIZING was supposed to be a pun, that Dis (another name for Pluto) had harmonized all the groups in Hades and stopped them from fighting little battles at all their borders. But disharmonizing is a real word and can be found in word lists. They wanted to know whether the answer could be brute-forced with fewer answers than we gave the testers. So I wrote a program to brute-force my own meta, and we found that if they just solved all the answers in two puzzles (so only knowing the color identities of those two puzzles) the answer just popped out, and other combinations suggested it might be gettable with even fewer answers. We didn't want teams to be able to bypass so much of the round, so we decided we had to scrap all this and start over with a new "non-Nutrimaticable" meta answer and new feeder answers. (Nutrimatic is a tool, beloved by some puzzlehunters, which lets you fill out partial answers with regular-expression-like syntax, sorting results based on commonality of phrases, with a huge corpus that includes all the text of Wikipedia.)

The new testers solved the new metapuzzle just as easily as the original using the final testing setup we had decided on, and we finally agreed this metapuzzle had passed testing.

At some point during this process, I was informed that, for puzzle story reasons, the round needed to be hosted in some American city. Our creative director, who was in this puzzle's discussion to try to figure out the art needs for this round, suggested Hell, Michigan. This is a real town, but in the real world a normal one, apart from being the butt of jokes about Hell freezing over. I agreed that it was reasonable to make a fictional portal there where we looked back into Hades and saw what was happening there.

Writing Feeder Puzzles

The normal result after a metapuzzle passes testing is that the answers become available to distribute to authors of feeder puzzles. My chief editor had been working with me on this puzzle and agreed that couldn't happen here. When he promoted the puzzle out of testing he immediately assigned all its answers to a placeholder puzzle, so I could write up more fully fleshed out placeholders for each of the seven feeder puzzles in the round, documenting all their required interactions, based on the detailed plans I had set up earlier. Once those were ready, my editor moved the appropriate answers into each puzzle and deleted his first placeholder.

I quickly wrote up the Blanks puzzle using the phrases I found while developing the final set of feeder answers, and then worked on the Akari. I had decided the interaction between the Akari and the Text Adventure was going to be that the Akari would have a theme of three central pillars labeled 1, 2, 3 (meaning they have lights adjacent to 1, 2, and 3 of their 4 sides, respectively) and in the text adventure solvers would encounter three numbered pillars with lights they could light up with text commands. When they matched the Akari solution, the game would give them the associated answer. This gave me a starting point for my Akari, so I wasn't simply writing any 15x15 Akari with no constraints, and I designed a puzzle I felt was of medium difficulty.

It was at this point that we gave both the Blanks and Akari to one of our factcheckers. Factchecking is so named because it often involves looking up bits of trivia that puzzles depend on, but it really means verifying that the puzzle works the way it should. For a logic puzzle like Akari, it means verifying the solution is valid and unique. For Blanks, it means making sure the phrases are spelled correctly and extract to the right answers. Usually we do this just before sending puzzles into test-solving. Because the next puzzles were going to depend on these puzzles working correctly, if there were any errors, we wanted to find them now. The Akari only used a Blanks phrase in its flavor text and would have been easily corrected, but the crossword was going to use one as a crossword clue, so the intended answer for this clue was also verified.

The chief editor I had been working with in the effort to get this round into the hunt was also a published crossword constructor (as opposed to myself being only an occasional amateur crossword constructor) and he volunteered to write the crossword with its odd variety of constraints. With that done and factchecked, I wrote the word search. I placed this last to construct of the three grid puzzles because I figured it had the most freedom.

I initially placed the letters in the grid needed for the two overlay extractions, and except for one Q saved for an unused-letter message, which I put near the end, I placed words in the word search to cover all those letters. I additionally added words to leave the right number of unused letters after the unused Q and the right total number of unused letters for the messages I was hiding. And then I put in the letters for the messages to fill the grid. I decided to make the word search a clued one, to give it a little more meat, and wrote some clues, and passed that on to factchecking.

There was a bit of a pause at this point, and when construction resumed, my chief editor had found a teammate Linus Hamilton eager to write a text adventure, and I'd found Alina Khankin eager to write a chemistry puzzle, and I explained to them the unusual constraints on each puzzle and the way they'd have to work together on one interaction.

Text adventure author Linus went all out, writing his game to present some classic puzzles like the towers of Hanoi and the river crossing but with constraints that made them not work. For instance, you were only given 20 moves to complete the towers with 5 discs, which normally takes 31 moves. You had to take advantage of aspects of the text adventure world or secrets revealed from other puzzles to cheat at each puzzle (in terms of its normal rules). Each puzzle was hosted by a particular historical figure, such as Genghis Khan at the towers. If you tried to cheat by putting the Hanoi discs somewhere other than the poles, you were prevented from doing so, and all-seeing Khan yelled at you in all caps THAT IS NOT A POLE or (for people) HE IS NOT A POLE, even if you were doing it in another room. But two of the historical figures, Copernicus and Chopin, were Polish, and hence Poles. Khan let you get away with using either or both of them to store discs for the puzzle, which made it possible to complete in the allowed moves. It took a lot longer to complete the text adventure than the other puzzles in this round, but it was also a lot of fun, and it was the single puzzle from the round most called out as solvers' favorite. Massive thanks, Linus!

Once I had the messages confirmed that the Matchmaker needed to send to these and all the other puzzles, I wrote the very overloaded Matchmaker. I had submitted another puzzle idea which could be implemented with matching, and not having any other good ideas for matching items to use in the

Matchmaker, after verifying I had enough items and the right letters to make it work, I decided to adapt this idea for Hell's Matchmaker. It involved works of fiction which uncannily predicted inventions of historical events, such as Morgan Robertson's *The Wreck of the Titan*, which tells of a shipwreck much like the Titanic, of the same size, on the same route, caused by hitting an iceberg, and with a similar name, but published 14 years before the Titanic sank. You had to match the works of fiction to the historical events, for which only the lengths of words and their initial letters and years were given.

Testing the Feeder Puzzles

Only once all the feeder puzzles were all constructed could we test any of them, because they all needed to be used together to get the answers. So we recruited some larger teams (compared with the usual pairs who would test individual feeder puzzles) with larger blocks of time. We also used this as an opportunity to test our post-production process (post-prod, for short).

The term post-production is borrowed from filmmaking, where it refers to the steps taken after scenes are filmed and audio is recorded, including editing, sound mixing, special effects, and the like. For our puzzles, it's the process between taking the puzzle the way it was presented to testers and reproducing it as a web page, consistent in style with our other puzzles. The team working on this had just gotten things ready to be able to post-prod puzzles, so these were used as guinea pigs, and the full-round test also tested our ability to host puzzles on a web site and to use the real answer checker as opposed to the one in our test-solving platform.

Fortunately, this went smoothly. Not only did the testers successfully solve the round, but they loved it. So did many solvers during the actual hunt, as they let us know through their cheerful and praising feedback. My success was built on making an ambitious goal, planning carefully, having the right insight to see how to make it work, and great teammates to help with planning, writing, testing, and factchecking.

Puzzle Food: More Food for Thought

By Rik van Grol, NL
Rvgrol@hotmail.com

Introduction

At G4G8 in 2008 I presented a puzzle food menu. With puzzle food I mean mechanical puzzles that look like food. Two examples are shown in Figure 1; an interlocking puzzle that resembles a hamburger and a 2×2×2 Rubik's Cube variant in the shape of a can Budweiser. In 2008 I presented a table with in total 88 examples of puzzle food that are in my puzzle collection. Some sixteen years have passed since G4G8 and I continued to look for Puzzle food. This paper is an attempt to bring you up to date.

Figure 1. Examples of puzzle food: a Hamburger and a Budweiser

Recently I have moved house, and all my puzzles are boxed up, so I am a bit handicapped. This means that the extension of the puzzle menu that I present is probably not complete. Also, while writing this update I have come across puzzle food that I did not yet have, so I can promise you this is unlikely to be the last update. You could say, this is work in progress.

The best puzzle food could be mistaken for real food when viewed from a distance. For most of the "antique" puzzle food this is the case. The puzzle food of recent years (let's say after 2000) is less uniform. Some recent examples of Puzzle food could, even at relatively close distance, be mistaken for real food, see Figure 2 left. Some other recent examples mimic very realistically the shape of real food, but to attempt is made to actually look as food, see Figure 2 middle. And then there it a group of puzzle food where no attempt is made to take the shape of puzzle food, but a maximum attempt is made to mimic the colours, see Figure 2 right. Most puzzle food puzzles can be classified to one of these examples.

Figure 2. Three examples of puzzle food that are quite different: near realistic (left), realistic in shape only (middle), realistic in print only (right)

In this paper an attempt is made to classify the listed puzzle-food examples by 5 characteristics:

- Is the **shape** realistic: blank=no or not applicable, 1=somewhat, 2=realistic, 3=very realistic
- Is the **size** realistic: blank=no or not applicable, 1=close, 2=optimal
- Is the **colour** realistic: blank no or not applicable, , 1=somewhat, 2=realistic, 3=very realistic
- Is the **package** realistic: blank no or not applicable, 1=somewhat, 2=realistic, 3=very realistic
- Is the **solution** related to the food: blank=no or not applicable, 1=close, 2=optimal

With **shape** the 3-dimensionsional proportions are meant. Together with the correct size real food could be replaced by puzzle food. The **size** is either far from correct, close to the realistic size or exactly the right size. The **colour** is either far from right (or irrelevant) or to a degree realistic. Some puzzle food cannot be realistic by itself, like liquids such as coca cola. However, the cola is in a can, its **package**, which can be very realistic. There is also solid food for which this applies. Some of the sweats come in a package that could easily be mistake for the real package. Finally, the **solution** could be related to the food, e.g., the way you eat a donut, or the way you drink a cup of coffee.

The subject of puzzle food can be dealt with in a serious tone, but I believe it lends itself more for a humoristic approach. Instead of going through the world of puzzle food by one of the well-known puzzle classifications. I will present it by going through a menu: breakfast, diner, fast food, deserts, beverages, etc. The mechanical puzzle classification, which I will adopt from Jerry Slocum[1], do enter the story, but as the "food groups". For a healthy puzzle food diet, you need to get balanced meal with something from every food group!

In this article I will restrict to presenting some samples of puzzle food, giving you an idea of what is available. Mind you, I will only show what I have in my own collection, which means that there is (much?) more to be found out there. I will not repeat what I showed before, only new puzzles I collected since G4G8.

Normally I would finish an article by pointing the reader towards further reading, but I believe that in this case there is hardly any further reading. The only fairly brief presentations of puzzle food can be found in two books from Jerry Slocum and Jack Botermans[2] and on the website from Rob Stegmann[3].

[1] Jerry Slocum and Jack Botermans, *Puzzles Old and New*, 1986, ISBN I-85336-018-X.
[2] Jerry Slocum and Jack Botermans, *The book of Ingenius & Diabilical puzzles*, 1994, ISBN 08129-2153-4 and *Het Ultieme Puzzelboek* (in Dutch), 2007, ISBN 978-90-5897-720-5.
[3] http://home.comcast.net/~stegmann/home.htm

Puzzle breakfast suggestions

The breakfast suggestions have been supplemented with lots of fruits and eggs. I just learned that the unstable eggs (dexterity puzzles) in Figure 4, series 1 and series 2 have been extended with series 3.

Figure 3. From left to right, Peach, apple, orange, lemon and watermelon

Figure 4. Unstable eggs: series 1 (left) and series 2 (right)

Lunch puzzle suggestion

The lunch suggestions (1) have been doubled. Figure 5 show a sandwich (left).

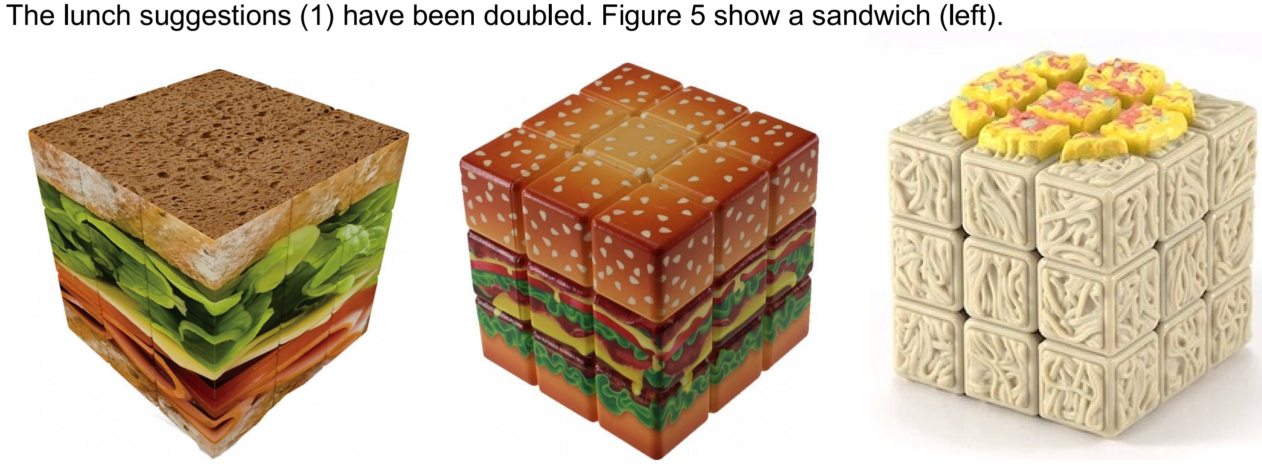


Figure 5. Two V-CUBE3 puzzles: Sandwich (left) and hamburger (right), and a noodle puzzle

Apéritif puzzle suggestions

No new puzzle available in this category.

Puzzle diner suggestions

For diner several new puzzles have been added. Figure 5 (middle and right) shows a hamburger and some noodles The examples in Figure 5 are quite realistic when it concerns the colour and textures, but when it concerns the shape they score low.

Salad bar puzzle suggestions

There is one addition, but momentarily no picture available.

Desert puzzle suggestions

The number of desert puzzles have almost tripled. Quite a lot of new cakes, see figures 6, 7 and 8. But also ice cream (not shown) and a banana (Figure 9).

Figure 6. Puzzle bon bons and two petit four puzzles from Perry McDaniel

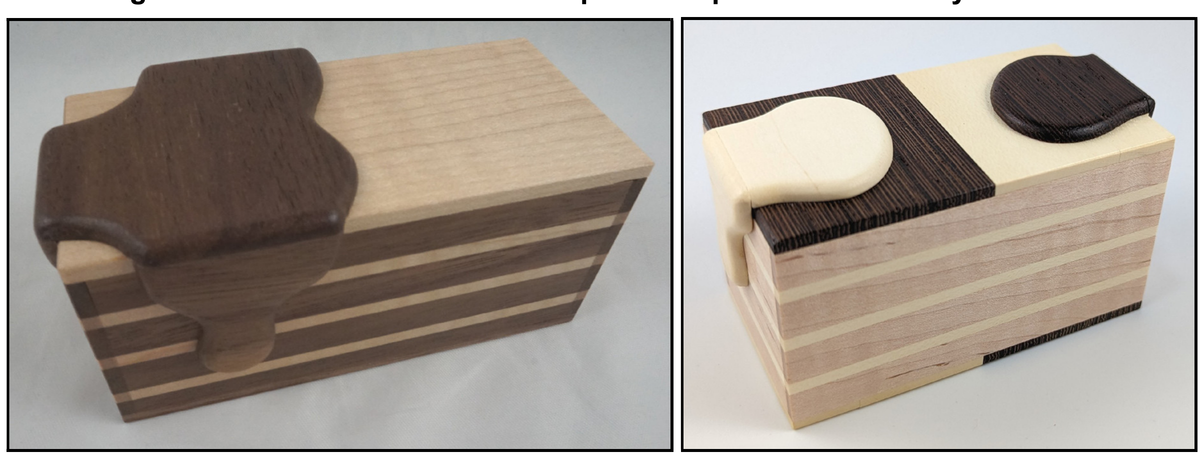


Figure 7. Blackjack cake (left) and Wedlock (right) by Perry McDaniel

Figure 8. Banana split from Lakeside

Figure 9. Four puzzle cakes by the Karakuri group

Figure 10. Three very realistic puzzle cakes by Pierre Hermé Paris

Cold beverage puzzle suggestions

There are a few additions, but for the moment no pictures.

Hot beverage puzzle suggestions

For the hot beverages I can proudly say that I have an important addition, which I knew existed in 2008, but which I did not own: a cup of coffee by Akio Kamei, see Figure 11. This is a unique puzzle. It is realistic in shape, but it is also realistic in its solution.

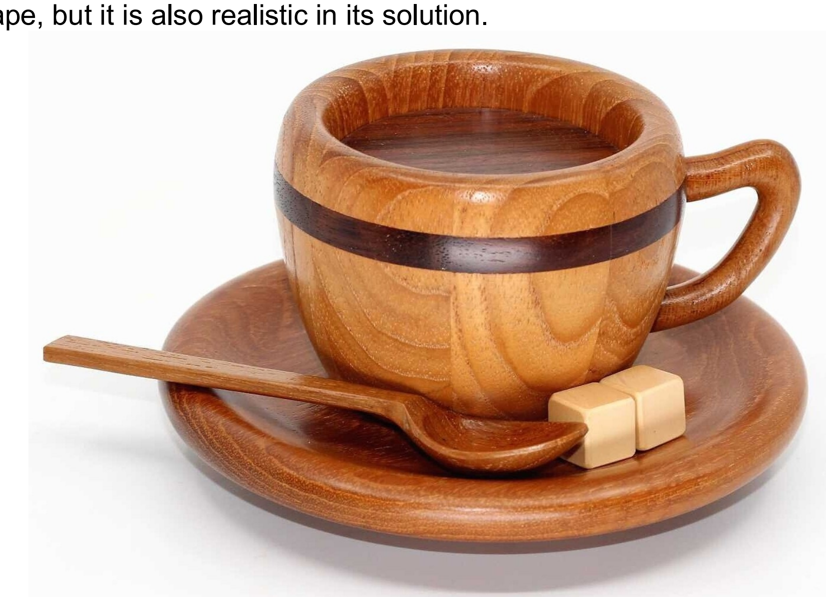


Figure 11. A cup of coffee by Akio Kamei

After diner puzzle suggestions

Alle the new after diner puzzle suggestions come from Japan. A few are shown in Figure 12.

Figure 12. Cupa Chups lolly (left), and chocolate sweets in the Meiji Seika puzzle series

Closure

The following table shows 159 puzzles. Of these puzzles 88 were shown in the 2008 article. The additional 71 puzzles were collected since. The table shows for each puzzle: a name, manufacturer / designer and date (as far as known to me), the puzzle-type category, the puzzle-food score, and whether it was in the original paper, or new in this paper.

I very much welcome information to improve the table. I would also very much appreciate information about puzzle food missing in this overview.

	Put Together	Take Apart	Interlocking	Disentanglement	Sequential movement	Dexterity	Puzzle Vessels	Shape	Size	Colour	Package	Solution	Puzzle type	Date	Manufacturer, Designer	2008	2024
Healthy Puzzle Breakfast suggestions																	
Pack the Orange	1							2	2	1			Packing	1990s	Toyo Glass	1	
Pineapple Delight	1							2	2	1			Packing	1990s	Toyo Glass	1	
Pack the Plums	1							2	2	1			Packing	1990s	Toyo Glass	1	
Pack the Rice-Crackers	1							2	2	1			Packing	1990s	Toyo Glass	1	
Apple (puzzle vessel)							1	1		1			Bottom fill			1	
Apple with worms	1							1	2	1			Packing		Lakeside	1	
Apple (wood)	1							1	2	1			3D jigsaw			1	
Adam's Apple (plastic)	1							1	2	1			3D jigsaw	1989	Mag-Nif	1	
Apple (plastic, red)	1							1	2	1			3D jigsaw	2004	Jeruel Industrial	1	
Apple (plastic, green)	1							2	2	1			3D jigsaw	2004	Jeruel Industrial	1	
Apple (plastic, red)					1			2	2	1			Rubiks Cube (3x3x3)	2014	Fanxin (Chinese)		1
Pear (plastic, green)					1			2	2				Rubiks Cube (3x3x3)	2014	Fanxin (Chinese)		1
Lemon (plastic, yellow)					1			2	2				Rubiks Cube (3x3x3)	2014	Fanxin (Chinese)		1
Orange (plastic, orange)					1			2	2				Rubiks Cube (3x3x3)	2014	Fanxin (Chinese)		1
Peach (plastic, peach)					1			2	2				Rubiks Cube (3x3x3)	2014	Fanxin (Chinese)		1
Banana (plastic, yellow)					1			2	2				Rubiks Cube (3x2x2)	2014	Fanxin (Chinese)		1
Water melon (plastic)					1					3			Rubiks Cube (2x2x2)	2024*	V-CUBE 2 (Verdes Innovations)		1
Scrambled Egg	1							2	2				3D jigsaw	1989	Mag-Nif	1	
Grapes (plastic, purple)	1							2	2				3D jigsaw		3D Christal jigsaw puzzle		1
Tomato (plastic)	2							2	2				3D jigsaw		3D Christal jigsaw puzzle (red & yellow)		2
Egg puzzle		1						2		1			Interlocking		Greenbrier International Inc.	1	
Egg (P-10-5)		1							1				Secret opening	2000	Akio Kamei	1	
Egg	1								1				3D jigsaw			1	
Unstable eggs						12		1	1				Balance		Etsi		12
Deviled Egg	1								1				3D jigsaw	1980s	Mag-Nif	1	
Half an Apple/Orange	1							2	1	2			3D jigsaw	1980s	Mag-Nif	1	
The Amazing Chewdini		1						2					Secret opening	1998	Binary Arts / Thinkfun (Ken Forsee)	1	
I Love Cheese Puzzle					1			2	1				Route finding	1996	Sjaak Griffioen (IPP16)	1	
Spongebob in Cheese					1			1	1				3D sliding piece			1	
Kase Puzzle	1							1	1				Packing	1995	Naef (Ulrich Namislow)	1	
Something Fishy (No 7001)	1							1	2				Packing			1	
Cracker Jack (J-10)	1								2		3		Packing	1980s	Synergistic Research Group	1	
Wheaties (B-300)	1								2		3		Packing	1980s	Synergistic Research Group	1	
Phony Balony	1							2	2	1			Packing	1970	Parker Brothers Inc.	1	
Small Baloney	1							2	2	1			Packing	?	?	1	
Cheerios	1								2		3		Packing	?	?		1
Puzzle Lunch suggestion																	
Bento -- Packet Lunch		1						3	2				Secret opening	2007	Hiroyuki Oka	1	
Sandwich (plastic)					1					3			Rubiks Cube (3x3x3)	2024*	VCUBE (Verdes Innovations)		1
Apéritif puzzles																	
Cocktail (wood)	1							2	2				3D jigsaw			1	
On the Rock (Wisky)	1							3	2	3			Packing	1990s	Toyo Glass	1	
On the Rock (Wodka)	1							3	2	3			Packing	1990s	Toyo Glass (IPP special)	1	
Puzzle Diner suggestions																	
Hamburger			1					2	1	2			Interlocking	1993	McDonald's Corporation	1	
Big Mag			2					2	1	2			Interlocking		McDonald's Corporation (limited)		2
Hamburger	1							2	2	2			3D jigsaw	2006	Beverly Enterprices Inc	1	
Burger Thing	1							2	2	2			3D jigsaw	1977	Reiss Games	1	
Prankfurter	1							2	2	2			3D jigsaw	1977	Reiss Games	1	
Fries	1							1	1	2			Packing	1993	McDonald's Corporation	1	
Japanese Rice Ball	1							3	2	2			3D jigsaw	2006	Beverly Enterprices Inc	1	
Rice Cube	1							2	2	2			3D jigsaw				1
Tamago sushi (egg sushi)	1							3	2	2			3D jigsaw				1
Sashimi sushi (meat sushi)	1							3	2	2			3D jigsaw				1
Pizza Puzzle	1							1	1	1			Packing	1997	Joe Becker (IPP17)	1	
Potato	1							2	2	1			Packing		Edi Nagate - private production		1
Onion	1							2	2	1			Packing		Edi Nagate - private production		1
Hamburger					1					3			Rubiks Cube (3x3x3)	2024*	V-CUBE 3 (Verdes Innovations)		1
Mihon and egg					1					1			Rubiks Cube (3x3x3)	2020	Megahouse		1
Sandfield Salt and Pepper Shakers		1						2	2				Secret opening	2002	Norman and Robert Sandfield and Perry McDaniel	1	
Salad bar puzzle suggestions																	
Pack the Broad Beans	1							2	2	1			Packing	1990s	Toyo Glass	1	
A-maize-ing	1							2	2	1			Packing	1990s	Toyo Glass	1	
Asparagus	1							2	2	1			Packing	1990s	Toyo Glass, Dick Hess	1	
Peter Piper's Fickle Pickles	1							2	1	1			Packing	1973	Stevens Manufacturing Company	1	
Whiskey sause	1							2	2				3D jigsaw				1

	Put Together	Take Apart	Interlocking	Disentanglement	Sequential movement	Dexterity	Puzzle Vessels	Shape	Size	Colour	Package	Solution	Puzzle type	Date	Manufacturer, Designer	2008	2024
Desert puzzle suggestions																	
Sunday chocolate sause (plastic)	1							2	1	1			Packing	1993	McDonald's Corporation	1	
Sunday strawberry sause (plastic)	1							2	1	1			Packing		McDonald's Corporation (limited edition)		1
Sunday (wood)	1							2	2				3D jigsaw			1	
Dovetail Bar - Walnut Flavour		1						2	2				Secret opening	2000	Norman Sandfield (IPP20)	1	
Dovetail Cherry Surprise Cake		1						2	2				Secret opening	2003	Norman Sandfield & Perry McDaniel (IPP23)	1	
Marbled Walnut Cheese Cake		1						2	2				Secret opening	2006	IPP26 (Perry McDaniel)	1	
Petit four series		4						3	2				Secret opening	2009	IPP29 (Perry McDaniel)		4
Bon Bon series		4						3	2				Secret opening	2013	IPP33 Perry McDaniel		4
Blackjack Cake		1						2	2				Secret opening	2017	IPP37 Perry McDaniel		1
Wedlock		1						2	2				Secret opening	2023	IPP40 Perry McDaniel		1
Karakuri Cheese Cake		1						2	2				Secret opening		Karakuri group Japan		1
Karakuri Marble Cake		1						2	2				Secret opening		Karakuri group Japan		1
Karakuri Chocolate Cake		1						2	2				Secret opening		Karakuri group Japan		1
Karakuri Fruit Cake		1						2	2				Secret opening		Karakuri group Japan		1
Pudding	1							3	2				Packing	1990s	Toyo Glass	1	
Hershey Kiss	1							2	2				Packing	1980s	Synergistic Research Group	1	
Banana split	1							1	2				Packing	?	Lakeside		1
Satine	1							3	2	3			Packing	?	Pierre Herme Paris 25 pcs		1
Macaron Ispahan Cake Puzzle Games	1							3	2	3			Packing	?	Pierre Herme Paris 35pcs		1
Carrement Chocolat	1							3	2	3			Packing	?	Pierre Herme Paris 30 pcs		1
Ice cream	4							1	2				Packing	?	Hanayama		4
Cake (wood)	1							2	2				3D jigsaw	?		1	
Cold puzzle beverages																	
Coca Cola (can)	1								2	3	3		Jigsaw	?		1	
Coca Cola - Magic Rotating Jug					1				2	3	3		Sliding piece	1990s		1	
Coca Cola (Sliding Piece - straight)					1				2	3	3		Sliding piece	1990s		1	
Coca Cola (cup)	1								1	3	3		Packing	1993	McDonald's Corporation	1	
Coca Cola Coasters	1												Pattern	?		1	
Pepsi Cola (can)	1								2	3	3		Pattern	1980s	Synergistic Research Group	1	
Pepsi Cola (Sliding Piece - slanting)					1				2	3	3		Sliding piece	1990s		1	
Diet Coke (Sliding Piece - straight)					1				2	3	3		Sliding piece	1990s		1	
Diet Coke (Sliding Piece - slanting)					1				2	3	3		Sliding piece	1990s		1	
Sprite (Sliding Piece - straight)					1				2	3	3		Sliding piece	1990s		1	
Sprite (Sliding Piece - slanting)					1				2	3	3		Sliding piece	?			1
7up (sliding Piece - slanting)					1				2	3	3		Sliding piece	1990s		1	
Budweiser (can)					1				2	3	3		Sliding piece	?			1
Miller - High Life (can)	1								2	3	3		Pattern	1980s	Synergistic Research Group	1	
Lager Beer (mug)	1								2	2	2		Packing	1990s	Toyo Glass	1	
Budweiser (can)					1				2	3	3		Rubik cube (2x2x2)	2000s		1	
Duff beer					1				2	3	3		Rubik cube (2x2x2)	?	From the American animated series The Simpsons		1
Beer (mug, wood)	1								2	1	2		3D jigsaw	2000s		1	
Ice cola (plastic with straw)	1								1	1	3		Packing	?		1	
Mint Liquor (plastic with straw)	1								1	1	3		Packing	?		1	
Pocari Sweet (Sliding Piece - slanting)					1				2	3	3		Sliding piece	1990s		1	
Hot beverages puzzle suggestions																	
Java Tea	1								2	2	3		Packing	1990s	Toyo Glass	1	
Cup of Tea (wood)	1								2		2		3D jigsaw			1	
Coffee Cup		1							2		3	2	Secret opening		Coffee with sucre cubes and spoon by Akio Kamei		1
Secret Tea Box		1							2	2	3		Secret opening	1997	Nanco Bordewijk (IPP17)	1	
After diner puzzle suggestions																	
Meiji Milk Chocolate Puzzle	1							2	2	1			Pentomino	2005	Hanayama	1	
Meiji Black Chocolate Puzzle	1							2	2	1			Polyomino	2005	Hanayama	1	
Meiji White Chocolate Puzzle	1							2	2	1			Polyomino	2007	Hanayama	1	
Puzzle Bar	1							2	2	1			Packing		Pentangle	1	
Sandfield's Dovetail Donut		1						2	2			1	Secret opening	1996	Robert Sandfield (IPP16)	1	
Chocolate puzzle	1							2	2	1			Pentomino			1	
Chocolate puzzle II	1							2	2	1			Pentomino	2001		1	
Meiji Caramel	1							1	2				Packing	2006	Hanayama (Wil Strijbos)	1	
Chiclets (spearmint)	1								1		3		Packing	1980s	Synergistic Research Group	1	
Chiclets (peppermint)	1								1		3		Packing	1980s	Synergistic Research Group	1	
Lifesavers (Cherry)	1								1		2		Packing	1980s	Synergistic Research Group	1	
Lifesavers (Pep-O-Mint)	1								1		2		Packing	1980s	Synergistic Research Group	1	
Chocolate Fix	1							2		1			Logic	2007	ThinkFun	1	
Cigar Puzzle		1						1	1				Secret opening	2003	Marcel Gillen (IPP23)	1	
Cigar Puzzle		1						1	1				Secret opening	2003	Bits and Pieces, Marcel Gillen	1	
Peanuts	1							2	2	1			Packing	1990s	Toyo Glass	1	
Meiji Marble Chocolate Puzzle	1							2	2	1	3		Packing	?	Chocolate sweets		1
Meiji Apollo Chocolate Puzzle	1							2	2	1	3		Packing	?	Chocolate sweets		1
TIROL chocolate	3							2	2	2	2		3D Packing	2007	Megahouse (TIROL, MILK, ??)		3
Chupa Chups lollipop			3					2	2	2			Interlocking	?	Shape lollipop - comes in three variations		3
Meiji Curl (Cheese Aji) Puzzle	1							1	2	1	2		Packing		Meijiseika Puzzle Series		1
Meiji Curl (Usu Aji) Puzzle	1							1	2	1	2		Packing		Meijiseika Puzzle Series		1
Meiji Takenoko no Sato					1			1	1	1	3		Sliding piece	2007	Hanayama	1	
Meiji Kinoko no Yama					1			1	1	1	3		Sliding piece	2007	Hanayama	1	
159	87	27	6	0	26	12	1									88	71

FIVE PROBLEMS

emrehan@halici.com.tr

(G4G15-2024)

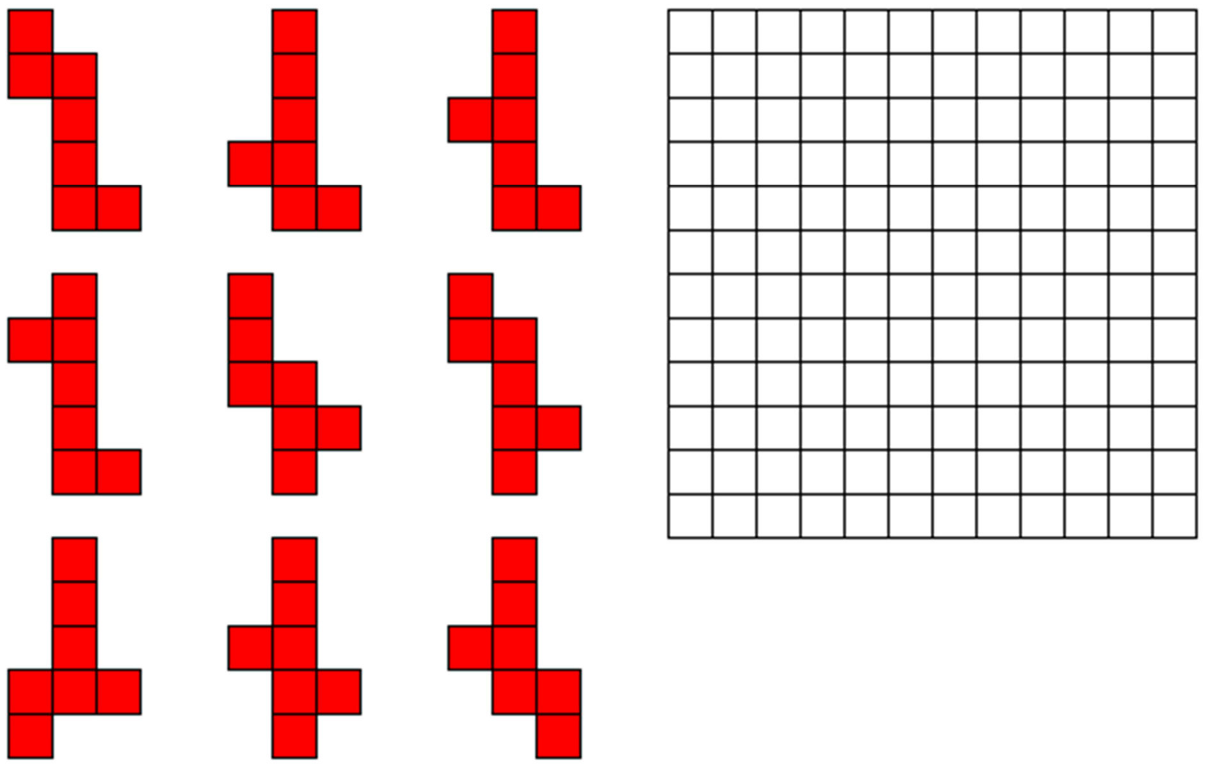
These problems are chosen from puzzleup.com 2018 (weekly puzzle competition prepared by Emrehan Halici).

1		**A Lottery Ticket** A 6 digit lottery ticket hits the jackpot. Tickets whose numbers differ from the winning ticket with only one digit (54 tickets) get consolation prizes. The jackpot winning number is a prime, and none of the consolation prize winners is a prime number. What is the largest "winning number" that satisfies these conditions ?
2		**Fourteen** The sum of the digits of a positive integer is divisible by 14. The sum of the digits of the next integer is also divisible by 14. Find the smallest such integer.
3		**11 Footballers** There are 11 footballers, having jersey numbers from 1 to 11. You will divide these football players into groups such that the sums of the jersey numbers of the footballers in all groups will be the same. In how many distinct ways can this be done? If the question was asked for 7 footballers the answer would be 5: (1-2-4-7, 3-5-6), (1-2-5-6, 3-4-7), (1-3-4-6, 2-5-7), (1-6-7, 2-3-4-5), (1-6, 2-5, 3-4, 7)
4		**Balanced Numbers** Let us define "balanced" number as a positive integer with distinct digits, such that half of its digits are odd and the other half are even. How many balanced numbers are there?
5		**The Number Cube** Let us place distinct positive integers to the edges of a cube (12 edges) such that, the products of three edges that intersect at any of the vertices (8 vertices) are equal. What is the minimum possible value for this product?

1)971767 - 2)5899999999999 - 3)79 - 4)4240125 - 5)240

Hazmat Cargo 2

by Carl Hoff (carl.n.hoff@gmail.com)



Hazmat Cargo 2 is the sequel to Hazmat Cargo, the puzzle which was the topic of my G4G13 presentation. That puzzle is discussed at length here:

https://www.gathering4gardner.org/g4g13gift/puzzles/HoffCarl-GiftExchange-FromUntouchable11toHazmatCargo-G4G13.pdf

The object of Hazmat Cargo 2 is to place the 9 pieces on the 12x12 board such that no 2 pieces touch, not even at a corner. The puzzle has 26226 near solutions or states where all 9 pieces are placed on the board with a single corner touch.

Good Luck.

Hints for Designing Puzzles

Iwahiro (Hirokazu Iwasawa)
iwahiro@bb.mbn.or.jp

From the perspective of a designer of various mathematical puzzles, including mechanical puzzles, this article carefully selects puzzles deemed suitable for discussion in this context, from those I have designed so far. It introduces these puzzles and shares the background of their creation. (This article is a translation, with minor revisions made to the original Japanese text, of a contribution originally published in the March 2023 issue (Volume 68, Issue 3) of *Operations Research*, a journal of the Operations Research Society of Japan. It was translated by ChatGPT and Iwahiro.)

1. Introduction

Let me begin with a brief introduction about myself, essential for understanding the purpose of this paper. I am somewhat known in the world of mechanical puzzle design. My recognition is partly due to my reasonably successful track record in the International Puzzle Design Competition (officially known as the Nob Yoshigahara Puzzle Design Competition) [1], including two top awards [2, 3]. Though I produce work sparingly, I am recognized by puzzle collectors worldwide for my extremely simple and unique designs rooted in a mathematical background. Therefore, I sometimes refer to myself as a "puzzle designer," though I am more of a mathematical puzzle researcher. I have written and translated numerous books and articles on mathematical puzzles and recreational mathematics. For several years now, I have also had the opportunity to pose a problem for the "Seeking Elegant Solutions" column in the *Mathematics Seminar* (数学セミナー) magazine, so Japanese readers of that column might be familiar with my name (the latest article at the time of writing this G4G15 paper is [4]. A book [5] collecting masterpieces from the same column also includes a problem I have posed).

While I don't like to refer to my activity of creating mathematical puzzles, including mechanical puzzles, as merely a "hobby," it's true that my main profession is different – I am an expert in actuarial science. Moreover, I don't create enough puzzles to consider it a full-time profession. However, even so, I have devised several puzzles that I consider masterpieces according to my own values, and I intend to continue doing so. A common feature of these puzzles is a certain kind of mathematical beauty, and I intend to create only those puzzles that embody this beauty. Occasionally, there are people who share a similar aesthetic sense to mine and who highly praise my puzzles. I understand that the invitation to contribute to this issue of *Operations Research*, which is a special feature on "The Conception of Puzzles," is a result of such connections.

In this article, I humbly present some stories about puzzle design from my perspective as a sparingly productive puzzle creator who is dedicated to some type of mathematical beauty. Through the following descriptions, I hope to convey some of the delights of devising new creations, and it would be extremely gratifying if this could serve as a reference or inspiration in any way.

2. Triangular Jam

2.1 Introduction of the puzzle

One of the mechanical puzzles I have designed is "Triangular Jam" (see Figure 1. While this puzzle has been commercialized several times under various names, the version shown in the figure is the original one I personally produced).

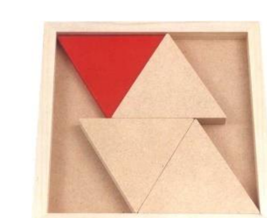
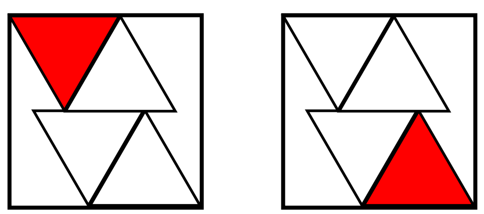


Figure 1: Triangular Jam

It would be quite challenging for readers to make

"Triangular Jam" themselves, so I will introduce a puzzle that is essentially almost the same. Even with this alternative puzzle, it's better to physically make and work with it, but I believe the enjoyment can still be conveyed reasonably well without making one.

The puzzle consists of a square frame and four identically sized equilateral triangular pieces. One of these four triangles should be distinguishable from the others, for example by painting it red. If the length of one side of the square frame (inner dimension) is set to 1, then the length of one side of the equilateral triangles is $1/\sqrt{3}$. The starting arrangement is as shown on the left in Figure 2, and the goal is on the right in the same figure (in the original puzzle, the goal is to remove the red piece, which is a thin board that can be extracted through a tunnel on the bottom edge of the frame).



Start Goal

Figure 2: A variant of Triangular Jam

As a puzzle, the objective is of course to move from the start to the goal, but during this process, the pieces must not be lifted at all. That is, the goal must be reached only by sliding the pieces, and in this respect, this puzzle is a "sliding puzzle." Additionally, of course, it's NOT a joke puzzle where one could simply rotate the frame or change their viewing angle without moving anything and claim "Look, I've solved it."

This puzzle is interesting because, from the solver's initial perspective, it's surprising that reaching the goal is possible given the seemingly too crowded arrangement of triangular pieces. The key move for solving it, which I won't disclose here to maintain the reader's enjoyment, can be quite inspiring for some. When discovered, it often brings a sense of satisfaction or delight to solvers.

In terms of dimensions, if the equilateral triangles are made even slightly larger (relative to the frame), moving the top-left triangle to the bottom-right corner would become impossible. Given that this limitation is almost self-evident, the specific design of the puzzle can be seen as a kind of optimization problem. To demonstrate that this length is optimal in the design, one simply needs to solve the puzzle.

2.2 The Origine of the Idea

How did I come up with such a puzzle, though?

At that time, I was aiming to create a "revolutionary" puzzle in the sense that it wouldn't fit neatly into traditional puzzle classifications. Triangular Jam, as mentioned earlier, falls into the category of sliding puzzles. Given this, it might seem not at all revolutionary. However, for puzzle enthusiasts, this is not the case.

Puzzle enthusiasts commonly use two classification methods for mechanical puzzles: the Slocum classification and the Dalgety-Hordern classification. In both, sliding puzzles are categorized as a sub-type of Sequential Movement Puzzles (SMP). SMPs are puzzles where each step in the solving process is clearly distinguishable from others, and the choices at each step are limited to a few simple options. This includes puzzles like the 15 Puzzle, the Tower of Hanoi, and the Rubik's Cube. Therefore, in principle, they are puzzles that can be solved with simple search algorithms (although there are cases where they become NP-problems, making them challenging for computers, or finding the shortest solution is difficult).

Then, I set my sights on creating a sliding puzzle that was not an SMP. This idea stands even apart from classifications only known within the puzzle community. In essence, since all traditional sliding puzzles could, in principle, be solved with search algorithms, my aim was to design something "revolutionary" by creating a puzzle that deviated from this norm.

The rest was simply a search for something that used as few types and as few pieces as possible, while still being mathematically simple and aesthetically pleasing, yet not too easy to solve, and possessing some element of surprise. In fact, using just one type of equilateral triangle was my first option, and four pieces seemed almost the bare minimum for a sliding puzzle, making the conception of this puzzle quite natural. Therefore, after I thought of it, it seemed so straightforward that it made me wonder why such a puzzle hadn't existed before, and it felt like it wouldn't have been surprising if it had been invented long ago whether its inventor is known or unknown.

In any case, this puzzle was a logical creation born out of the ambition to devise something revolutionary. Furthermore, the sentiment that "it would not be surprising if it had been invented long ago" is an important criterion (though not necessarily essential) for my satisfaction with a puzzle I have designed.

3. Rectangular Jam

3.1 Introduction of the Puzzle

Believing that it would be interesting to have a genre of sliding puzzles with features similar to Triangular Jam, I devised several puzzles of this type. Among these, the one I consider to be my greatest masterpiece is "Rectangular Jam" (see Figure 3. While there are various versions of this puzzle, the version shown is the original one I personally produced, marked with a branding iron. Depicted nearly in its goal state).

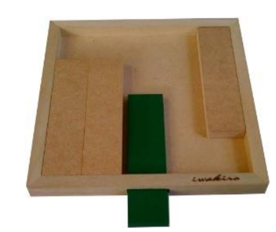
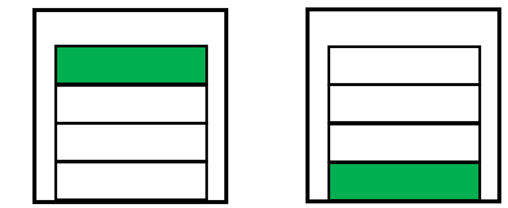


Figure 3: Rectangular Jam

This one would also be difficult to make oneself, so I will introduce an outline of a variant. I say "outline" because I will not provide the precise dimensions of the pieces, in order to preserve the reader's enjoyment (as will be mentioned later).

The puzzle consists of a single square frame and four identically sized rectangular pieces, each with an aspect ratio of 1:4. One of these four rectangles should be distinguishable from the others, for example by painting it green. The starting arrangement is as shown on the left in Figure 4, and the goal is on the right in the same figure. The remaining rules are exactly the same as those for the variant of Triangular Jam (the differences from the original puzzle are also similar to those in the case of Triangular Jam).



Start Goal

Figure 4: A variant of Rectangular Jam

3.2 The Origin of the Idea

The process of conceiving this puzzle was quite different from that of Triangular Jam. I say this because, initially, I did not expect that a puzzle with such a simple arrangement could be challenging. Hence, it was not conceived in a logical way, as was the case with Triangular Jam.

I came up with this puzzle when I was fiddling with rectangular pieces and noticed an interesting movement. This notice led me to think, "What if, for 1:4 rectangles, this movement is the optimal among all possible movements? Then, a puzzle in the same genre as Triangular Jam could be created with a remarkably beautiful arrangement?" Even with this idea, I experimented skeptically, thinking, "It can't be that straightforward." But to my surprise, it turned out that this movement was indeed the optimal one. The moment I realized this was truly a joyous one.

I believe this puzzle was born as a result of combining my constant pursuit of interesting movements that could potentially become a puzzle, with the idea of creating a puzzle in this genre always lingering in the back of my mind.

3.3 Design Method and Unsolved Problem

I am often asked about how I designed the dimensions of this puzzle. The question varies in meaning depending on who asks it, but it can be interpreted as asking how I solved the following mathematical problem:

Problem: Find the largest possible dimensions of the rectangle with a 1:4 aspect ratio used for four pieces, such that they can reach the goal from the start position as shown in Figure 4, by moving only through sliding.

This problem can be said to have been solved, of course, but since I have never written down a precise mathematical proof, it remains an "unsolved problem" in that respect. I would definitely encourage you to give it a try.

Solving this puzzle itself involves a key movement, and determining the largest dimensions for which this movement works using numerical calculations with a computer is not too difficult (for most readers of this article). Of course, to actually design it, one must first identify the key movement. Finding this movement without physically manipulating the actual puzzle can be quite challenging.

3.4 Solvable by Computer?

There is another problem that I think might interest the readers of this article. This issue is related to the purpose for which Triangular Jam and Rectangular Jam were designed, so I will introduce it here. The problem is as follows:

Problem: Although the number of possible states during the process of solving Rectangular Jam is infinite, can Rectangular Jam be solved by a computer using a search algorithm in a broad sense?

The intention of the creator of Triangular Jam and Rectangular Jam was to devise puzzles that could not be solved by search algorithms. Therefore, solving this problem is, in a sense, a true challenge to these puzzles. If you are interested, I would like to encourage you to take up the challenge. There is also some prior research [6].

4. Sliding Coin Puzzle: "Four Coins"

4.1 Common Rules

Let's take a brief detour from mechanical puzzles and introduce a single problem (not counting the example problem for explaining rules) from a category known as "sliding coin puzzles." These puzzles use only ordinary coins instead of specialized tools. The problem is quite simple, but before delving into it specifically, it's necessary to first understand the common rules of sliding coin puzzles.

The following example problem is a classic sliding coin puzzle.

Example Problem: Arrange six coins as shown on the left in Figure 5. Starting from this arrangement, move one coin per step and reach the configuration shown on the right in Figure 5 in three steps.

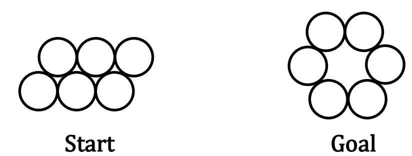


Figure 5: A sliding coin puzzle

These sliding coin puzzles follow these rules:

Rule 1: Each step involves moving only one coin, ensuring no other coins are moved, either directly or indirectly.

Rule 2: Lift no coins; only slide them.

Rule 3: Move each coin to a position where it becomes adjacent to at least two other coins, thus determining its placement.

Rule 4: The goal diagram represents the coins' relative positions; their exact location and orientation may vary.

4.2 Introduction of the Puzzle

The problem I devised, which I consider a masterpiece, is as follows:

Problem: Arrange four coins as shown on the left in Figure 6. White represents heads-up coins, and black represents tails-up coins. Starting from this arrangement, move one coin per step and reach the configuration shown on the right in Figure 6 in three steps.

Figure 6: "Four Coins"

Since it can be solved in just three steps, it shouldn't be too difficult. However, some people feel that it is unsolvable. Additionally, many who have solved it once often forget how they did it after a while. Solving it in four steps is not very easy either, so I suggest starting by trying to solve it in four steps.

4.3 The Origin of the Idea

Is it an exaggeration to say that it took decades to develop this puzzle?

The basic idea for this sliding coin puzzle has been on my mind since my teens. I didn't come up with it myself; I heard it from someone, though I don't remember who. Although I always thought the idea was interesting, directly turning it into a puzzle didn't produce an elegant question and was not satisfying. After considering it for many years, I suddenly realized that using the heads and tails of coins was the ideal way to implement this idea as a puzzle, leading to the creation of the problem mentioned above.

Therefore, I believe the birth of this puzzle resulted from long contemplating an idea I always found interesting. I still have many other ideas that I've been considering for a long time, but it's challenging to turn them into actual puzzles. Later, I will introduce another example where a long-contemplated idea was successfully realized.

4.4 A Hint

Providing hints might seem superfluous, yet I understand that for some who couldn't solve the puzzle, it might seem like a joke. To address this concern, I will offer a hint below, though I still won't reveal the solution.

I wrote earlier, "I suggest starting by trying to solve it in four steps." However, if you can solve in four steps and your solution is relatively straightforward, trying to reduce it to one fewer step might not help you solve the puzzle. Of course, since it's only three steps, exploring all possibilities should quickly reveal the solution (considering symmetries, the number of cases isn't so large). But, if you don't pay enough attention to Rule 4, you might almost reach the goal but fail to realize it. That is, you must be fully aware that the overall goal configuration can be rotated compared to what is shown on the right in Figure 6.

5. Probability Puzzle: "Coin Toss Challenge"

5.1 Introduction of the Puzzle

The area of mathematical puzzles in which I specialize the most is probability puzzles. I have introduced many problems in various articles and books. I have even authored a book [7] solely dedicated to probability puzzles. Here, I will introduce just one problem.

This problem, which I created relatively recently, is one I'm particularly fond of. I introduce it partly because I thought it was perfectly suited for the readers of *Operations Research*. I recommend approaching it not as a puzzle at first, but as if you've encountered essentially the same problem during some research process.

Problem: Two players each have five fair coins. Since there are ten sides in total for the five coins, each player writes numbers from 1 to 10 (using each number only once) on them as they like. Once they're ready, both players toss their five coins, and the player with the larger product wins (if the products are the same, they keep tossing until a winner is determined). How should the numbers be written on the coins to maximize the chances of winning? If both use the best strategy, the game naturally results in a fifty-fifty chance of winning, but is there a way to ensure a winning probability of at least fifty percent regardless of the opponent's strategy? If such a way exists, what is it? If there are multiple equivalent ways, list them all.

5.2 Enhancing Problem Experience

This problem may seem like a simple exercise in finding the optimal strategy. Indeed, it is likely a good practice problem. Since each player has five coins, the number of cases to consider is considerable (though too many for a human to handle easily), but it's not too daunting with the use of a computer. Or, since the goal is just to find the optimal solution, you might not want to start with a full search immediately. Instead, it might be wise to think of a few seemingly good strategies and do some simulations first. In any case, I feel this problem is particularly well-suited as a practice problem for students studying in fields related to operations research.

Additionally, while it may not guarantee the optimal strategy, as a first step to get a rough idea, you might try finding a strategy that maximizes the expected value of the product. This in itself is an interesting mathematical exercise and, in fact, leads to a solution with a beautifully symmetrical result.

I believe that this problem can be enjoyed as various kinds of challenges, as described above. Also, no matter how one approaches it, upon finally arriving at the answer, I believe it will be quite surprising for many people. On the other hand, and perhaps "of course," since this is a puzzle, there is also a quick solution without the use of a computer. Unlike the other puzzles introduced in this article, the solution to this puzzle will be provided within this article. Therefore, I recommend thoroughly enjoying it before looking at the answer.

5.3 The Origin of the Idea

I came up with this problem after hearing the following problem from someone:

Problem I Heard: Prepare three standard six-sided dice (without any bias) and one twenty-sided die (also unbiased), numbered from 1 to 20. When rolling these dice, compare the sum of the numbers on the three six-sided dice with the number on the twenty-sided die. The higher number wins. Which has the advantage in this game?

For this problem (I will provide the answer immediately in this paragraph, so those who want to solve it themselves should be cautious), I later came up with an elementary and elegant solution myself. However, when I first heard it, I quickly realized the answer using a rather abstract and general principle. The answer is that “neither has an advantage; it is fair.” The principle I used at that time was that “if there are two symmetrical probability distributions with the same mean, then for any two independent random variables X and Y following their respective distributions, $P(X < Y)$ equals $P(X > Y)$.

While I quickly realized the answer, I thought it better to avoid using advanced principles when explaining it to others. So, I pondered over the mathematical aspects of this problem and discovered several intriguing points. I endeavored to encapsulate this interest in the form of a puzzle, aiming to make it engaging in various ways, as described in 5.2. The main innovation in posing this puzzle was to maintain its simplicity while ensuring the answer was not easily guessable, and the inherent symmetry was not overtly obvious.

When I encountered an interesting idea and tried to turn it into a puzzle, it worked out quite well. In that respect, I think the creation of this puzzle was similar to the development of Rectangular Jam.

5.4 Answer and Solution

I believe the charm of this problem won’t be fully conveyed unless the solution is also presented, so I’ll describe it below.

Before the solution, let me first reveal the answer, which might be surprising if you haven’t yet arrived at it yourself: “you can write the numbers any way you like, as long as the rules are followed.” In other words, “no matter how the numbers are written, there is no advantage or disadvantage.” The phrase “you can write any way you like” doesn’t mean that you need a random selection like choosing a hand in rock-paper-scissors. In fact, even in a situation where you are allowed to write your numbers after seeing your opponent’s, it’s still impossible to write them in a way that creates any advantage or disadvantage.

There is a straightforward solution to understand why this is the answer. No matter how the numbers are written, as long as the rules are followed, the product of all numbers on both sides of the coins is the same. Therefore, in any pair of coin tosses, if one player wins in a game viewed from the top, they are losing when viewed from the bottom. Since the probability of winning when viewed from the top is naturally the same as from the bottom (due to symmetry), no matter how the numbers are written, the probability of winning equals the probability of losing, which is 50% (note that if the products are the same, the game continues until a winner is determined).

6. Card in the Bag

6.1 Introduction of the Puzzle

As the last puzzle to be introduced in this article, I present another mechanical puzzle named "Card in the Bag." Though it is a mechanical puzzle (meaning it requires specialized tools), this puzzle, as explained below, can be easily made by anyone. I highly encourage you to make it yourself and then give it a try.

Prepare a credit card or a card of the same size, and a plastic bag with internal dimensions of 10 cm in width and 5 cm in length. That's all you need for the setup (see Figure 7). When you try to insert the card directly into the bag, it will protrude slightly. The goal of the puzzle is to completely enclose the card inside the bag without deforming the card in any way (including during the process of solving the puzzle). Of course, cutting or tearing the bag is also not allowed.

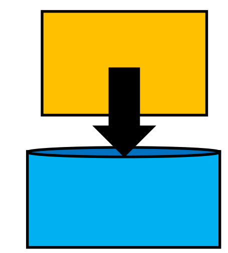


Figure 7: Card in the Bag

The fascination of this puzzle truly comes to light only when one actually tries it. Some people may think they have “solved it” just by thinking about it, but often they haven't figured out how to implement their solution in reality. If you think you've solved it, it's best to either try it with an actual card and bag or, at the very least, carefully verify the exact steps of your proposed solution.

6.2 Tips for Properly Making It Yourself

Since I strongly encourage you to create and try this puzzle yourself, I will provide some tips and cautions for properly making it.

Various materials are used for poly bags, so avoid those that tear easily or irreversibly stretch with little force. If you have a poly bag exactly 10 cm wide, cutting it to a length of 5 cm will suffice. However, be cautious: even if a product is marked as 10 cm, the internal width might be shorter, so ensure that the internal width is indeed 10 cm. If it's slightly larger, up to about 1 mm more is acceptable.

The length is mentioned as 5 cm, but this is primarily for the simplicity of a round number and ease of remembering. Actually, it can be about 2 mm shorter than this. Therefore, there's no need to be overly precise with the length; as long as it's close to 5 cm and not longer, it should be fine.

6.3 The Origin of the Idea

For this puzzle as well, I will not provide the solution in this article. However, I'd like to caution readers regarding the timing of reading the following section, as it significantly hints at the solution within the story of its origin.

I have always found a certain special property of some tetrahedra fascinating since my teens. It was only relatively recently that I tried to articulate this property in words. While a technical term may already exist in some field unfamiliar with me, I have informally called it "Untaperedness (寸胴性)."

Untaperedness is defined as follows:

Definition: A solid is said to be untapered if there exists a plane such that, when the solid is sliced by any plane parallel to this plane, the perimeter of the cross-section is always the same.

Defined this way, it's obvious that cuboids, or more broadly parallelepipeds and cylinders, are untapered. But interestingly, regular tetrahedra and octahedra are also untapered. Moreover, there exists a rich class of untapered tetrahedra.

Since I started designing mechanical puzzles around 20 years ago, I have been consciously focusing on the intriguing aspect of untaperedness, even though I wasn't using that specific term at the time. In fact, a few years after I began, I successfully incorporated this concept into several puzzle designs. Among these, the one that stood out to me was based on the untaperedness of a degenerated tetrahedron.

However, I wasn't entirely satisfied with its realization in that form and continued to embrace the basic idea of untaperedness. About a decade after I started designing mechanical puzzles, I suddenly came up with the idea for this puzzle, though I've completely forgotten the direct trigger. To be precise, the idea of using a credit card-sized card to make it easily replicable by anyone occurred to me a bit later. Initially, what I had realized was a mathematical discovery: any rectangular plate could be used to create essentially the same puzzle, provided a bag of suitable dimensions is prepared.

I don’t introduce the solution to Card in the Bag in this article, but the essence of its solution is included in Japanese patent P5131793. Those who are interested might want to consult it.

7. Conclusion

When I came up with Card in the Bag (or, more accurately, an essentially the same puzzle), I felt even more strongly than with my other puzzle creations why such a puzzle hasn't existed before and that it wouldn't have been surprising if it had been invented long ago. The level of satisfaction of this feeling was truly gratifying.

On the other hand, when I thought calmly, I felt it's understandble why this puzzle hadn't been thought of before. The puzzle, once created, seems to be just a combination of very familiar materials, and it looks unrelated to any specialized mathematical concepts like untaperedness. However, for me to conceive this puzzle, I had to go through the concept of untaperedness (even though I didn't use that specific term). Moreover, contemplating that idea for an extended period was likely essential. When I eventually found a way to realize it without any atmosphere of complicated concepts, the result was, as I describe it with modesty aside, a masterpiece that appears as if it could have existed before long, yet had remained undiscovered.

All the puzzles I've discussed in this article seem to share something in common with what I've just described. Each began with an idea that I found very interesting from my own mathematical aesthetic perspective. Then, whether it took a long time or not, when I was fortunate enough to realize that this idea could be implemented through something very simple and commonplace, then, a "masterpiece" of a puzzle was born.

I would like to mention two final points. First, as I stated at the outset, I hope my descriptions have conveyed some of the delights of devising new creations. It would be extremely gratifying if this could serve as a reference or inspiration in any way.

Second, I have a request. If you have any ideas that could be interesting if transformed into a puzzle, but you have no idea how to make it happen, please do share them with me. I would value such concepts, contemplating them and hoping to one day turn them into fascinating puzzles.

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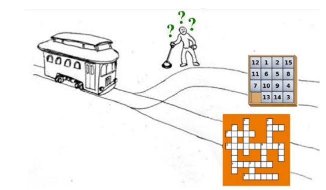


The Trolley Puzzles

Introduction and Puzzle 1

You have walked this way before, hearing the din of machinery lower down. But this time, something urges you to take a closer look. You disregard the ‘Danger: Do Not Enter’ sign and slip through to where you might get a view of what exactly is going on below. When you find a suitable place for looking down on the railyard, you are surprised to see an old woman standing there, surveying the scene. She smiles and winks at you conspiratorially. Below, you see a jumble of tracks, with train cars, trolleys, and shipping containers scattered about.

“So, you had to come see it, too,” she says. You nod. “I know the feeling. I’ve been coming here for many, many years. I don’t know what keeps me coming back. You know, a very long time ago, this place inspired me to develop some philosophical ideas you may have heard of. Are you aware of the ‘trolley problem’?”

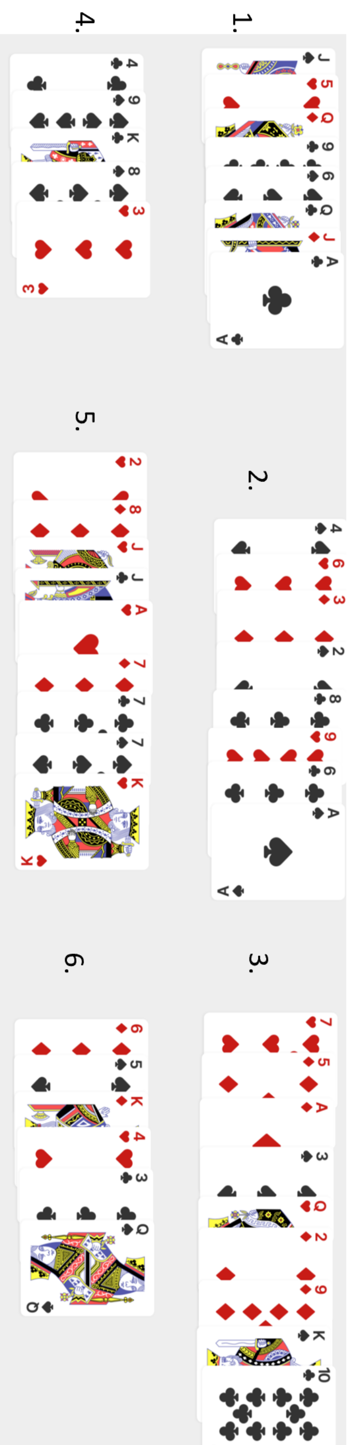
“Oh, yes,” you say. “Isn’t that the one where, to save the lives of five people on one track, you have to divert a trolley to a different track where it will instead kill one person?”

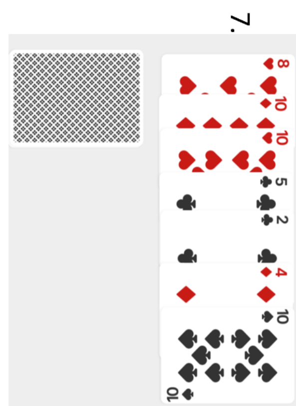
“That’s the basic idea,” she says, “though the analysis of it is what matters. And there are quite a number of variations on the theme. But I don’t know how much you want to hear me natter on about philosophy: you just got here, and you probably want some time to take a look around at all this. Would you like to borrow my binoculars?”

You accept her kind offer, and find the binoculars amazingly strong: you can even make out the playing cards on a table in a game that seems to have been abandoned by some of the railyard workers. “But I can’t figure out what game they might have been playing: it seems very strange.”

“The game isn’t what matters,” the old woman says. “What’s more interesting is what it tells you. If you look closely and think about it, you’ll find that there’s a hidden message there: a secret word. Can you figure out what it is?”

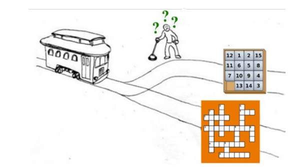
Can **you** figure out the secret word? See for yourself!





1. Where the British can enjoy First Dates
2. Interceded to help resolve conflict
3. ______ Valley Cultural Society (1970s act whose members are now hard to fool)
4. Trunk of the body
5. Locals who spend their winters down in Florida, for instance, or Canada's air demonstration squadron
6. Reddish-brown horse, or Oxalic plant
7. Cattle thief

Answer: ______



The Trolley Puzzles

Puzzle 2

You solve the old woman's puzzle and tell her that you'd be interested to hear a little bit about philosophy. "But I have to warn you, I don't know that much about it," you add. "Just a little bit about that 'I think, therefore I am' guy – what was his name?" She tells you that it's 'Rene Descartes', and you remember seeing it written down somewhere, but the way she says it isn't at all the way you thought it would sound. In your notebook, you make a note of the pronunciation she uses: **wren – aid – ache- art.** Feeling whimsical, you replace these four words with their definitions:

Small brown songbird / assist / dull pain / skill or craft.

Your new friend tells you about some of the philosophers she knows at Rutgers. Eventually, she looks at your notebook and sees your unusual way of keeping track of their names. She smiles mischievously and adds some mysterious numbers to the end of each of your paraphrases, so that the page in your notebook ends up looking like this:

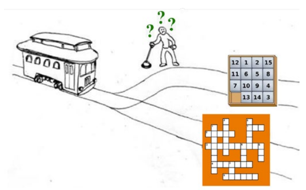
Decorative vase _ _ _ / *Moray, for example* _ _ _ / *Backtalk* _ _ _ / *Rowing need* _ _ _ **(5)**
Mind _ _ _ _ / *Little devil* _ _ _ / *Corral* _ _ _ / *Receive as profit* _ _ _ _ **(2)**
Tolerate _ _ _ _ / *Fishing rod attachment* _ _ _ _ / *One in the red* _ _ _ _ **(4)**
Belonging to me _ _ / *Select from a group* _ _ _ _ / *Quaker rolls them* _ _ _ _ / *Arab market* _ _ _ _ / *"I see!"* _ _ **(1)**
Wise bird _ _ _ / *Defeat* _ _ _ _ / *Frighten* _ _ _ _ _ / *Pointer* _ _ _ _ _ **(7)**
Challenge _ _ _ _ / *Morty's grandfather* _ _ _ _ / *2022 Blanchett film* _ _ _ / *Social insect* _ _ _ **(1)**

"And those numbers mean… what exactly?", you ask.

"Well, you know their last names…" she begins, trailing off.

"Don't tell me there's another secret word here for me to find!"

There is. And the secret word is ________________________________



The Trolley Puzzles

Puzzle 3

"That was tricky," you say. You see that the old woman is now staring toward an array of shipping containers, four containers high and five containers wide.

"Doesn't that just show how *near* everyone is to everyone else, in a sense, today? Look at that: items from Wales, Oman, Thailand, Ecuador, and so many other places right next to each other! I remember when things were not so open, long ago. It makes me so happy just to see this!"

"But how can you tell which container comes from where?", you ask. "I guess I don't know the signs as well as you do. You've clearly been coming here often."

She nods. "Yes, I suppose so… well, I can give you a few hints, if you're interested."

"1. The first letters of the shipping containers' nations spell out the final answer to Puzzle 2, if you begin at the upper left corner and move only along a path between containers that are vertically, horizontally, and diagonally adjacent, never touching the same container twice.

"2. The opposite edges of the array (right and left) both have crates from Thailand, and both have crates from Rwanda. The Rwandan and Thai crates are in the same relative positions on both edges.

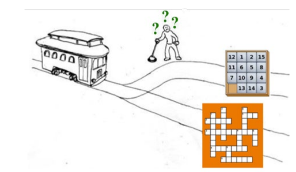
"3. The order of three consecutive crates in one row, from left to right, is China – Lebanon – Uganda.

"4. One pair of distant corners (that is, corners that don't share a row or column) is occupied by containers from Haiti and Norway. The two Norwegian containers are a knight's move apart from each other, as are the two Haitian crates, and as are one Haitian and one Norwegian crate. The only other pair of containers from one country that are a knight's move apart are the two from Oman.

"5. Either the Australian container or the Chinese one is somewhere above the one from France, not necessarily in the same column.

"6. A diagonal somewhere in the open array of containers, four containers long, extends from one Ecuadorian container to another, with a Welsh container and a Norwegian container in between. And if one of the Danish containers were one container lower, the two Danish crates would be as far apart as the two Ecuadorian ones are."

Answer: ____________________



The Trolley Puzzles

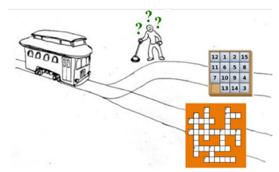
Puzzle 4

In another part of the railyard, you see some posts sticking out of the ground in a grid-like pattern. In the square spaces between some of the posts, you see some road cones with numbers on top. You draw a diagram of it in your notebook and ask the old woman about it.

"You and I know that we shouldn't be here,", she says with a grin, "but we're safer up here than down there! Those posts form the basis for a fenced-off area within the railyard, to keep people out of the *really* dangerous areas. Management has already mapped out where the fence is meant to go. Soon, some workers will come out and put chain-link fences between some of those posts. They have maps of where to put the fence, naturally; but the road cones with numbers are there to help them double-check. You see, the numbers in the squares indicate the number of sides of the square that should have a fence. The workers all know that the fence never touches or crosses itself, and that it forms a complete loop. So the chance of an error is low. Here…" (she draws some letters in your diagram with a laugh) "… Can you figure out where the fence goes? The letters in squares whose edges touch exactly two links of fence at a 90-degree angle will tell you something!"

2	2	E	2	O	3	2	1	N	2	2
2	1	3	2	2	R	0	U	3	T	2
2	0	S	1	I	2	A	3	I	2	L
2	H	1	2	O	3	L	1	3	I	3
T	2	D	D	2	2	1	1	H	1	G
O	2	A	2	2	3	3	E	S	2	E
3	S	2	2	T	1	E	2	F	I	2
O	1	D	3	L	1	T	3	T	2	3
2	L	0	T	2	S	1	2	1	2	1
2	2	S	2	2	0	I	2	N	3	1

Answer: _____________



The Trolley Puzzles

The Meta-Puzzle, Part I

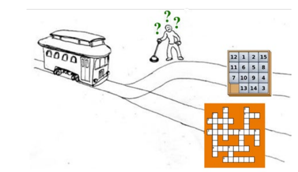
You have thoroughly enjoyed making friends with this unique and interesting person, but your brain feels as though it has had its fill of puzzling for the day. She shakes hands and shows you a shortcut home across the railyard. You climb down the structure the two of you were both standing on, and walk toward the secret exit she has shown you. As you walk, you keep watching a sort of switching area in which the train cars and trolleys become hitched and unhitched.

You find that you can't stop thinking of the world around you as a puzzle. You now imagine all the four secret words you learned by solving the 'trolley puzzles', lined up like trains from shortest to longest, each letter being a different train car. The first one goes straight through the switching yard without any alteration. The second loses its first letter (the train's engine, which is reserved for some other train, you presume). The third answer either has its first letter switch with an A, or its second letter switch with a B, or its third letter switch with a C, and so on. And the fourth answer is replaced by a different word with exactly the opposite meaning. Each of these transformations leaves you with a common English word.

Suddenly, you realize something interesting: all four of these words (the three new ones, plus the one that never switched) can be placed before or after another common English word – the very same word for each of the four – to make a common phrase! Two of the words have the new word going in front, and two have it going in back.

Can you think of the common word that has this property?

Answer: ________________________



The Trolley Puzzles

The Meta-Puzzle, Part II

Suddenly, you notice that two parents and their three children are cutting across the railyard unsafely. You realize that you're hardly one to talk – you did, after all, ignore the warning sign up above – but these people seem altogether more reckless, especially given that they are leading their children into danger. In fact, they are going right through the most dangerous area, with the road cones and numbers, and are walking the narrow line between two long lengths of fence that have already been put up. As you look behind you, following the track that the family is walking along, you see, to your horror, that a heavy trolley is rushing toward them. There is no time for them to escape the danger. But what can you do? Immediately, you see beside you an emergency switch for stopping anything coming down the track.

In an instant, you realize how the switch works: it swings a very thick and heavy bar onto the railyard, big and sturdy enough to mechanically stop even a trolley like this one from going further when it smashes to the ground. But that bar is… just the place where you and the old woman were just standing! You see her still there, looking through her binoculars.

What should you do?

To know the answer, you will need to know the difference between two things. The words for these two things both appear in common phrases that also use the secret word you discovered in Part I of this Meta-Puzzle. The first phrase means 'following a course likely to succeed'. The second phrase means 'a poor neighborhood', and uses the secret word in the plural.

What must you know? ____

Father & Son Sudoku

Multi-generational explorations in a 9x9 grid

Graham & Gabriel Kanarek, G4G15, Feb 21-25 2024

Like many other folks in the G4G community, over the past few years, we Kanareks have become involved in the variant sudoku creation scene, thanks to the popularity of Cracking the Cryptic. The two of us collaborate frequently, and in that vein, we've set three puzzles for this conference. All three of these puzzles can be solved online using the SudokuPad online app (with answer checking), or of course with traditional paper & pencil. We hope you enjoy!

Puzzle A

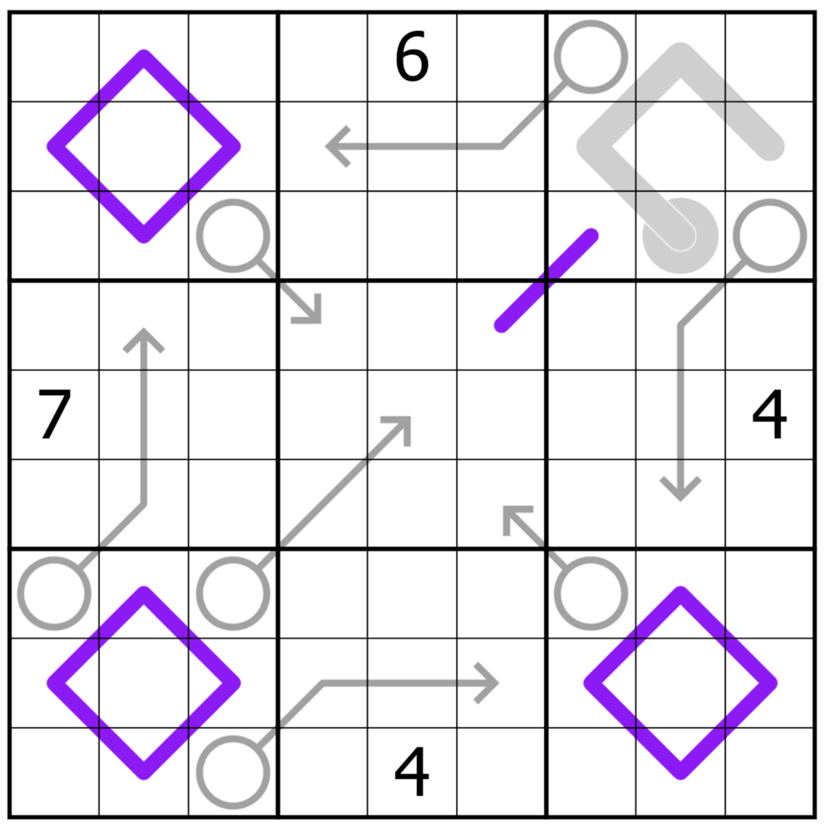
Normal sudoku rules apply (i.e., fill the grid with the digits 1-9 such that each row, column, and 3x3 box contains each digit exactly once).

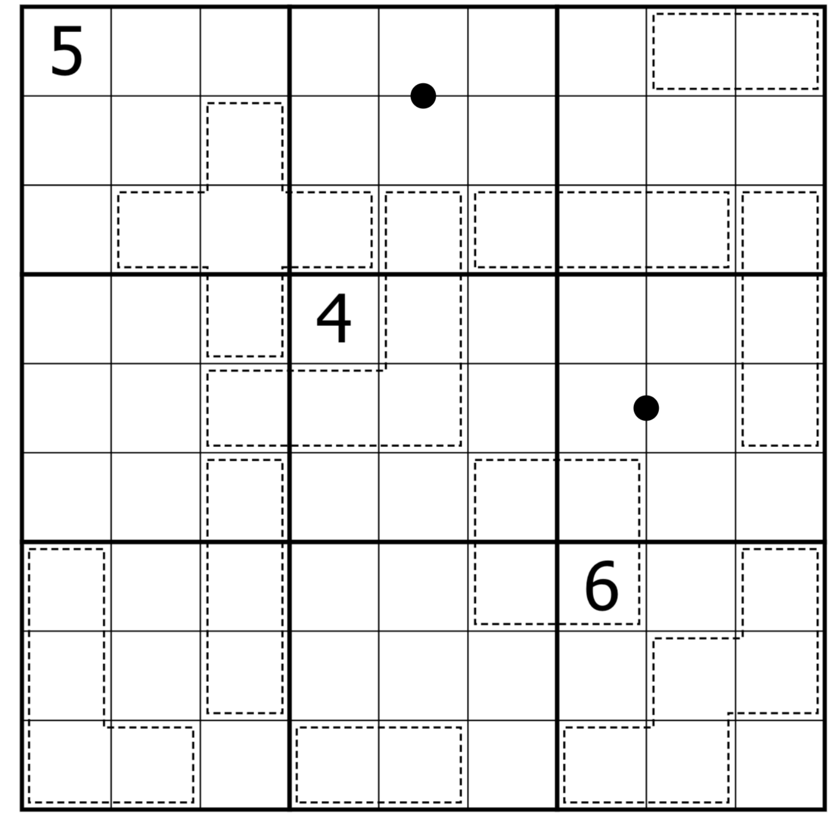
Digits along a purple line must form a set of consecutive digits without repeats, in any order.

Digits along an arrow must sum to the digits in the associated circle.

Digits along a thermometer must increase from bulb to tip.

Solve online!
https://sudokupad.app/wyljb3k736





Puzzle B

Normal sudoku rules apply (i.e., fill the grid with the digits 1-9 such that each row, column, and 3x3 box contains each digit exactly once).

Digits in cages must sum to 15, and may not repeat.

Digits separated by a black dot are in a 2:1 ratio.

Solve online!
https://sudokupad.app/1zig1g1s95

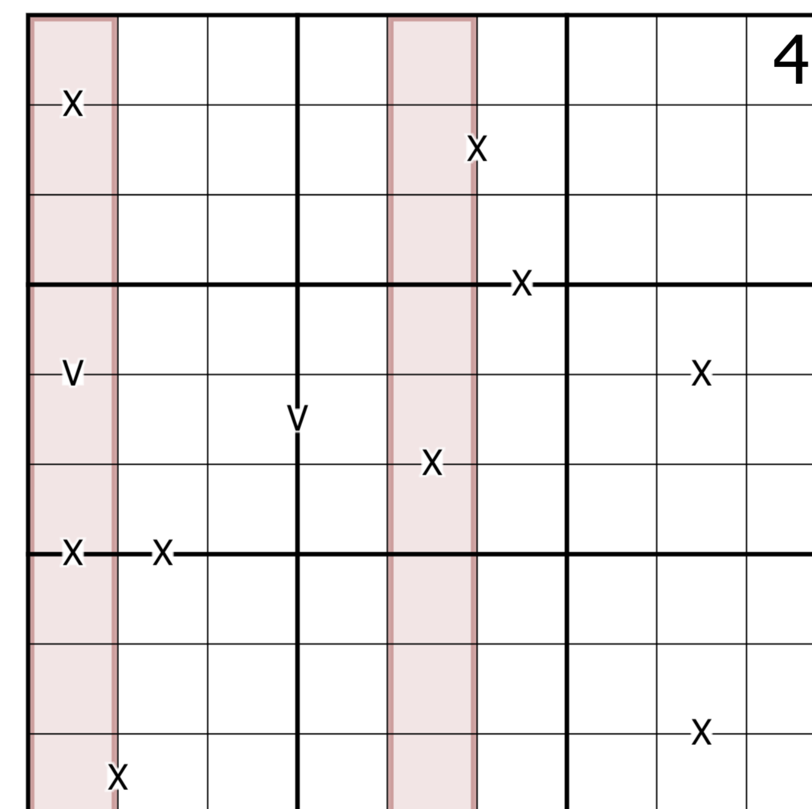
Puzzle C

Normal sudoku rules apply (i.e., fill the grid with the digits 1-9 such that each row, column, and 3x3 box contains each digit exactly once).

Digits separated by an X sum to 10, and digits separated by a V sum to 5.

A digit in column 1 indexes the position of 1 in that row, and a digit in column 5 indexes the position of 5 in that row. (For example, if row 2 column 5 is 6, row 2 column 6 is 5.)

Solve online!
https://sudokupad.app/03b92n18cd



Elevating Tessellations into Three Dimensions: Approximating Geodesic Domes

Sara Kapasi

Georgia Institute of Technology

December 2, 2021

G4G15 Gift Exchange Paper

skapasi3@gatech.edu

I. INTRODUCTION

The intrinsic and extrinsic beauty of shapes and their vast applications present themselves everywhere in our everyday lives. By combining their 3D equivalents, polyhedra, with their interlocking patterns appearing in tessellations, combining the two with both visual artistry and architectural applications in mind extends that underlying symmetry further. Integrating polyhedra and tessellations optimizes material efficiency in construction, and material efficiency in tile creation and building design in architecture is increasingly vital to sustaining a changing planet. When constructing a spherical building, many times only flat sheets of materials are available. Pouring molten metal over a cast of a building prototype is not feasible, so using small flat tiles to create the structure of a building is the closest approximation to a sphere. A sphere is ideal with its maximum volume per unit surface area, which saves on material costs. Integrating tilings provides both visual and practical appeal since making minimal cuts to create the tiles saves on cutting and gluing costs to assemble the 3D structure.

Tessellations have been present since antiquity in many designs, from the floors of the Alhambra in Spain to more recently in the works of MC Escher, the famous graphic artist and mathematician (Kaplan and Salesin 2000). They appear as the bridge between architecture and mathematics with their visual appeal and versatility (Deger and Deger 2012). A general tessellation is defined as the two-dimensional plane consisting of repeating shapes with no gaps or overlaps. Many tessellations that mathematicians work with are regular tessellations, tessellations composed solely of regular congruent polygons (Deger and Deger 2012). Semi-regular tessellations, the other class of tessellations can be made up of more than one type of shape. When constructing a tessellation, four types of transformations are used to create symmetry in the tessellation pattern. These transformations are rotations, translations, reflections, and glide reflections. One specific program used to create tessellations with the knowledge of symmetry is Microsoft Paint (Deger and Deger 2012), as well as more recent programs like Conway's Magical Pen (Bakker et al. n.d.).

When looking at polygons connected in three dimensions, polyhedra are the main focus of analysis. Polyhedra are composed of polygonal faces and their edges and vertices. These faces connect without overlap on their edges, with three faces meeting at a vertex. Convex polyhedra are generally classified into two types: Platonic solids and Archimedean solids. Archimedean solids are analogous to semi-regular tessellations, in that Archimedean solids are made up of more than one type of polygon, whereas Platonic solids are made up of one type of regular polygon (Akiyama et al. 2010).

Some of the vast appearances of polyhedra in nature include molecular structures, viruses, and various physics and architectural concepts (Koca, Al-Ajmi, and Koç 2007). However, many other applications of polyhedra extend elsewhere. In drug delivery, metal-organic polygons (MOPs) act as metallo-therapeutics which can selectively deliver drugs and dyes to tissues. Metal complexes self-assemble out of copper, palladium, platinum into structures that contribute to an inhibitory effect in tumors. The complexes sometimes assemble into tetragonal prisms, trigonal prisms, and cuboctahedron complexes, which have numerous anticancer and antitumor effects, with applications in chemotherapy and photodynamic therapy (Samanta and Isaacs 2020). Creating new polyhedra derivatives could potentially improve drug delivery efficiency through a complex able to hold more volume.

In architecture, combining Archimedean solids to create a building can reduce structural material through a unit cell. This unit cell, self-supporting and able to distribute weight equally, is found in existing structures like the Hearst Tower in New York City and The Gherkin in London (Obradović et al. 2013). Improving this unit cell by making it more spherical can support more weight with less material. Polyhedra also have applications in sustainable water solutions as electrodes in the process of capacitive deionization. Due to the ordered and open structure of carbon nano-polyhedra, they have high permeability, which contributes to good electrochemical performance. Transmission electron microscope (TEM) and field emission scanning electron microscope (FESEM) images found the structure of the carbon nano-polyhedra to be rhombic dodecahedrons, which held their shape after multiple rounds of deionization (Xu et al. 2019). While rhombic dodecahedrons exhibit structural strength, polyhedra derivatives closer to a sphere exhibit the highest structural strength due to their likeness to a sphere, which could improve the efficiency of capacitive deionization. The durability and versatility of polyhedra in nature cannot be understated.

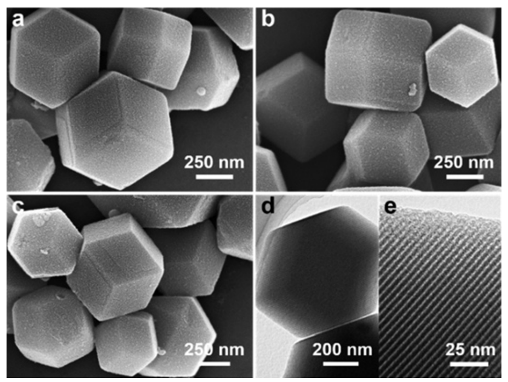


Figure 1: Carbon nano-polyhedra (Xu et al. 2019)

Figure 2: Hearst Tower, NYC ("Hearst Tower" n.d.)

Geodesic domes further complexify polyhedra, but with vast applications in architecture and material storage. The Epcot Ball at Walt Disney World, parts of the Dali Museum in Figueres, and greenhouses, are all examples of geodesic domes present in public spaces. Popularized by Buckminster Fuller, geodesic domes were based on polyhedra and had icosahedral symmetry. Subdividing their surface produced a lightweight material framework, producing triangles most commonly, but also hexagons, pentagons, and rhombi. They were applied to material systems due to their high weight-strength ratio, and Fuller was awarded multiple patents in the design and application of geodesic domes (López-Pérez 2020).

In its original context, sphericity, or "degree of true sphericity," described how close rock particles approximated a sphere (Wadell 1933) in geology. However, Wadell also hinted at sphericity's use in classifying geometric solids, signifying its cross-disciplinary usage. Wadell gives the formula for sphericity as s/S, where s is the nominal surface area of a solid (the surface area of a sphere with the same volume as the solid) and S is the actual surface area of the solid. Sphericity is an intrinsic property of a solid, or rather a dimensionless property independent of the scale of the solid due to it being a ratio with a maximum value of one. Multiple studies have used sphericity to investigate the properties of polyhedra, in the context of polyhedral complexity (Balaban and Bonchev 2005) as well as in polymer structures (Lee, Leighton, and Bates 2014). However, no studies found by the author have integrated sphericity in architectural applications.

By combining tessellations, polyhedra, and sphericity, optimizing material efficiency translates into optimizing sphericity values of polyhedra with nets of tessellations. Especially without directly renewable building materials, designing the next generation of buildings must take into consideration material efficiency, as well as preventing unnecessary cutting or welding costs. On large scales, material inefficiency is especially hurtful towards company budgets, so designing optimal nets can redirect funds towards other crucial developments. Energy efficiency for the next generation of buildings is also needed concerning global warming, to prevent excess heat from being lost to building surroundings.

In this project, how to best approximate the net of a polyhedron with maximum sphericity using 2D tilings was investigated.

II. MATERIALS AND METHODS

The physical part of the project was split into two parts: creating the 2D tessellations and creating the 3D models. Seven tessellations were made, four of which were regular and three of which were semi-regular. To create the tessellations, Adobe Illustrator was used. Tessellations were derived by drawing in and then deleting lines of symmetry, then forming shapes around the symmetry lines with the Adobe Illustrator Polygon Tool. The command Path Length was used to find the specific centimeter side lengths, which were input into Mathematica, the program used for the 3D models. Physical copies of the tessellations were printed on cardstock and cut out using an X-Acto Knife.

For the 3D models, the objective was to elevate as many 2D tessellations as possible with the help of Mathematica. In Mathematica's integrated command library, the volume and surface area of polyhedra corresponding to the 3D models could be easily found to calculate the sphericity later. For more common polyhedra, the command PolyhedronData[], with specific commands for "SurfaceArea," "Volume," and "Net" produced the respective values for the commands listed. To ensure consistent sphericity values, the side lengths of the physical 3D models were measured and used to calculate the respective sphericity values. Adjacent to quality control, calculating the sphericity of the physical model compared to the computer model served to make sure the physical representations were accurate in assembly.

III. RESULTS AND DISCUSSION

2D Tessellations

When creating the 2D tessellations, the first class of tessellations created were the regular tessellations. Four tessellations were created, three of them qualifying as tilings since they were made up of solely regular polygons.

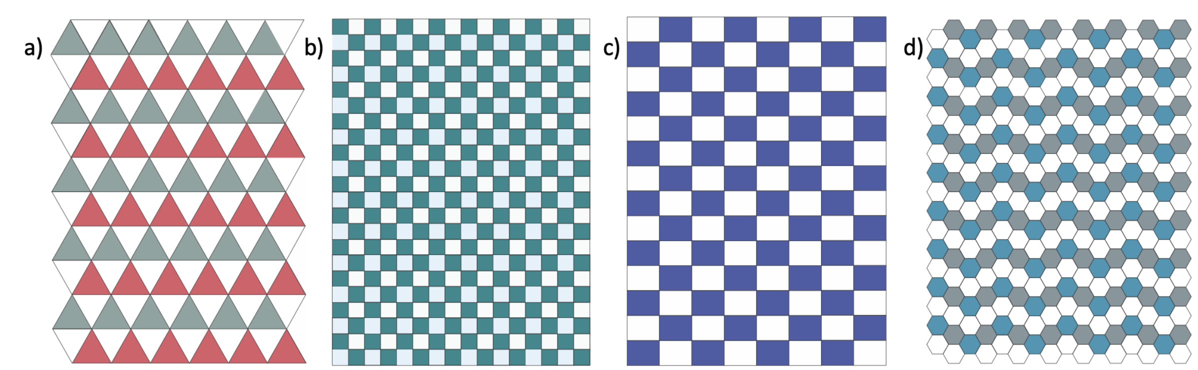


Figure 3: Regular tessellations a) Triangular tiling b) Square tiling c) Rectangular tessellation d) Hexagonal tiling

The three tilings were the triangular tiling (Figure 3a), the square tiling (Figure 3b), and the hexagonal tiling (Figure 2d), and the rectangular tessellation (Figure 3c) was included for the sake of incorporating non-regular polygons.

The second class of tessellations created were the semi-regular tessellations. Each used two colors for visibility purposes. None included solely regular polygons.

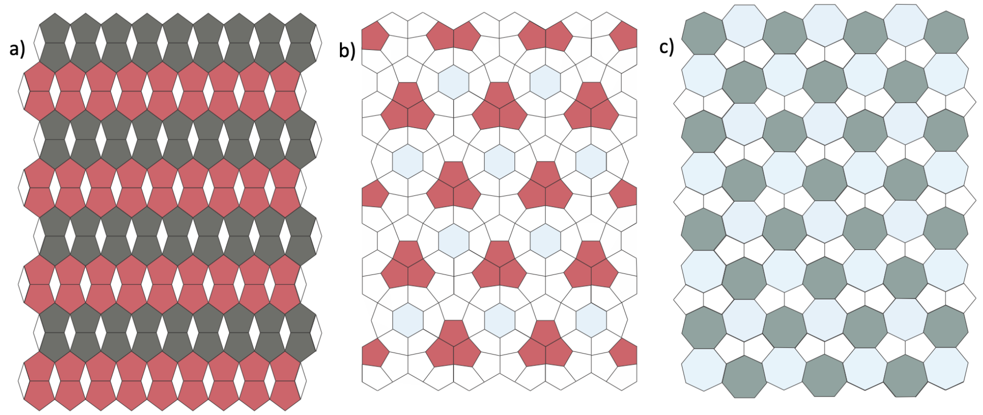


Figure 4: Semi-regular tessellations a) Pentagon-rhombus tessellation b) Pentagon-hexagon tessellation c) Pentagon-heptagon tessellation

The three semi-regular tessellations were the pentagon-rhomb tessellation (Figure 4a), the pentagon-hexagon tessellation (Figure 4b), and the pentagon-heptagon tessellation (Figure 4c).

3D Models and Corresponding Nets

When creating the 3D models, net simplicity took priority, which led to the prioritization of regular tessellations. However, due to the large interior angle of regular hexagons, the hexagonal tiling was excluded as a contender for a potential 3D model, since overlap would be inevitable when folding into 3D space. The remaining tessellations serving as nets included the triangular tiling, the square tiling, and the rectangular tessellation.

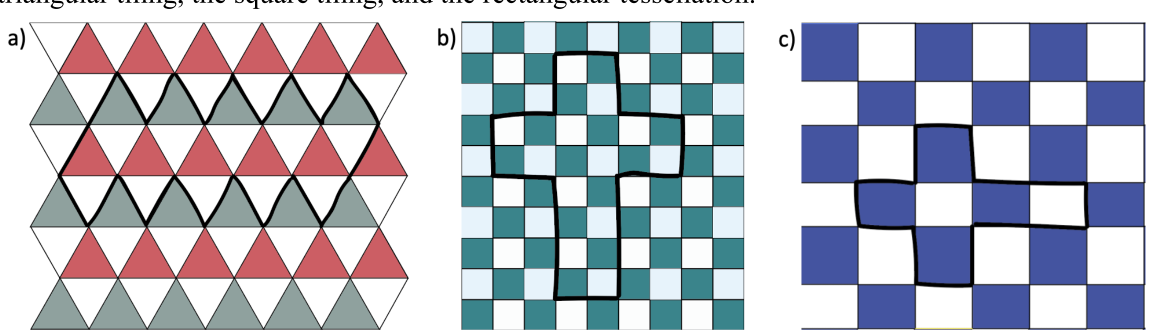


Figure 5: Potential nets directly from tessellations a) From triangular tiling b) From square tiling c) From rectangular tessellation

Icosahedron nets could be created directly from the triangular tiling (Figure 5a), and prism nets could be created directly from the square tiling (Figure 5b) and the rectangular tessellation (Figure 5c).

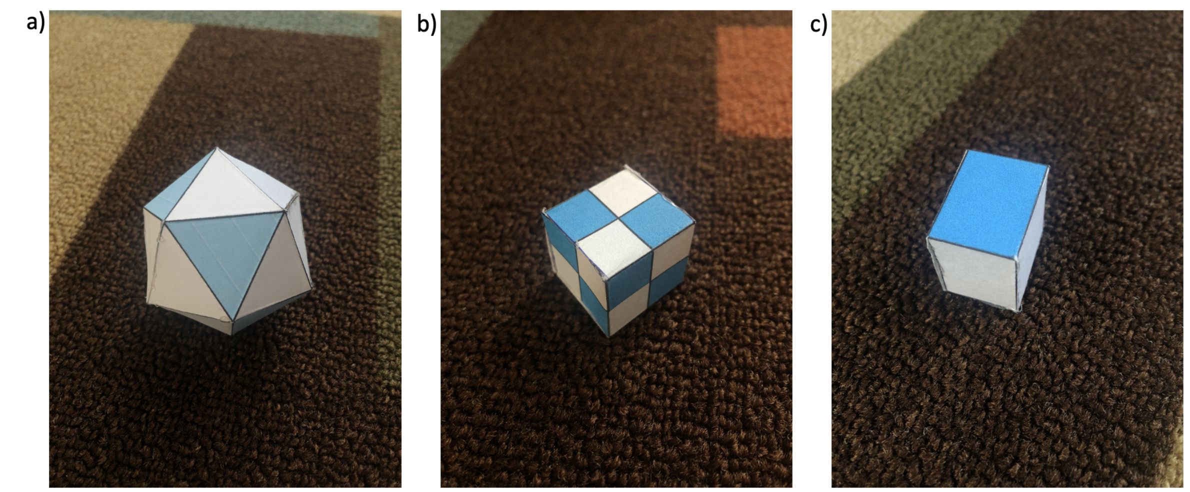


Figure 6: 3D models created directly from regular tessellations a) Icosahedron from triangular tiling b) Cube from square tiling c) Rectangular prism from rectangular tessellation

When folded upwards, the nets took the formation of an icosahedron (Figure 6a), cube (Figure 6b), and rectangular prism (Figure 6c), respectively, as predicted.

When attempting to elevate the semi-regular tessellations, three things were prioritized: the edges, center, and branches of the net. The branches are rooted in the center of the net and fold into space around each other. However, individual shapes of the net should be as connected as possible while still keeping the net in 2D. A potential net from the pentagon-heptagon tessellation was constructed, with edges to cut highlighted and branches marked in black.

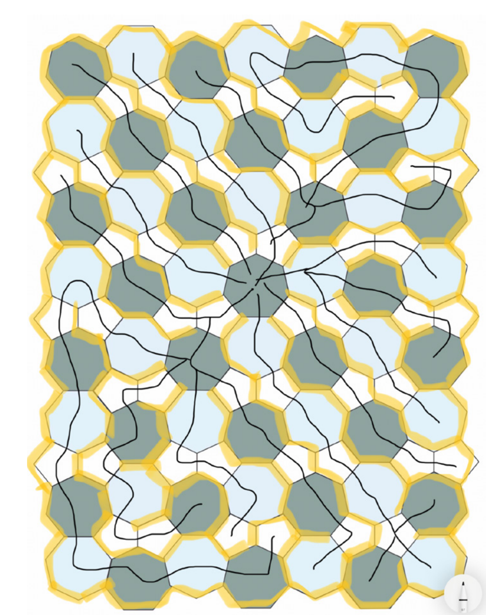


Figure 7: Pentagon-heptagon potential net, edges, and branches marked

The wings around the center of the net (Figure 7) act as supports for the surrounding wings and ideally exhibit no overlap folded upwards. However, some tessellations, including the pentagon-heptagon net, cannot directly act as nets without modification to the original tessellation, since the shapes overlap when folded upwards. By instead creating nets with the same class of shapes, rather than the same class and same configuration of shapes as the tessellation, greater flexibility for model creation appears.

To best represent the pentagon-rhomb tiling, already existing polyhedra with pentagons and triangles emerged as candidates for the approximate result of elevating a pentagon-rhomb tiling, in this case, a snub dodecahedron.

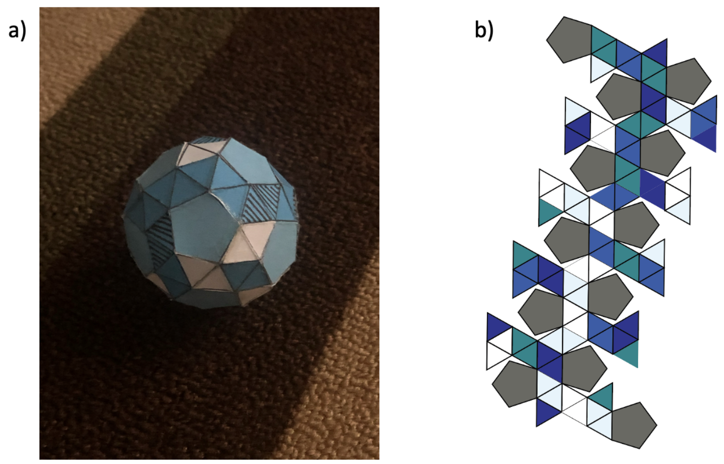


Figure 8: Snub dodecahedron a) Folded 3D model b) Color-coded net using six colors

Grouping adjacent triangles in the snub dodecahedron net (Figure 8b) by color formed rhombi of six colors so that the shape composition of the net was still pentagons and rhombi, like in the original tessellation. The choice of six colors was made to have no rhombus touching another of the same color throughout the 3D model (Figure 8a).

When elevating the pentagon-hexagon tiling, already existing polyhedra with pentagons and hexagons emerged as candidates for the approximate result of elevating a pentagon-hexagon tiling, in this case, a truncated icosahedron.

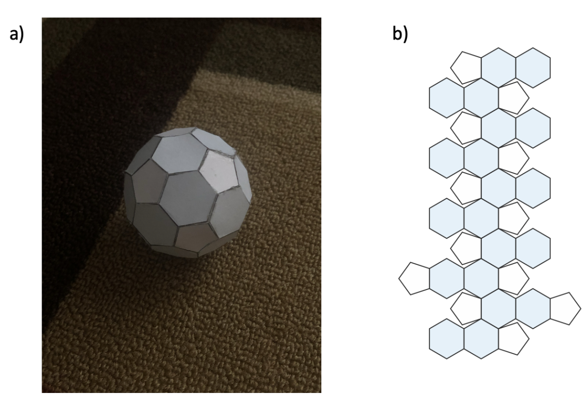


Figure 9: Truncated icosahedron a) Folded 3D model b) Color-coded net using two colors

While both the pentagon-hexagon tessellation and the truncated icosahedron model and net (Figure 9a, 9b) are made up of pentagons and hexagons, only the polygons in the truncated icosahedron are regular. Both, however, are made up of shapes with the same number of sides.

When elevating the pentagon-heptagon tiling, the existing polyhedra with pentagons and heptagons did not seem to have predicted high sphericity values. Instead, creating a novel model and the novel corresponding net emerged as a new possibility. Since the attempt to create a direct net out of the pentagon-heptagon tessellation failed, similar compromises over net space efficiency were made, like the net of the snub dodecahedron and the truncated icosahedron.

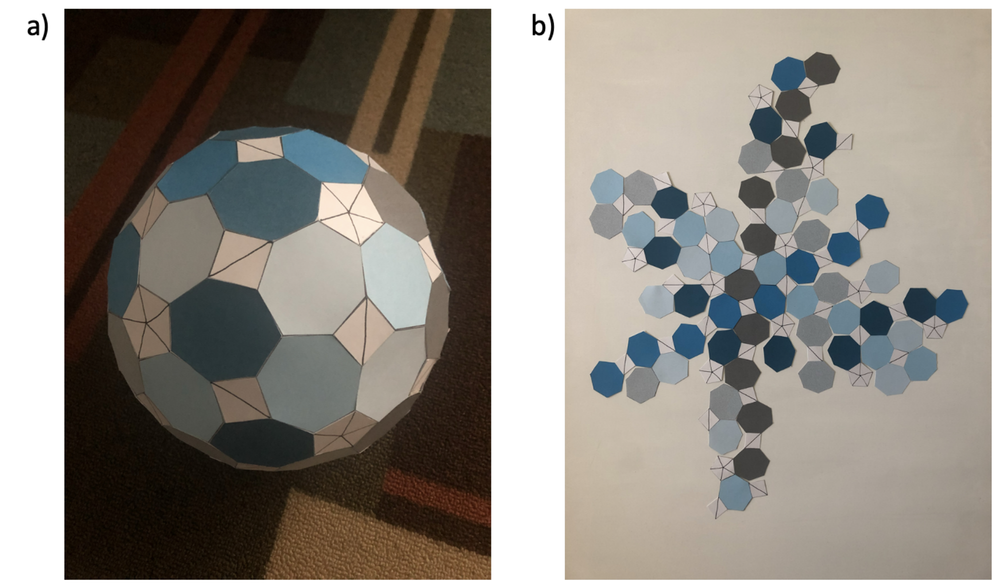


Figure 10: Novel triangle-heptagon model a) Folded 3D model b) Color-coded net using seven colors

The triangle-heptagon model (Figure 10a, 10b) has 102 faces (sixty equilateral heptagons, thirty rhombi, twelve regular pentagons), held together with hot glue. Through incorporating pentagons, the model was predicted to exhibit dodecahedral symmetry with six different “rings” around the model, leading twelve base pentagons to be used, and groups of three heptagons met at a 120º angle looking straight down. Once the model was assembled with the pentagons and heptagons, rhombus-shaped gaps appeared between the heptagons. The dimensions of the gaps were input into Adobe Illustrator and printed out for model construction usage. To keep the model limited to two types of shapes like the others, subdividing the rhombi and pentagons into triangles made the model a triangle-heptagon model, instead of a pentagon-heptagon model. Since the model was novel, surface area, volume, and thus sphericity measurements were not calculated. The known sphericity values calculated for the other models are listed below. All volume and surface area measurements were from the cm values in Adobe Illustrator.

Model	V	SA	Sphericity
Triangle	62.2115	80.8317	0.9393
Square	15.9725	38.0540	0.8060
Rectangle	12.3261	32.2964	0.7990*
Pentagon-rhombus	98.0235	104.6938	0.9820
Pentagon-hexagon	115.8653	118.9040	0.9666
Triangle-heptagon	?	?	?

Table 1: Sphericity as a function of volume (V) and surface area (SA) of the model.

Volume (V) and surface area (SA) of the model are related to sphericity (s) by the given equation $s = (\frac{3V}{4\pi})^{2/3} \times \frac{4\pi}{SA}$. The sphericity of the rectangular prism is designated by an asterisk (Table 1) since changing the side lengths of the prism changes the sphericity value, and thus a general sphericity value for a rectangular prism does not exist.

IV. CONCLUSIONS

Direct tessellations do not produce 3D models with the highest sphericity values, as the top two polyhedra with the highest sphericity values were both corresponding to semi-regular tessellations. The model with the highest calculated sphericity was the snub dodecahedron, corresponding to the pentagon-rhomb tessellation, and the model with the second-highest sphericity value was the truncated icosahedron. However, the unresolved sphericity value of the triangle-heptagon model demonstrates the triangle-heptagon model's promise as an alternative to the snub dodecahedron through its high visually-inferred sphericity. To quantify the sphericity of the triangle-heptagon model, future calculations for volume and surface area will be assessed by splitting up the model into prisms and using triple integrals to quantify the lines of symmetry pervasive through the model. Further improvement of the nets of novel models, like the triangle-heptagon model, is also needed to increase the space efficiency of the nets in the context of material sheets, since space is still present compared to the direct tessellation-derived nets. Extending this study would involve creating 2D and 3D models with different shape compositions - i.e. investigating what types of hexagons are best for tessellating, and what types of hexagons are best for elevating into polyhedra, as well as incorporating polygons with more sides, like octahedrons and dodecagons.

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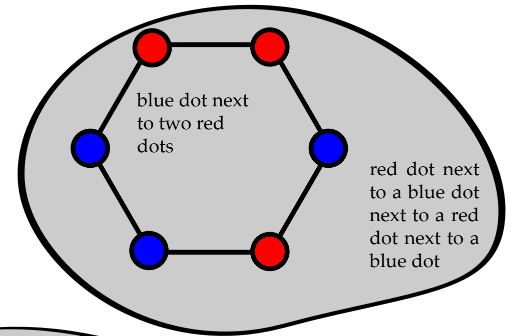
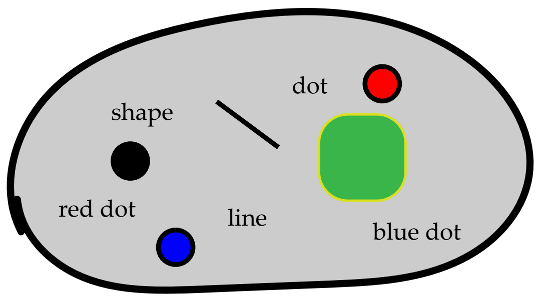
Determine the Dots

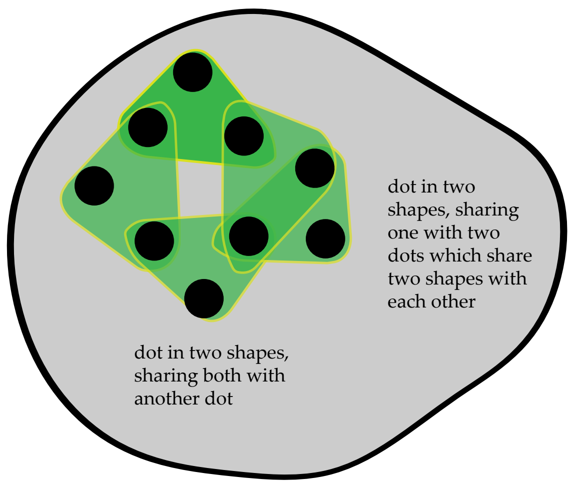
Josh Laison

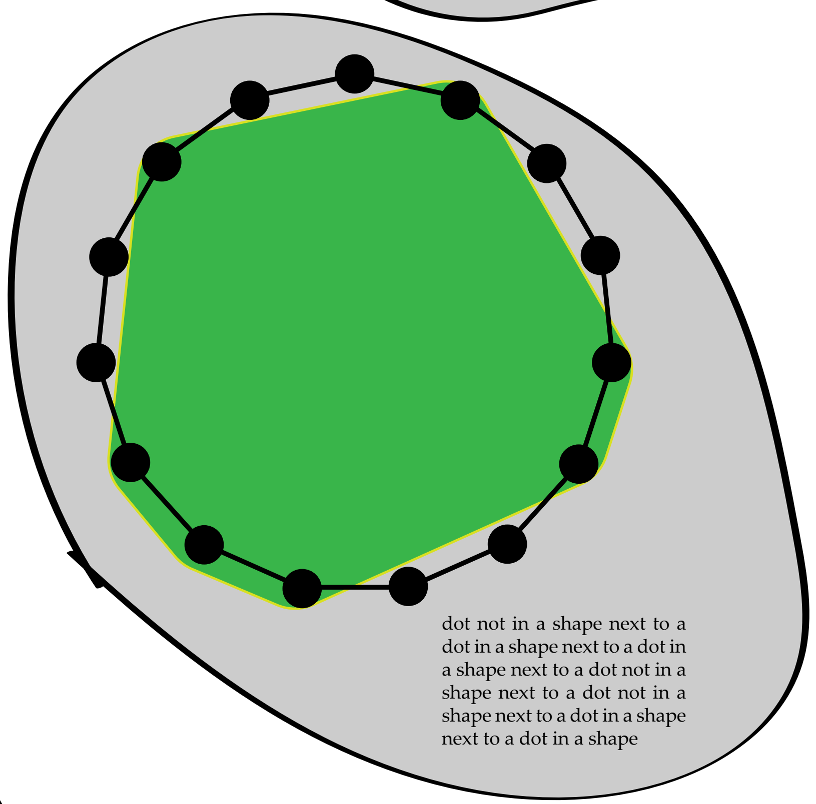
Developed from Graph Stamping: Art-Inspired Mathematics, Hazel, Laison, Kerkhoff, 2021

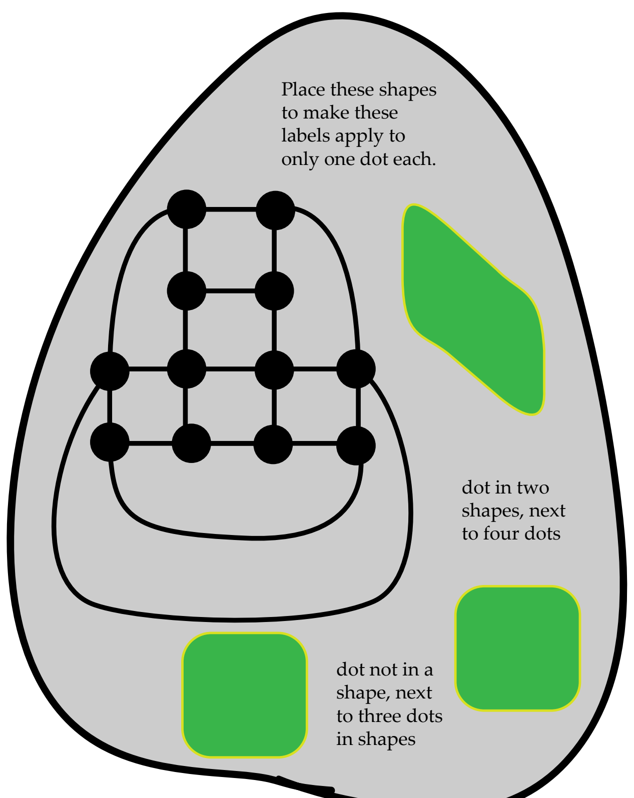
Which labels match which objects? Can all the objects be unambiguously labeled?

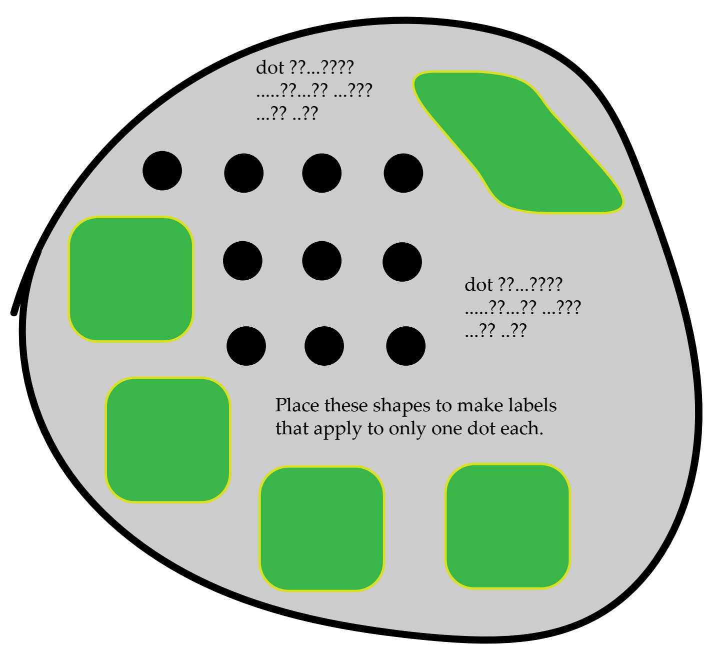








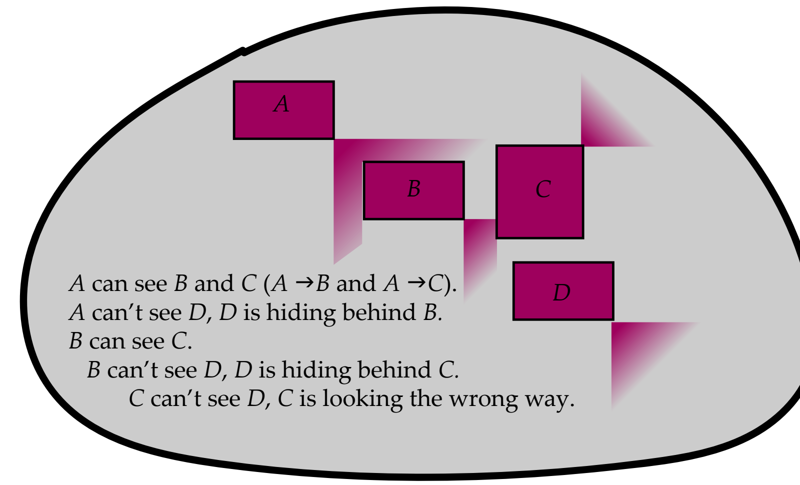


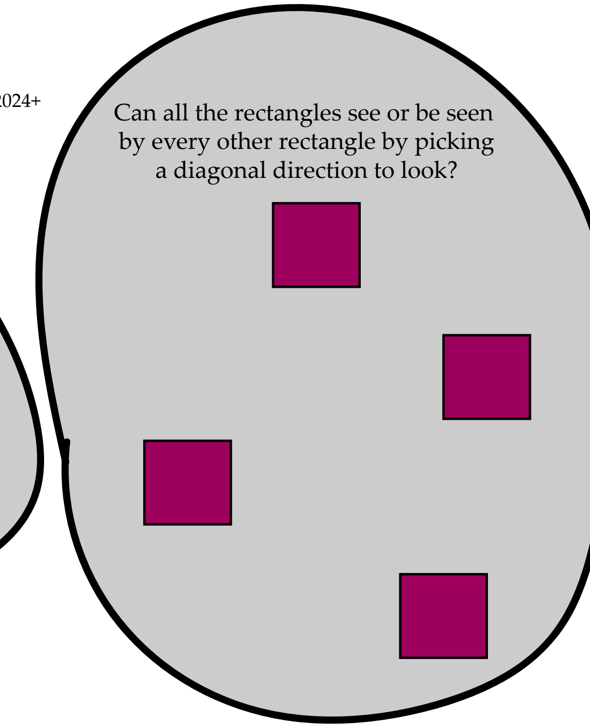


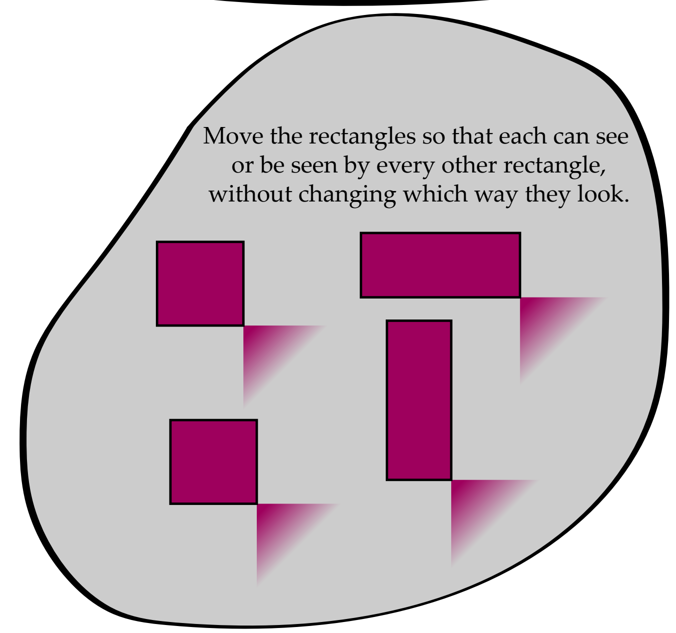
Looking Diagonally

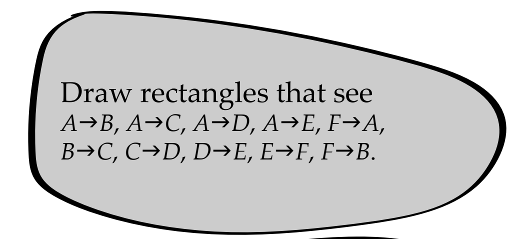
Josh Laison

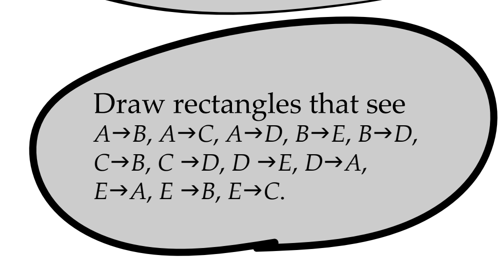
Developed from Corner Rectangle Visibility Graphs, DeYoung, Laison, Li, Southern, 2024+

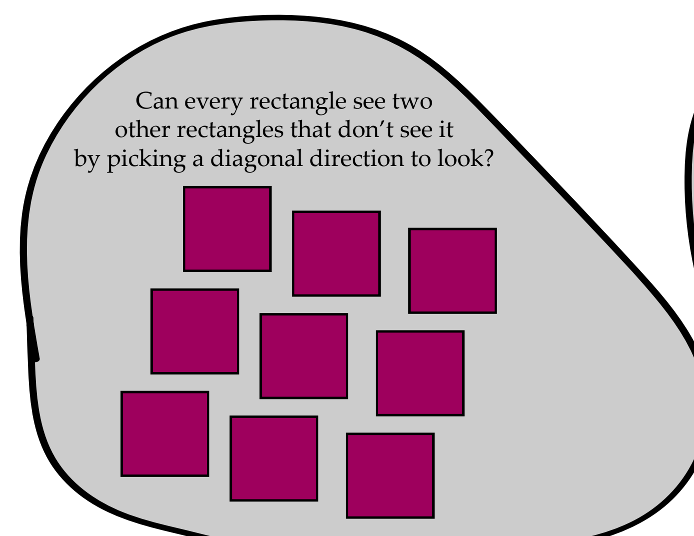


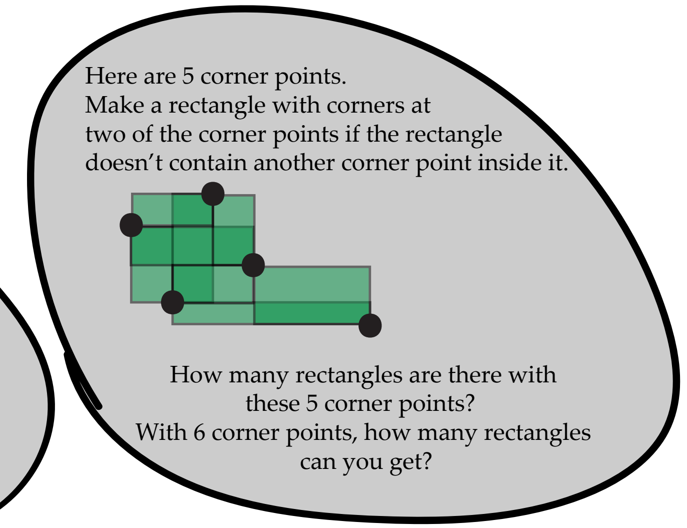










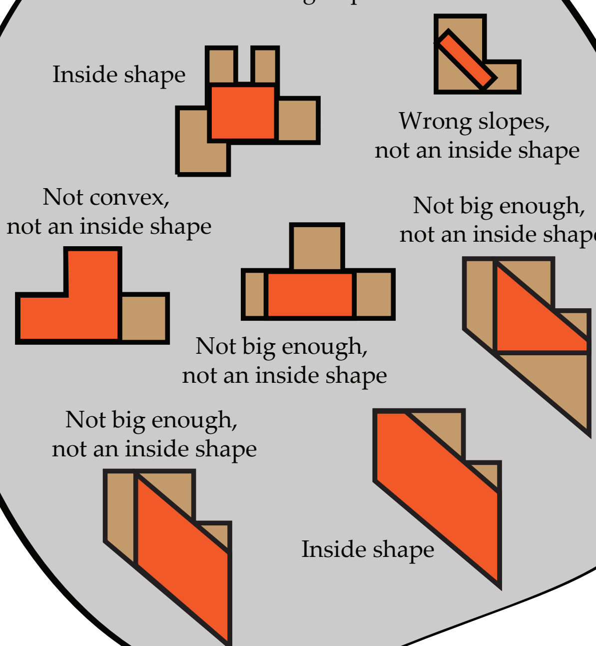


Inside Shapes

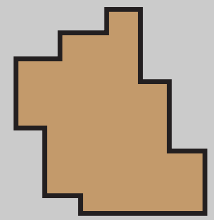
Josh Laison

Developed from Intersection Graphs of Maximal Sub-polygons of *k*-Lizards, Daugherty, Laison, Robinson, Salois, 2023

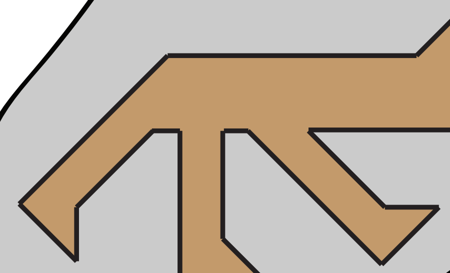
Inside shapes are convex,
have sides parallel to the big shape,
and are as big as possible.



How many inside shapes does this shape have?

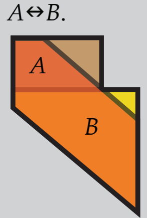


How many inside shapes does this shape have?

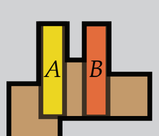


Find a shape with 10 inside shapes.

These inside shapes intersect $A \leftrightarrow B$.



These inside shapes don't intersect $A \not\leftrightarrow B$.



Find a shape with inside shapes intersecting
$A \leftrightarrow B$, $B \leftrightarrow C$, $C \leftrightarrow D$, $D \leftrightarrow A$, $B \leftrightarrow D$
and not intersecting
$A \not\leftrightarrow C$
and no other inside shapes

Find a shape with inside shapes intersecting
$A \leftrightarrow B$, $B \leftrightarrow C$, $C \leftrightarrow D$, $D \leftrightarrow E$, $E \leftrightarrow A$
and not intersecting
$A \not\leftrightarrow C$, $C \not\leftrightarrow E$, $E \not\leftrightarrow B$, $B \not\leftrightarrow D$, $D \not\leftrightarrow A$
and some other inside shapes.

How to Shorten Waiting Lines

An Excerpt from *More Sex is Safer Sex*
by Steven E. Landsburg

You spend too much time waiting in lines. That's not some vague value judgment; it's a precise economic calculation. The people in front of you are wasting your time, and none of them cares. That's a recipe for a minor disaster.

Standing in front of you in line is just like dumping leaves on your lawn or ordering dessert when you're splitting the check. Because I don't feel all the costs, I'm sure to do too much of it. If I spend half a minute drinking while ten people wait behind me, I've imposed five minutes worth of costs on others. What are the odds my drink was really worth that much? Would I have stuck around for a drink if it had cost five minutes of my *own* time?

In principle, there's a market solution to this problem. If I'm in front of you, you can pay me to leave, or take up a collection among the people behind you and *then* pay me to leave. But you don't because the negotiations are a hassle, or because you're worried about "free riders" mooching off your investment, or because you don't want to look like some kind of econ geek. So you and I miss out on a mutually beneficial exchange. That's unfortunate.

Here's a different solution: Change the rules so each new arrival goes to the front of the line instead of the back. Then people near the back will give up and go home (well, actually they'd leave the line and try to re-enter as newcomers, but let's suppose for the moment that we can somehow prevent that.) On average, we'd spend less time waiting and we could all be happier.

If that sounds crazy, try an example. Imagine a water fountain in a city park with a steady gaggle of equally thirsty joggers running by. Each jogger looks at the line and decides whether it's worth joining. Because they're all equally thirsty, they all have the same cut-off for how long a line they'll join; let's say the cut-off is 12. As long as there are 12 people in line, joggers run right on by. Whenever the line length falls to 11, someone instantly joins and bumps it back up to 12.

That's disastrous. It means the line is always at the maximum length anyone will tolerate. The people in line can't be any happier than the people who look at the line and jog on—if they *were* happier, the line length would grow even longer. Since the water fountain doesn't make anyone any happier, it might as well not be there in the first place.

But what if we send newcomers to the *front* of the line? Then—because we've assumed a steady stream of new arrivals—the second guy in line never gets to drink; by the time it's his turn, someone else will cut in front of him. So as long as someone's drinking, you might as well jog right by. But if you're lucky enough to arrive just as someone else is finishing, you immediately take his place.

That's a great outcome, because nobody ever wastes time in line. You might think it has the offsetting disadvantage that a lot of people never get to drink, but that disadvantage is an illusion. Under the traditional system there are also a lot of people who never get to drink— namely the ones who never join the line because it's too long. Under *either* system the fountain is in constant use, so either system serves exactly the same number of drinkers. The only difference is the line length.

Now let's tweak the example to make it more realistic: Suppose the newcomers arrive not in a steady stream but sporadically and unpredictably.[1] Then, since newcomers go to the front of the line, it's always worth stopping for a drink. But if someone else comes along before you're finished, you'll get pushed back. If you get pushed back far enough, you'll leave.

That keeps the line short, which is good.[2] In fact, it's better than good: It's ideal. We'll always have *exactly the right line length* and here's why: Entering the line is a no-brainer. The only hard decision is whether to *leave* the line. And that decision is made by the guy at the back, who doesn't hurt anyone if he stays and doesn't help anyone if he goes.

[1] Note to the terminally geeky: To make this argument precise, you'll want to assume that both the arrival times and the amount of time it takes to drink are Poisson distributed.

[2] On the other hand, we wouldn't always want a line length of zero, because then the fountain might end up sitting idle.

In other words, the decision maker feels all the costs and benefits of his own actions! And that's exactly the prescription for a perfect outcome.

Now, there are a lot of assumptions here. I've assumed people have enough informationn to know when to bail out. That means they know both the current line length and the expected frequency of new arrivals. I've also assumed that everyone is equally thirsty; without that assumption we'd get bad outcomes when less-thirsty newcomers replace their thirstier counterparts.[3] And I've assumed there's a way to prevent people from leaving the end of the line and re- entering at the beginning—just as the traditional system assumes there's a way to stop people from cutting in.

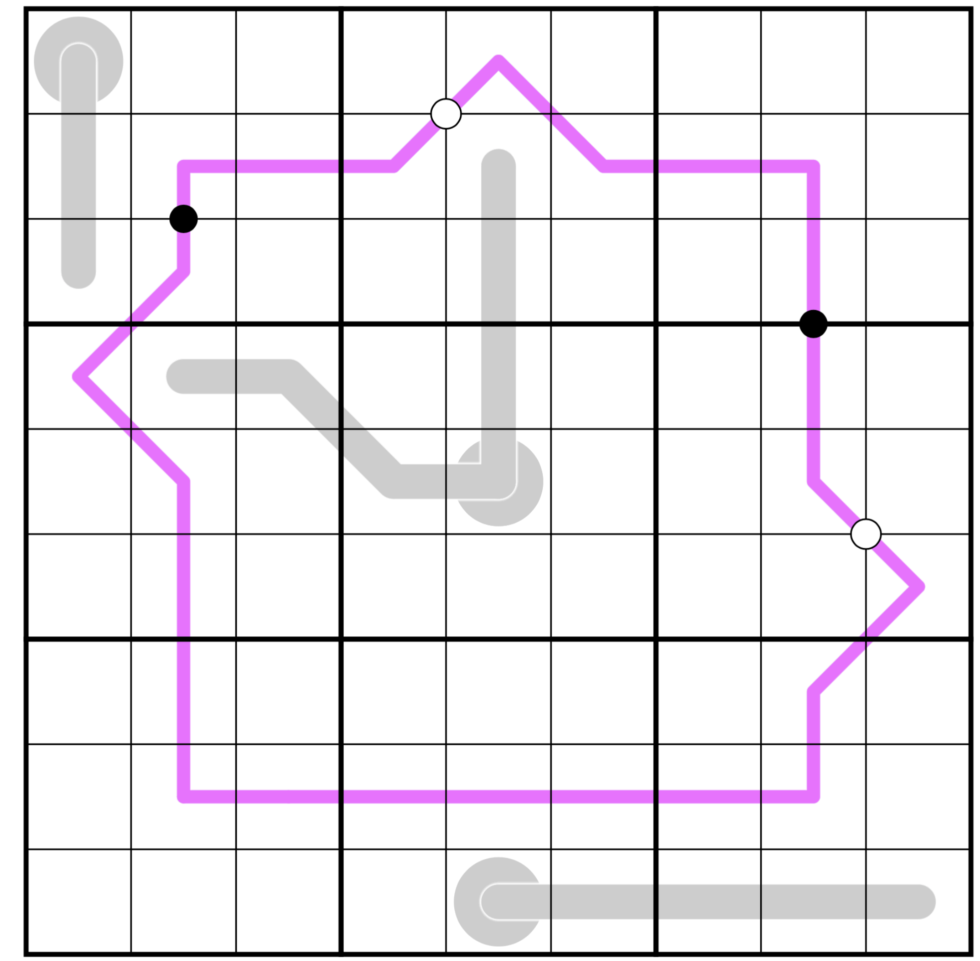
Those assumptions can all be tolerably well approximated in the queues for telephone customer service. Here's how it would work: You call Microsoft for help installing Windows. An initial recording announces the average frequency of calls and explains that each new call will be placed in front of yours. Every minute or so, a new recording tells you how far back in the line you've been pushed. If you hang up and call back, Caller ID makes sure you can't get through. And for those with true emergencies (like those desperately thirsty customers at the water fountain) there can be a separate queue that you pay to join.

Sound crazy? Partly that's because you're probably not thinking about how much shorter the waiting time would be on average. It might just be crazy enough to work.

[3] So if some customers are thirstier than others, the go-to-the-front system falls short of ideal. But it's probably still better than the system we've got now.

Ten O'Clock

Cris Moore, moore@santafe.edu



Each row, column, and 3x3 box contains the digits 1-9.

The purple loop is a "10-line." It can be divided into segments, each of which sums to 10. Within each segment, digits cannot repeat, and must increase in the clockwise direction: for instance, if a segment is 235, these digits occur in that order clockwise. No two segments use the same set of digits.

Digits on thermometers increase starting at the bulb. Adjacent cells on the loop connected by a dot are consecutive if the dot is white, and in a 1:2 ratio if the dot is black.

You can play online at Sven's SudokuPad: https://tinyurl.com/48nskkuk .
Enjoy and THANK YOU for making my first G4G wonderful!

Jeu de Taquin: The Fifteen Puzzle in research mathematics

Tom Roby (`tom.roby@uconn.edu`)

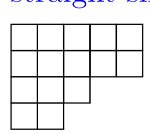
February 2024

In French the *Fifteen Puzzle* [Sl06] is known as *Jeu de Taquin* ("The teasing game"). It inspired the French mathematician Schützenberger [Sch61] to create combinatorial *jeu de taquin*, which has somewhat different rules. The sliding of numbers around each other turns out to be a very useful tool in algebraic combinatorics, with applications to representation theory and algebraic geometry.

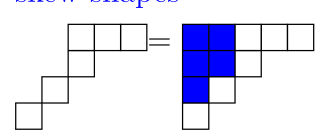
Schützenberger [Sch72, Sch73] and others later generalized much of the theory from labelings of squares in the plane to any finite partially ordered set. See the survey [Sta09] by Stanley for more on the history and for further references. Here we just explain a few fun accessible facts about *jeu de taquin* (in the research sense, which is the main usage of this phrase in English, since we already have a name for the "fifteen puzzle"). The references provide a starting point for more on the history, details, and proofs.

Our slides take place in left- and top-justified

straight shapes or skew shapes



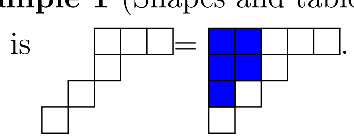
where the rows weakly decrease in length



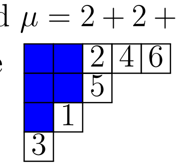
which are the difference of two straight shapes.

Putting numbers in the squares of our shape so that they are **always** increasing from top-to-bottom and from left-to-right turns it into a tableau. For any skew shape σ with n squares, we denote by $\mathrm{SYT}(\sigma)$ the set of all tableaux of shape σ with exactly the numbers $\{1, 2, \ldots, n\}$, each used once.

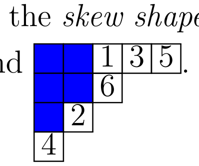
Example 1 (Shapes and tableaux)**.** For $\lambda = 5+3+2+1$ and $\mu = 2+2+1$, the *skew shape* λ/μ is



. Two *tableaux* of this shape are

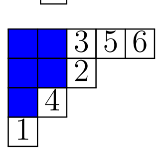


and



.

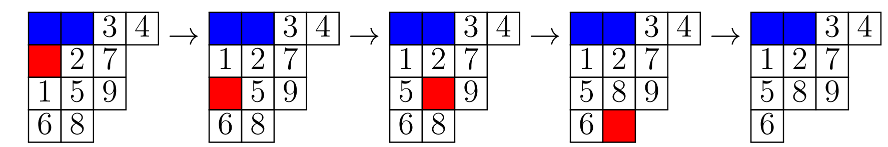
But



is **not** a tableaux because the 2 and 3 are out of order.

The next example shows some valid jeu de taquin slides. For mathematical jeu de taquin, once an empty square has been identified, each following individual slide is deterministic because of the ordering constraint. This is unlike the fifteen puzzle, where slides can result in elements being out of order, allowing more than one slide to be legal into an empty square.

Example 2 (Jeu de taquin move from one skew tableau to another)**.** Here we show the succession of individual slides that takes the chosen red square from the NW boundary of the skew shape to the SE boundary. At each step, the *smaller* of the adjacent squares (below or to the right) slides into the red square, and the red (empty) square moves to the original location of that smaller number. We call this a jdt move.



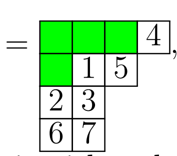
Note that every sliding move preserves the property that numbers increase from top to bottom and from left to right (though the intermediate tableaux are not necessarily of skew shape). The tableau on the right is the result of jeu de taquin applied to the initial skew shape and the selected red square. In some cases, we want to track where the red square ended, in which case the penultimate tableau better represents the output. Both points of view are useful.

From the mathematical point of view amazing things happen!

Rectification of skew tableaux

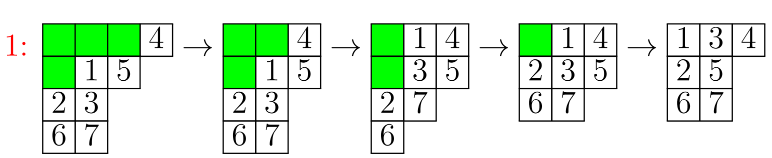
One natural (and useful) procedure is to use jeu de taquin to move a skew tableaux to a straight shape, a process called rectification. Even though each jeu de taquin move is deterministic, there is in general more than one order in which the moves can be performed to reach a straight shape.

Example 3 (Rectifying a skew tableau)**.** If we start with the tableau: Q =

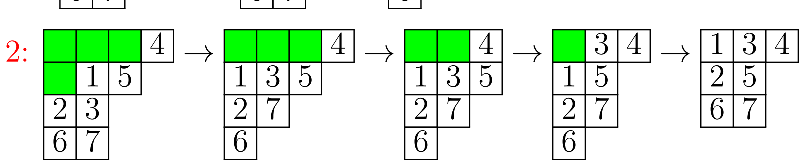


, then our first jdt move can begin with a slide into the rightmost green square in either the first row, or in the second row. Here are two (of the three possible) sequences of jeu de taquin slides.

Sequence 1:



Sequence 2:



Note the that at the penultimate step, we have different tableaux (of distinct shapes), even though we have done jdt moves into the same *set* of boxes. The next theorem says this cannot happen if we slide all the way to a straight shape (so rectification is well defined).

Theorem 1 (Confluence)**.** *Any sequence of jeu de taquin moves leading to a* ***straight*** *shape gives the same result (regardless of the order of the moves).*

The above examples illustrates this theorem.

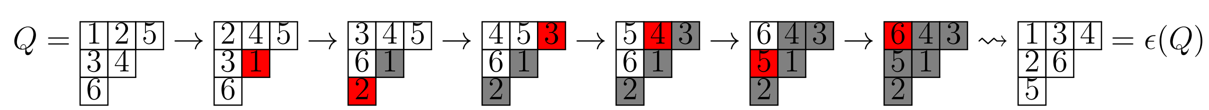
Lauren K. Williams of Mercyhurst University has a number of useful applets, including one that animates this type of jeu de taquin slide.

https://www.integral-domain.org/lwilliams/Applets/discretemath/jeudetaquin.php

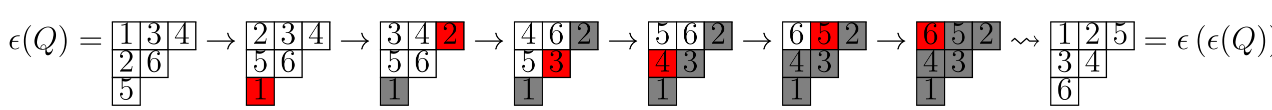
Evacuation of tableaux

Iterating jdt moves leads to an operation on skew tableaux called evacuation. We treat the lowest numbered square as empty, perform a jdt move, and keep track of where it ends up on the boundary. We then "freeze" that square and entry, so it does not participate in successive slides. Repeat this with the new lowest-numbered square, and continue until each square in the shape has undergone jeu de taquin (in the decreasing shapes). In the last step, reinterpret the result as a tableau (by reversing the ordering of the labels). This gives a map $\epsilon : \mathrm{SYT}(\lambda) \to \mathrm{SYT}(\lambda)$.

Example 4 (Evacuating a tableau)**.** For the straight tableau Q below we compute $\epsilon(Q)$ using six successive jeu de taquin moves. The red squares indicate the square that has just undergone a jdt move, and the gray squares are frozen.



If we repeat this procedure on $\epsilon(Q)$, we get back to our original shape. The next theorem shows that this is not a coincidence.



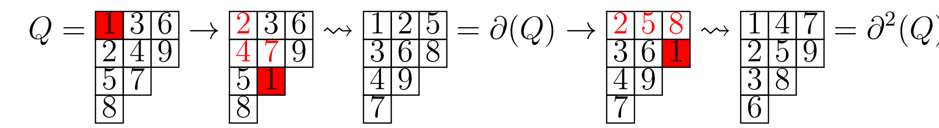
Theorem 2 (Evacuation is an involution)**.** *For any tableau Q, $\epsilon(\epsilon(Q)) = Q$, so $\epsilon^2 = \mathrm{id}$.*

This theorem continues to hold for skew-shaped tableaux. In fact, the sliding process, jeu de taquin moves and evacuation can all be generalized to labelings ("linear extensions") of *any* partially ordered set, and evacuation continues to be an involution [Sta09, Thm 2.1].

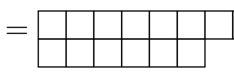
Promotion

Another (related) operation on $Q \in \mathrm{SYT}(\lambda)$ via jeu de taquin is called promotion. Perform a jdt move starting at the box with the lowest label until it gets to the boundary, where it becomes the new largest element. Then decrement the other labels by 1 to get the tableau $\partial(Q) \in \mathrm{SYT}(\lambda)$.

Example 5 (Promotion operation on a tableau)**.** Entries in red below show the sliding path for the jdt move.



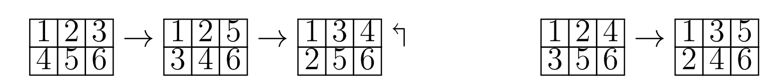
A natural question is how large is the period of this map on $\mathrm{SYT}(\lambda)$. That is, what is the minimum number of times we need to apply ∂ that guarantees we end up where we started, no matter what tableau we start with? In general the orbit structure is quite disregulated, and the period of promotion is quite large, as the next example shows.

Example 6 ([SW12])**.** Promotion has order 7,554,844,752 on $\mathrm{SYT}(\lambda)$ for $\lambda =$ , which has only 14 squares.

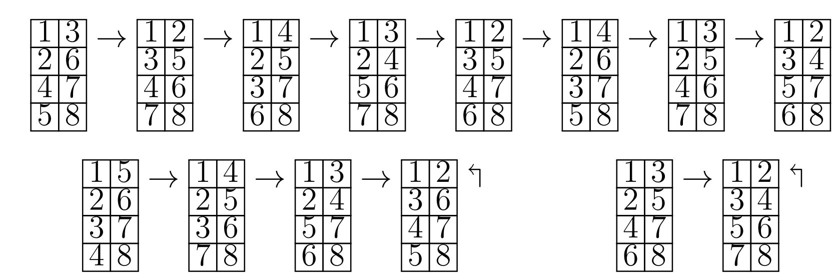
But for certain special shapes, the order of promotion is exceptionally small.

Theorem 3 (Order of promotion on rectangular tableaux)**.** *Promotion on* $\mathrm{SYT}(a \times b)$ *has order* ab.

Example 7 (Promotion orbits for tableaux of rectangular shape)**.** For the 5 tableaux of shape 2×3, we get one orbit of size three, and one of size two, so the order of promotion is $\mathrm{LCM}(2, 3) = 6 = 2 \cdot 3$, agreeing with the theorem.



For the 14 tableaux of shape 4×2, we get orbits of size eight, four, and two, so the order of promotion is $\mathrm{LCM}(2, 4, 8) = 8 = 4 \cdot 2$.



There are a few other special shapes where the order of promotion is small [Sta09, Thm 4.1].

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Jekyll and Hyde Chess Tours

Karl Schaffer
DanceAndMath@gmail.com

Here are some puzzles for G4G15 based on an upcoming paper for the 2024 Bridges conference titled "Artsy Pseudo Hamiltonian Tours" [2]. We define a *k*-Hamiltonian tour or cycle of a graph to be a sequence of the vertices of the graph, each vertex appearing once, such that each successive vertex in the sequence is graphical distance *k* from the previous vertex, and such that the end vertex in the sequence is the same as the starting vertex. For the cycle C_n, if *k* and *n* are relatively prime, a *k*-Hamiltonian tour is also known as the {*n*/*k*} star polygon consisting of one path that visits each vertex once. {*n*/*k*} is known as the Schläfli notation. An (*a*,*b*)-step Hamiltonian tour or cycle of a graph is a sequence of the vertices starting and ending at the same vertex such that the distances between successive vertices alternate between *a* and *b*, see Figure 1(b), where we extend the Schläfli notation to {*n*/(*a*,*b*)}. See [4]. Such designs sometimes appear in a variety of art forms, for example, Figures 1(c) and (d). Figure 1(d) duplicates the design of a three-person string loop octahedron. Each vertex is held by one hand; see a video at [1,"String Quartet"] showing how it is created. This could also be a design for six dance partners who each move through each of the six positions, in which under curves precede over curves, or for juggling patterns.

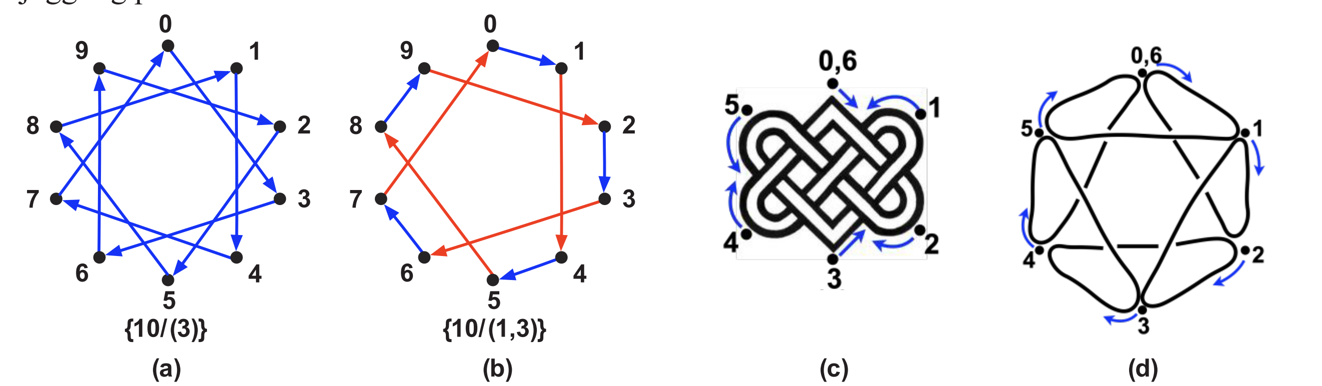


Figure 1: *(a) The {10/3} star polygon. (b) The {10/(1,3) tour, or the (1,3)-step Hamiltonian tour of C_{10}. (c) Celtic knot could represent {6/(3,2,2,3,-2,–2)} pattern; or six dancers following arrows, undercurves preceding over curves. (d) {6/(1,2)} string figure creating the octahedron or representing a dance or juggling pattern.*

Figure 2 : *Lace shirtwaist buttons by Laurel Shastri exhibiting several of the {m/(a,b)} patterns.*

However, for G4G15, it seems that a puzzle based on these ideas might be appealing. Can we, for example, find a tour of the 8 by 8 chess board by a chess piece that alternates its moves between those of two standard pieces? There are lots of possibilities, but here are some puzzles in which the touring piece alternates between a bishop and a knight. We'll call that piece the Jekyll and Hyde chess piece. Because the knight alternates colors on a checkerboard and the bishop stays on the same color, a tour can only work on a board with a multiple of four squares. See Figure 3, in which the first move is a knight move. If the first move is a bishop move, the number of squares must similarly be a multiple of four.

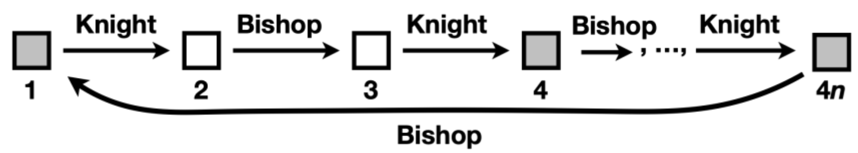
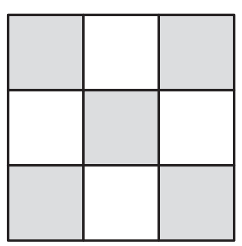
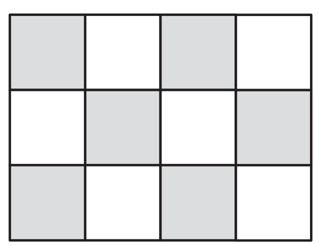
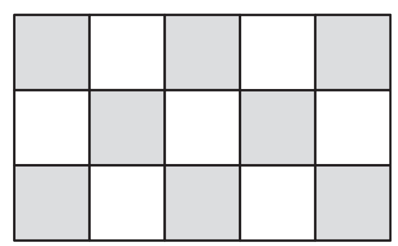


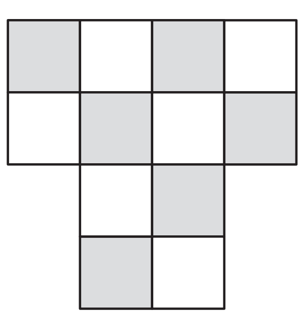
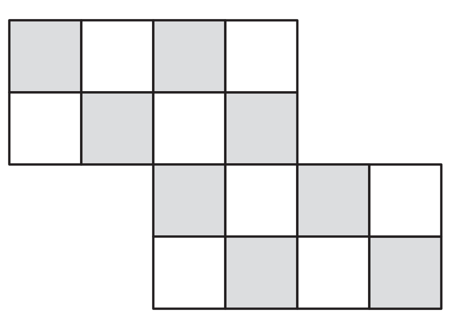
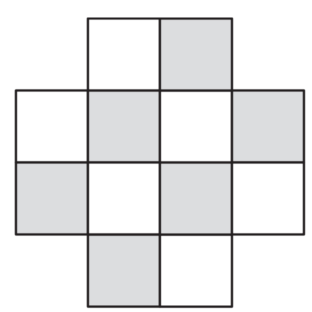
Figure 3: *A Jekyll and Hyde (alternating bishop and knight moves) tour of a checkerboard colored board must have a number of squares that is a multiple of four.*

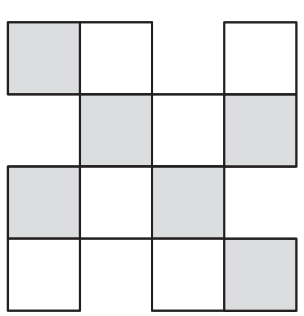
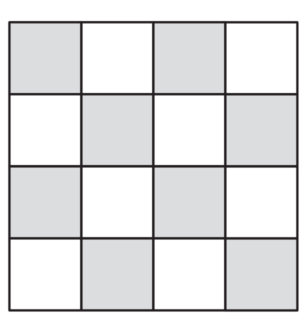
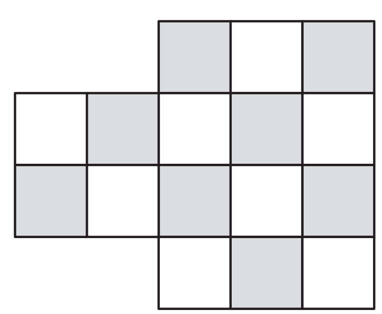
If the board is composed of a number of squares that is not a multiple of four we might instead try to find a pseudo Hamiltonian path that visits every square but does not return to the starting square. The first few puzzles below are easier. Do try to find a Jekyll and Hyde knight/bisohop Hamiltonian tour of the 8 by 8 chessboard. Example solutions (I have not attempted to tabulate all solutions for each puzzle!) are at [3]. Make up some similar puzzles of your own. For example, try a Jekyll and Hyde piece that alternates knight and king moves. Or what about a piece that alternates the moves of three pieces (an (a,b,c)-step Hamiltonian tour).

The 3 by 3 square below has a Jekyll and Hyde (bishop and knight) Hamiltonian path, but not a tour. The 3 by 4 rectangle has a genuine alternating Hamiltonian tour. The 3 by 5 rectangle has what we might call an "improper" tour, since it has an odd number of squares but there is a tour that starts with a knight move and ends in the starting square with a knight move rather than a bishop move. To record your solution you might write 1 in the starting square, 2 in the next etc. If your first move is a knight move then all the even numbers will be in a square that is the target of a knight move, and the bishop moves will end in odd numbered squares. Example solutions are at [].

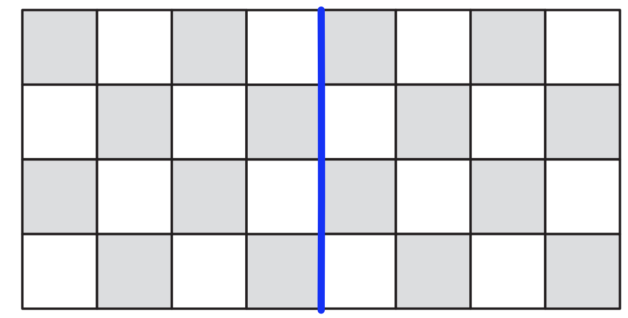
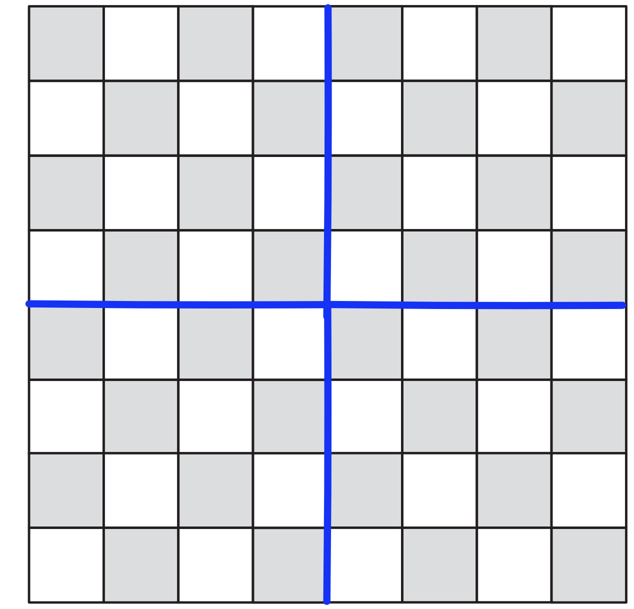




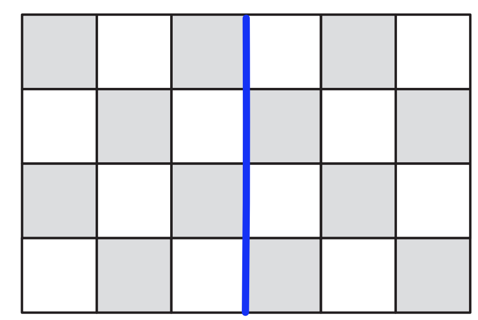
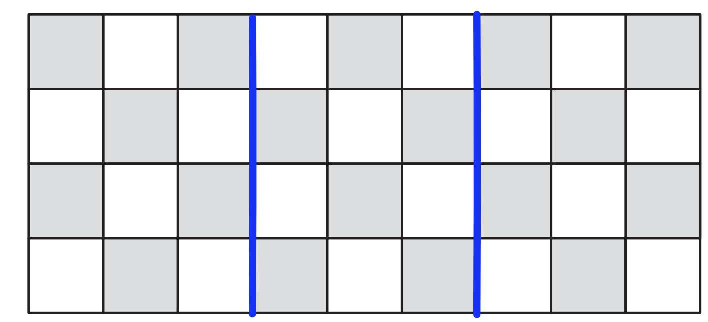
The following all have genuine Hamiltonian Jekyll and Hyde tours.

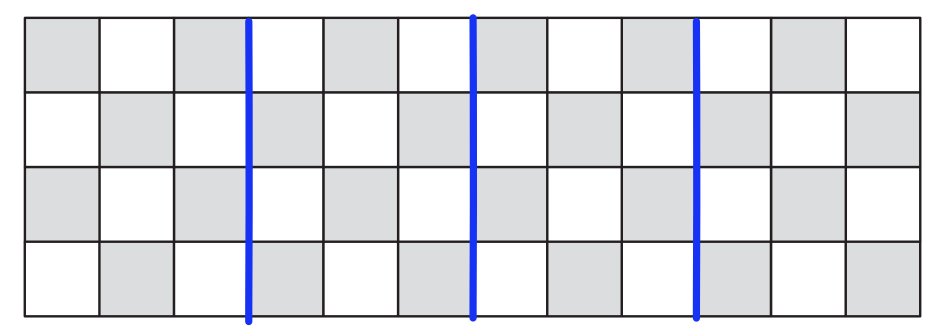




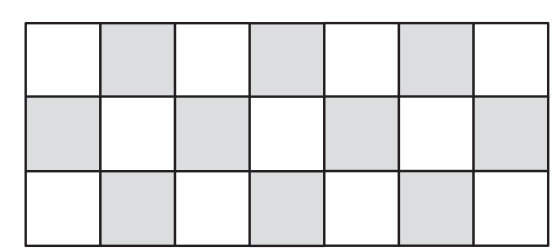




Can you use your solution to the 4 by 4 to find a solution to the 4 by 8 and the 8 by 8? You might be able to use your solution to the 3 by 4 rectangle to solve the 6 by 4 rectangle. Can you use the 3 by 4 and the 6 by 4 to extend and find a solution to the 9 by 4, in fact to all $3k$ by 4 rectangles?
The 3 by 7 has an improper tour.





References.

[1] G. Csicsery. “Short films on string polyhedra with Scott Kim and Karl Schaffer.“ 2009. http://mathdance.org/html/CsicseryShortFilms.html.

[2] K. Schaffer and M. Nathan. “Artsy Pseudo Hamiltonian Tours.” Bridges Richmond. 2024. Submitted.

[3] K. Schaffer. Solutions to these puzzles!. G4G15. 2024. https://www.movespeakspin.com/pdf/2024JekyllHydeChessSolutions.

[4] G.-C. Lau, S.-M. Lee, K. Schaffer, S-M Tong. “On (a,b)-step Hamiltonian Graphs. In progress. https://movespeakspin.com/pdf/2024_01_31_a_b__Step_Hamiltonian_Graphs.pdf.

G4G15 - PATHWORDS

Pathwords are minimized word searches. The goal is to find the words hidden within each grid. In this first set, there are three five-letter words hidden in each grid. Each word is found by “snaking” a path through the grid; i.e. connecting 5 adjacent squares to make a pentomino. For all puzzles, each letter in the grid is used exactly once. For each grid, all three words are themed by a single clue. Have fun, and good luck!

Example — FRUITS

L	E	P	■
P	Y	E	A
P	R	R	C
A	B	E	H

#1: MATH

E	M	I	■
S	L	T	E
M	A	U	Q
I	N	U	S

#2: I'VE GOT THE BLUES

M	L	A	Y
I	A	■	O
N	Z	U	R
E	D	R	E

#3: THAT'S YOUR Q

P	■	E	E
I	Q	U	Q
A	P	I	U
Q	U	A	S

#4: ANAGRAMS 1

E	L	E	W
■	B	L	O
B	O	W	B
E	L	O	W

#5: HEADWEAR

Y	T	■	B
B	E	R	E
R	A	R	T
E	D	A	I

#6: DRESS WARM

R	F	E	V
A	■	P	O
C	S	A	L
A	K	R	G

#7: ANAGRAMS 2

■	N	I	B
G	I	N	G
E	E	B	E
B	I	N	G

#8: SHE'S MY JEWEL

R	A	E	P
L	E	R	T
M	B	P	O
A	Z	A	■

#9: ANIMALS

E	O	U	S
L	M	L	E
G	A	E	M
■	E	C	A

#10 ANAGRAMS 3

T	A	L	R
E	R	T	E
■	A	L	A
T	R	E	L

#11:LET'S MAKE COOKIES!

R	U	R	A
S	O	L	G
P	■	F	U
I	H	C	S

#12: CHAMBER ORCHESTRA

T	E	L	E
U	O	L	C
L	F	I	P
O	N	A	■

#13: FORGIVE ME, FATHER

D	I	R	P
E	H	T	■
R	A	O	L
W	T	H	S

#14: SIMPLE MACHINES

E	D	L	E
W	G	E	V
C	R	■	E
S	E	W	R

#15: WHT? N VWLS?!

H	P	M	Y
S	H	C	N
N	■	Y	S
M	Y	H	P

G4G15 - PATHWORDS

In this second set, there is one 15-letter word per grid.
The word always starts adjacent to the black space.

#1: CV BOOSTERS

S	E	M	
T	N	H	A
L	I	S	C
P	M	O	C

#2: A WAVE OR A NOD

O	N	K	C
W	L		A
D	E	T	N
G	E	M	E

#3: CAMERA OPERATOR

R		C	I
E	H	P	N
G	R	A	E
O	T	A	M

#4: SHAKEN

D	E	L	U
	T	A	B
D	C	O	O
I	S	M	B

#5: CHANGING DIAPERS ALL OVER AGAIN?

	D	O	O
G	R	T	H
N	A	N	E
D	P	A	R

#6: MONKEY BUSINESS

N	S	U	O
E		E	V
S	M	I	H
S	I	S	C

#7: IN ANY CASE

O	N		G
T	W	I	N
S	H	T	I
T	A	N	D

#8: EYE DOCTOR

A	L	M	O
H	T	O	L
	H	G	T
O	P	I	S

#9: UNPLANNED

O	V	I	S
R	P	T	A
I	M	I	O
	L	A	N

#10: HOW “ln x” GROWS

R	I	M	I
A	T	H	C
G		Y	A
O	L	L	L

#11: GIVERS

H	T	N	A
R	T	S	L
O	S	H	I
P	I	P	

#12: I’LL CLUE YOU IN LATER

A	R	R	P
S	C	O	
T	N	O	I
I	N	A	T

#13: MANUAL SECTION

T	I	N	G
O	R	O	U
O	T		B
H	S	E	L

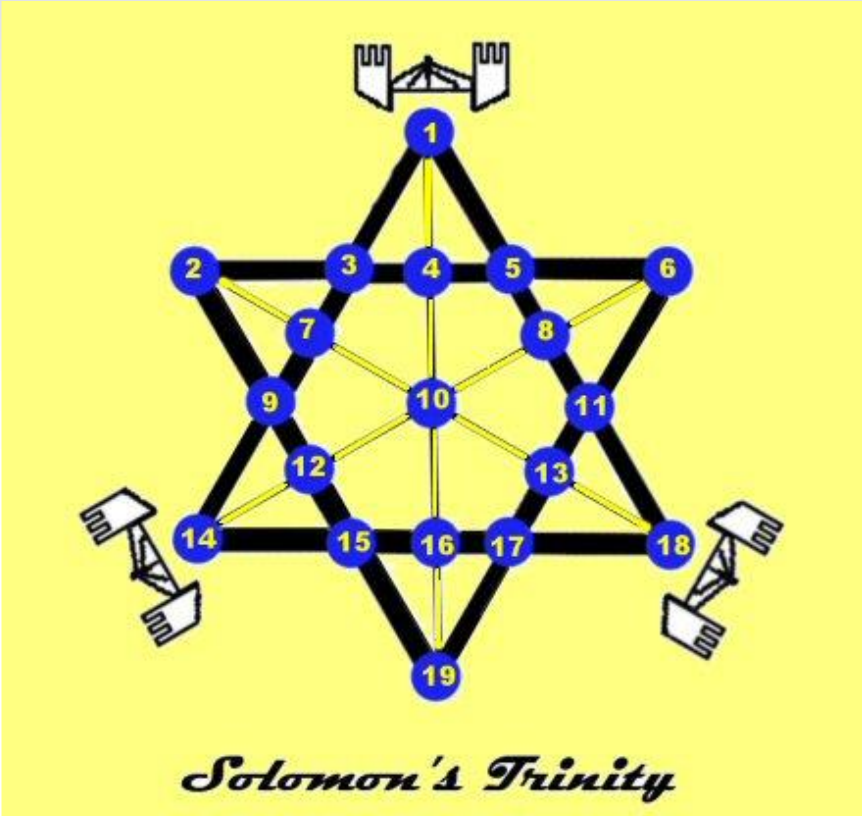
#14: LCD AND PCP

S	C	I	N
A	H	G	E
L		O	N
L	U	C	I

#15: CANDY BAR OR DOODAD

I	T	C	A
L	L	A	M
	A	T	A
W	H	C	H

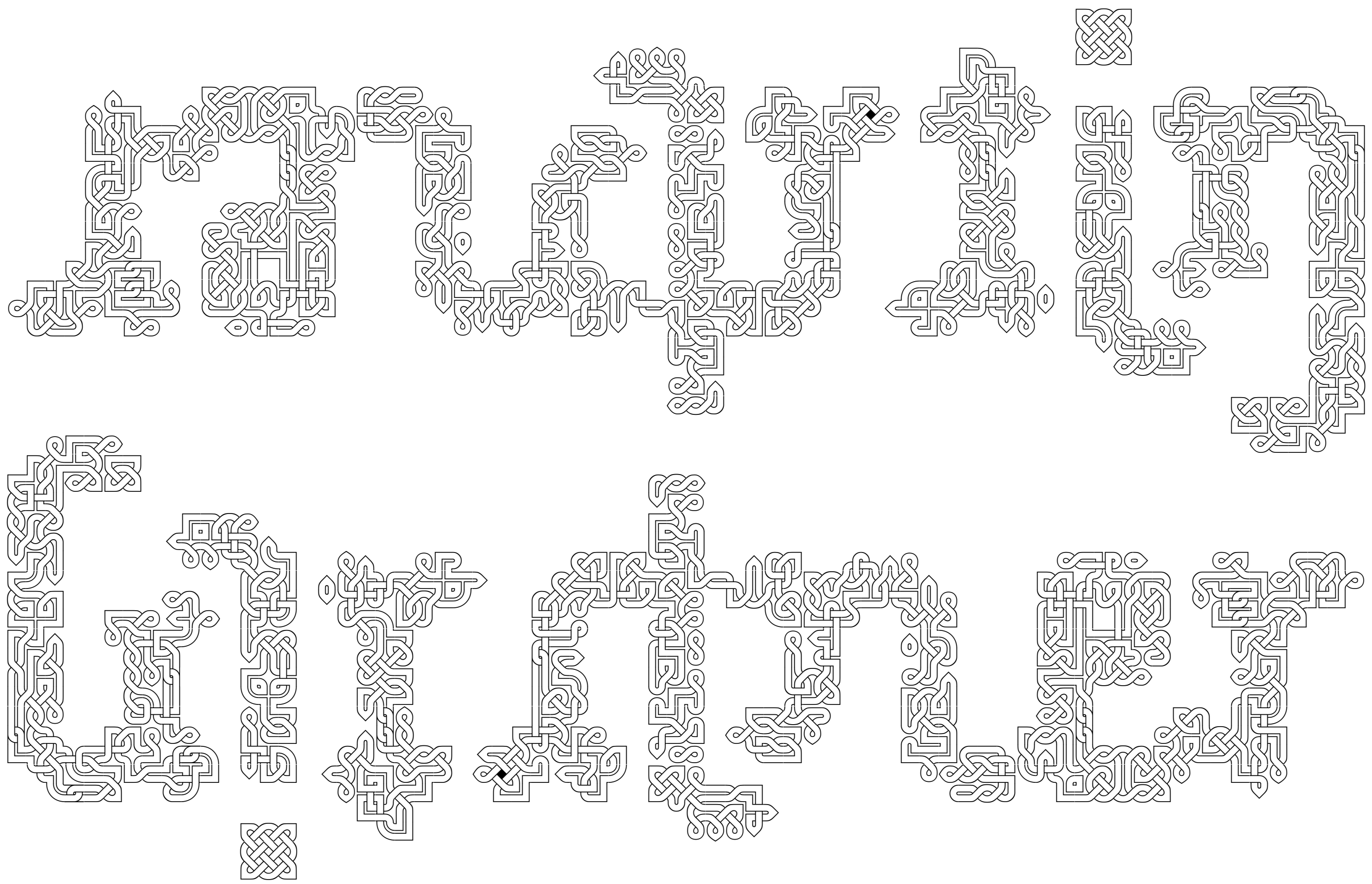
LEGACY



Solomon’s Trinity | Kate Jones | Page 324

Martin Gardner Randomly Generated Celtic Knot

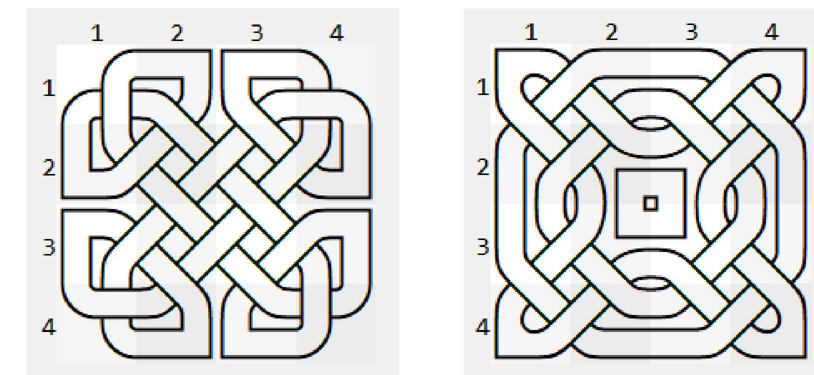
Braden Ganetsky



“Bask3twork” G4G15 Martin Gardner Knot #1/160 Braden Ganetsky 2024

“Bask3twork” Celtic Knot Generator G4G15 Gift

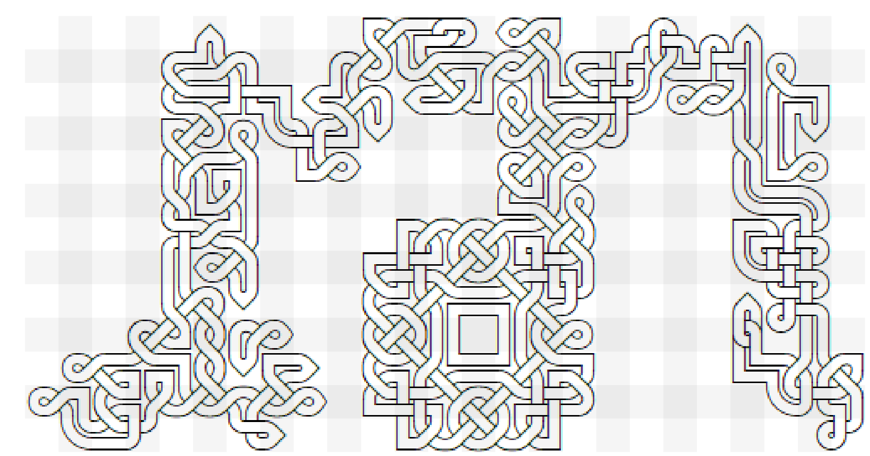
My application “Bask3twork” uses Daniel Isdell’s Celtic Knot font (clanbadge.com) to procedurally generate Celtic knots with any subset of D4 dihedral symmetry, including no symmetry at all. The program checks boundary conditions of each glyph to always generate valid knots, where the connections of neighbouring glyphs line up.



Pictured on the right are two 4x4 knots with full symmetry, i.e. D4 symmetry. Note that my claims of “mirror symmetry” are actually referring to 2-fold rotational symmetry in the third dimension.

For my gift, I adapted Masayoshi Iwai’s “Martin Gardner” 2-cycle Game of Life oscillator ambigram. (http://www.iwai-masaka.jp/54892.html)

With the “2-way Rotational” symmetry option, I used Bask3twork to generate 160 unique Celtic knots in this pixelated “Martin Gardner” ambigram. I generated each knot in small piecewise segments to ensure the result was appealing. I added a few elements of local symmetry, along with the global C2 symmetry.



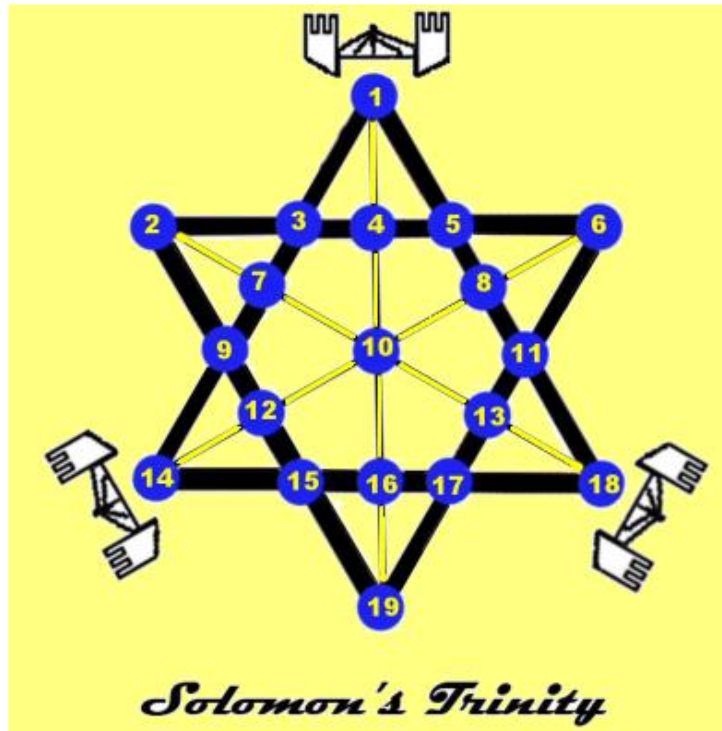
Pictured on the right is an example “M”, that doubles as the tail of the “n” and the “er” when viewed upside-down.

Braden Ganetsky 2024 **https://ganets.ky/**

SOLOMON'S TRINITY™
Sequential Movement Puzzle

Created by Kate Jones – ©2024 Kadon Enterprises, Inc.
Made in Pasadena, MD – G4G15 edition – 2024

Contents:
- Stylized Star of David rules with grid
- 3 coins as "deities": nickel, dime, quarter
- 18 "stones": pennies



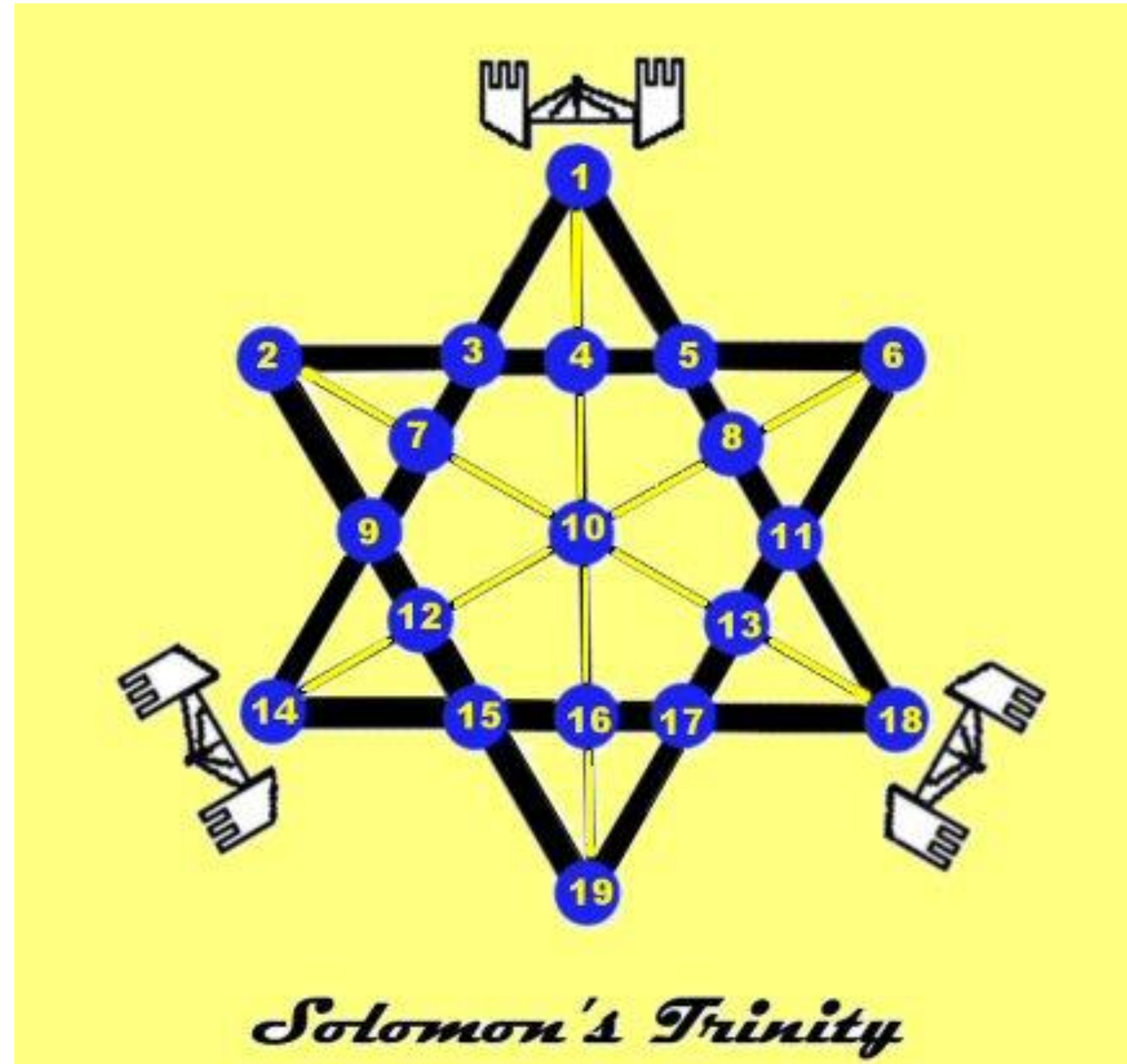
Goal of Solitaire:
Move the 3 silver coins (N, D, Q) in turn to fill all remaining spaces on the grid with pennies. For G4G15, fill only 15 spaces, block 1 space.

Goal of Strategy Game:
For 2, 3, or even more players: make the last possible move, even if empty spaces remain.

Concept:
The Middle East is the birthplace of the world's three major monotheistic religions. The three silver coins on the grid represent the deities of those three religions. The grid is their Universe.

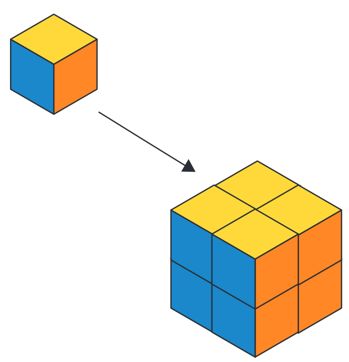
Process (solitaire or game):
- Place the three silver coins on their starting spaces: nickel-1, dime-14, quarter-18. Have 15 pennies off the grid in a supply pile for future use.
- The three silvers will take turns, N-D-Q, and thereafter move in the same sequence.
- On a turn, move the current silver along a straight open line to a vacant space. Place a penny from off the grid onto the space the silver just vacated.
- If a silver has no vacant space to move to, it may jump over an adjacent penny if the space immediately on the other side of that penny is vacant. The silver may jump over an adjacent silver only if there is no penny to jump. Move the jumped piece to the vacated space.
- If the silver isn't able to move nor to jump, it stays, and the next silver takes its turn.
- If two silvers are blocked, the third silver takes as many turns as it can. If all 15 pennies can enter the board, you win. If more than 1 empty space remains when no silver can move (not all 15 pennies on the board), you lose. Clear the board and try again. Many solutions exist.
- As a game, any number of players take turns moving the next silver. Last silver able to move wins. The players don't "own" a silver; they move the next silver in turn that can still move.

The Star of David gameboard was designed by Martin Gardner for the *Game of Solomon*, made by Kadon Enterprises, Inc., since 1985. See www.gamepuzzles.com/histfun.htm#GS. We adapted the original *Solomon* board to give *Solomon's Trinity* triple symmetry.



CUBES

The color cubes were first introduced by Percy MacMahon in 1893. There are 30 ways to color a cube with 6 colors so that all 6 colors appear on the cube's 6 faces. To the right is a representation of one of the cubes, with the yellow face on top. The four side faces are green, orange, blue, and red. The 6th face is hidden at the back, but must be purple.

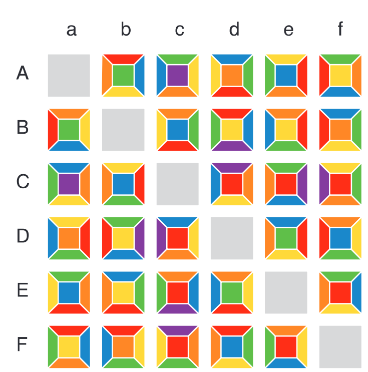


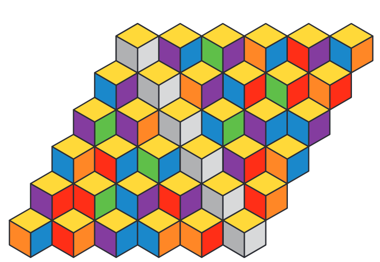
PROBLEM

Start by randomly selecting one of the 30 cubes as the target. The challenge is to find 8 cubes from the remaining ones that can be assembled into a double cube with its outer faces matching the target. In addition, each pair of adjacent faces on the interior of the double cube must match in color. For every target, only one set of 8 cubes can do this, but in two different ways.

SOLUTION

A method to solve the Double Cube problem was presented by Martin Gardner in *Fractal Music, Hypercards, and More*. To the right is a matrix developed by John H. Conway with **15** of the cubes on the upper right, and their mirror cubes on the lower left. Select a cube as target, go to its mirror in the matrix. The 8 cubes lying in the same row and column will provide the solution.





ARTWORK

I have created a digital print based on the Conway matrix, using the Tumbling Blocks quilt pattern for its structure. The matrix morphs into a rhombus, the cubes are oriented with one color-side facing up, and only 3 cube faces are displayed. Three such rhombi identify the cubes, and generate the final artwork.

Margaret Kepner
Insert for Exchange Gift
G4G15 -- 2024

5 Barriers to Spreading Joyful Mathematics

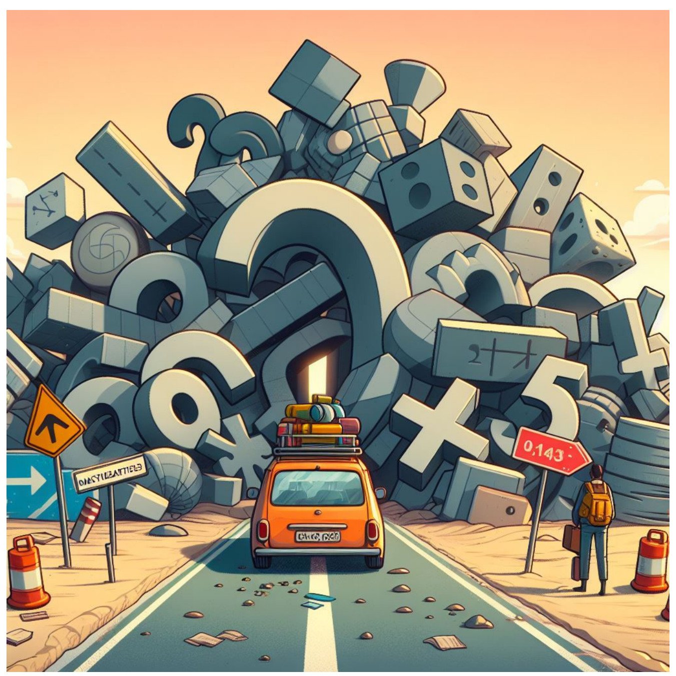
And how to overcome them

Scott Kim, February 2024

In his Aug 1998 article "A Quarter-Century of Recreational Mathematics" in Scientific American magazine, Martin Gardner wrote

> *For 40 years I have done my best to convince educators that recreational math should be incorporated into the standard curriculum. It should be regularly introduced as a way to interest young students in the wonders of mathematics. So far, though, movement in this direction has been glacial.*

Martin Gardner was a hero to me and many other budding mathematicians. He opened the door to the joys of mathematics for more people than anyone else in history. So to hear him despair at his inability to influence educators is sobering.



But not unexpected. All of us here understand the deep joy that doing mathematics can bring. And we love sharing that joy with others. But if you ever tried to bring that joy into schools, you know how resistant schools are to change.

And it's not just schools. Society at large is stubbornly mathphobic. If you tell someone at a party that you are a mathematician, you will hear stories of pain, embarrassment, and fear.

As Justin Reich chronicles in his book "Failure to Disrupt," education reformers have tried and failed to budge the school system for over a century. Every new technology, from television to VR, has been touted as the thing that will change schools forever. They've all failed to make progress because education is not a tech issue; it's a social issue.

So, if we want to spread joyful mathematics and change attitudes at a societal scale, we need to understand what we're up against to figure out what new actions we need to take.

In this article, I'll break down the barriers to the widespread adoption of recreational mathematics in education and discuss what it will take to overcome them.

Recreational Mathematics

Regarding recreational mathematics, Gardner writes:

The line between entertaining math and serious math is a blurry one. In general, math is considered recreational if it has a playful aspect that can be understood and appreciated by nonmathematicians. It encompasses mind-bending paradoxes, ingenious games, and bewildering topological curiosities such as Möbius bands and Klein bottles.

Gardner gives several examples of the playful mathematics he featured in his Mathematical Games column, which ran from 1956 to 1981. Check out the original article for details. His examples include

- A **magic trick** involving a matrix of numbers — understanding why you always end up with the same sum involves a surprising moment of insight.
- A notorious **paradox** that has been dubbed the Monty Hall problem — it's not a logical paradox but rather a crisis of intuition, where a seemingly simple probability problem sparks fierce debates, even among mathematicians.
- A classic 3D **puzzle** called the Soma Cube, invented in 1933, is in which 7 pieces must be assembled to form a 3x3x3 cube or other interesting shapes. It's a perenially popular toy enjoyed by kids and adults alike.

These experiences have four qualities that set them apart from typical classroom exercises:

- **Entertaining**. All these experiences hook the audience with a riveting premise that keeps you on the edge of your seat. In contrast, typical classroom experiences give you little reason to care other than "it will be on the test."
- **Exciting**. These experiences evoke curiosity, awe, and excitement — emotions which open you up to learning. Typical classroom experiences evoke boredom, anxiety, and fear of failure.
- **Participatory**. Curiosity leads naturally to classroom discussion, where students are eager to understand what is happening. Conventional math education overexplains what you are supposed to learn, leaving no room for natural curiosity or divergent opinions.
- **Approachable**. Finally, these experiences never overload the audience with difficult prerequisites. The elements of the problem are familiar and easy to understand. In the case of the Soma cube, the pieces are physical, making them pleasurable to touch and handle.

Together, these four qualities make magic tricks, paradoxes, and puzzles ideal ways to engage learners at the start of a lesson. Teachers are rarely trained in the art of creating captivating experiences, but as every teacher knows, their first responsibility is to engage the students' attention.

Now, let's look at the five barriers to the widespread adoption of recreational mathematics.

1. Lack of Exposure

Barrier: most people have never been exposed to joyful mathematics. That includes both students and teachers. Their only experience of mathematics is through school, and most of that is boring, anxiety-provoking, and meaningless. When older kids and adults finally experience joyful mathematics, they are often pissed — why didn't I know this sooner?

Solution: This is the most basic barrier, and it's the easiest to overcome. We must create more joyful mathematics experiences and get them in front of kids.

That's what the Julia Robinson Math Festival is doing with its in-school festivals and what the Museum of Mathematics in New York City is doing with its museum and many outreach programs for kids and adults. It's what Thinkfun is doing with its wonderfully designed puzzle toys for families and books like The Number Devil, and You Can Count on Monsters are doing for young readers.

Which leads us to...

2. Lack of Distribution

Barrier: we need to reach more people, especially those scared off by the word "math."

Solution: distribute through the most widely viewed channels, like YouTube. And use language and presentation styles that appeal to people who don't like math.

The most successful example of excellent distribution for an inventive educational experience is Sesame Street, which exploded on public television in 1969.

These days, math YouTubers like Grant Sanderson of 3 Blue 1 Brown and Derek Muller of Veritasium are making a serious dent in math education by producing seriously entertaining and highly personal videos on deeply mathematical topics.

YouTuber Vi Hart has done an outstanding job of reaching young women who don't love math by leading with a voiceover about how boring math class is.

But much great recreational mathematics needs better distribution. I recently hooked up with Alex Rosenthal at TEDed, which has produced a spectacular video library of over 100 mathematical riddles. With over 19 million subscribers, they have certainly reached a large audience. But like many nonprofit educational institutions, their funding is used primarily to pay their small staff, which means their beautifully produced material is less well known than it should be.

I think the lack of attention to distribution and marketing stems from the fact that teachers teach to a captive audience and thus don't have to market themselves. In contrast, the market forces on YouTube, where everyone competes for the viewer's attention, force content creators to design videos that hook and hold the viewer's attention — the best YouTube math videos tell great stories.

Next, we have the active barriers — forces that actively resist change.

3. Mental Model of Math

Barrier: if you believe the narrow definition of mathematics taught in school, you will reject anything playful as "not real mathematics."

Solution: widen people's understanding of what mathematics is.

When I take math games into classrooms, students and teachers have a good time and generally understand the value of what they are experiencing. But then, class goes back to "normal". Teachers treat recreational mathematics as a brief break from real mathematics rather than an integral part of good education.

The underlying reason for this disconnect is that school trains people to believe that mathematics consists only of memorizing and accurately reproducing canned formulas. Under that definition, any experience that involves ambiguity, asking questions, or creativity does not qualify as mathematics.

A similar situation once existed in English class. When I went to elementary school, I learned the rules of grammar and how to write an outline before writing a paper. Completely missing from this curriculum are the much messier things that real writers do — searching for topics, drawing mind maps, and most importantly rewriting. Thank goodness school now teaches the complete writing process, through programs like the widely adopted Readers and Writers Workshop programs, published by Heinemann.

We need to do the same for mathematics. And indeed, Heineman also publishes a "cognitively guided instruction" math program with the same philosophy as Readers and Writers Workshop. Doing math is more than memorizing formulas and computing correct answers. Doing math also includes noticing patterns, being curious, asking questions, trying things out, being wrong, and trying again. Under that much broader definition of mathematics, everyone is already a mathematician.

4. Curriculum Standards

Problem: curriculum standards leave no room for recreational mathematics

Solution: revise the standards and build recreational mathematics into the core curriculum.

Teachers resist including more recreational mathematics in their teaching — even the ones who love it — because they don't have time. Teachers are under increasing pressure to stick to state-mandated curriculum standards that prescribe precisely what to teach, and when. As a result, teachers race through a curriculum that is "a mile wide and an inch thin", without pausing to make sure that kids understand what they are learning.

California — often the leader in social change — is now embroiled in a highly contentious effort to revise state math curriculum standards. Parents and teachers are wary of new standards, and for good reason — every math reform I've lived through has been royally botched. For instance, the recent Common Core standards were originally drafted by state legislators who had no expertise in education (educators swooped in to triage the damage), and implemented without any funds or plans for producing revised textbooks or retraining teachers.

To move things forward, we need successful examples of progressive mathematics education that meet accepted standards. I'd like to see radically different approaches to teaching math that produce twice the results in half the time, with greater engagement and retention.

Finally, we come to the most stubborn barrier of all.

5. Societal Norms

Barrier: the widespread belief that math is a painful subject that must always be taught in the same way.

Solution: start a social movement that lets people heal math abuse and reclaim mathematical empowerment.

Math anxiety is a generational trauma passed on from one generation to another by parents and teachers who never learned to love math when they were kids. Math empowerment needs to become a society-wide movement. People need to heal from mathematical trauma, reclaim their right to a healthy relationship with mathematics, and know that mathematics can be joyful.

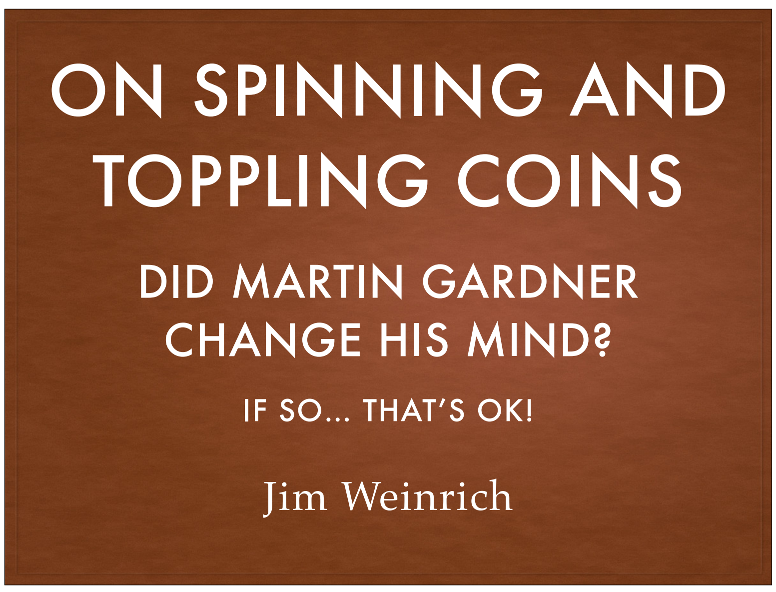
This movement is already happening with STEM, but somehow, math gets left out in the excitement of promoting science, technology, and engineering. M may not be as obviously flashy as STE, but you can't have STE without M.

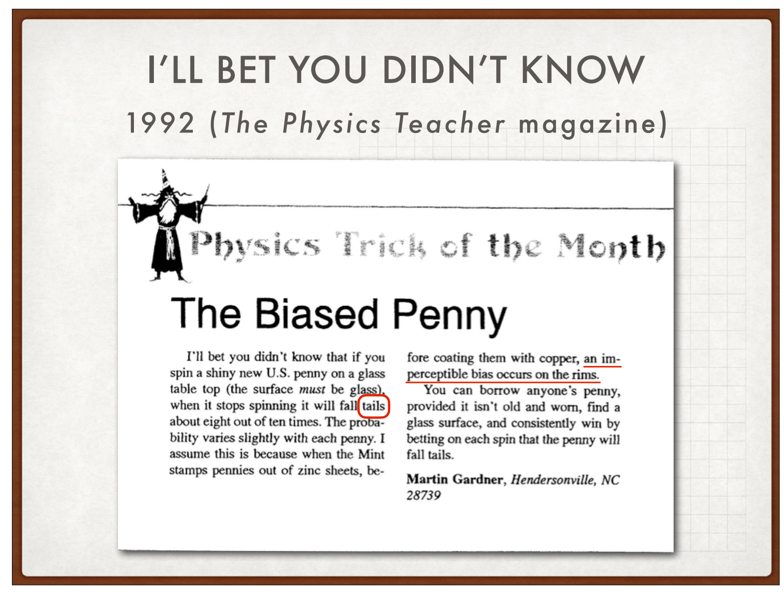
Starting a social movement is a grassroots effort that requires many leaders. I look to other examples of social change movements, like abolishing slavery and allowing women to vote, for inspiration. Within education, society already transformed literacy from a skill possessed by a few scribes to a skill enjoyed by the entire population. We can do the same for math.

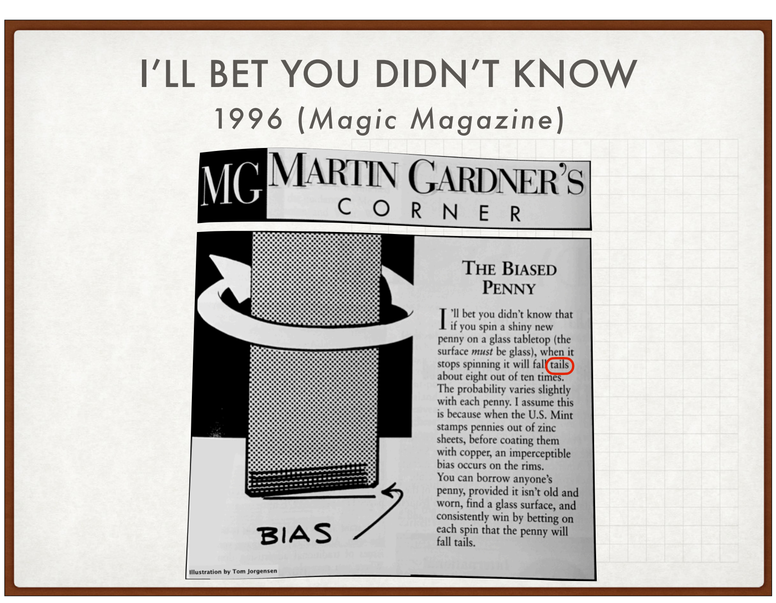
Don't be discouraged by the scale of the problem. Instead, be energized by the scale of the opportunity. As Margaret Mead once said, "Never doubt that a small group of thoughtful, committed citizens can change the world; indeed, it's the only thing that ever has."

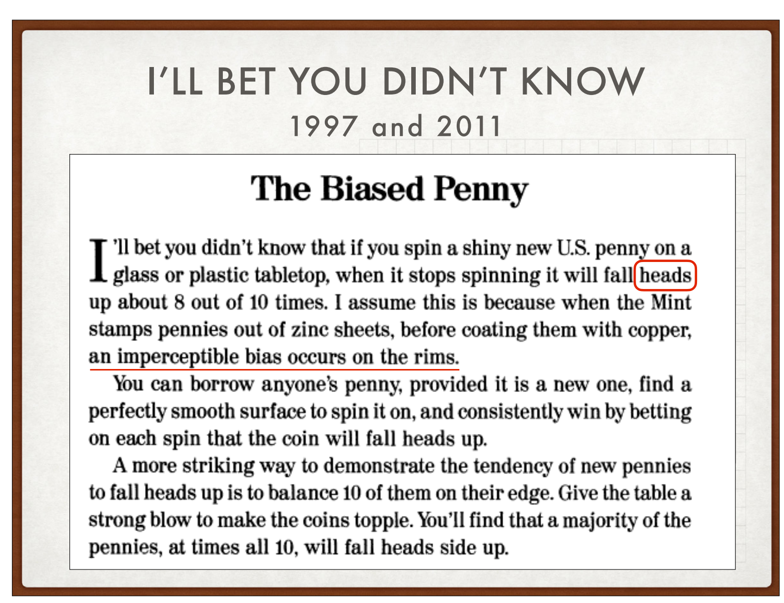

G4G 15
ON SPINNING AND TOPPLING COINS

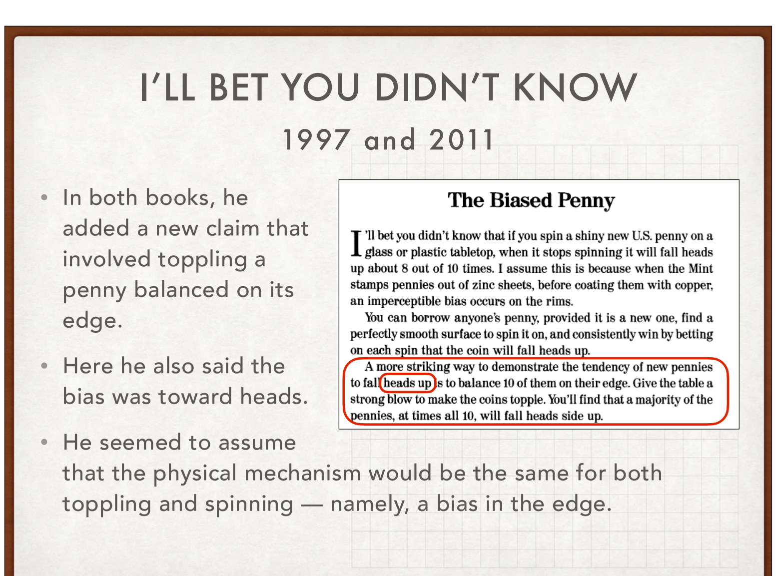

ON SPINNING AND TOPPLING COINS
DID MARTIN GARDNER CHANGE HIS MIND?

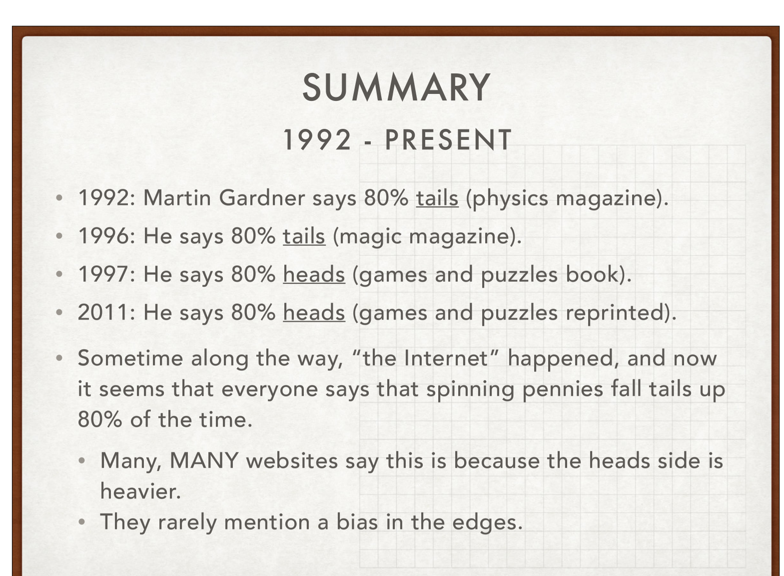

ON SPINNING AND TOPPLING COINS
DID MARTIN GARDNER CHANGE HIS MIND?
IF SO… THAT'S OK!
Jim Weinrich


I'LL BET YOU DIDN'T KNOW
1992 (The Physics Teacher magazine)
Physics Trick of the Month
The Biased Penny
I'll bet you didn't know that if you spin a shiny new U.S. penny on a glass table top (the surface must be glass), when it stops spinning it will fall tails about eight out of ten times. The probability varies slightly with each penny. I assume this is because when the Mint stamps pennies out of zinc sheets, before coating them with copper, an imperceptible bias occurs on the rims.
You can borrow anyone's penny, provided it isn't old and worn, find a glass surface, and consistently win by betting on each spin that the penny will fall tails.
Martin Gardner, Hendersonville, NC 28739


I'LL BET YOU DIDN'T KNOW
1996 (Magic Magazine)
MG MARTIN GARDNER'S CORNER
THE BIASED PENNY
BIAS


I'LL BET YOU DIDN'T KNOW
1997 and 2011
The Biased Penny
I'll bet you didn't know that if you spin a shiny new U.S. penny on a glass or plastic tabletop, when it stops spinning it will fall heads up about 8 out of 10 times. I assume this is because when the Mint stamps pennies out of zinc sheets, before coating them with copper, an imperceptible bias occurs on the rims.
You can borrow anyone's penny, provided it is a new one, find a perfectly smooth surface to spin it on, and consistently win by betting on each spin that the coin will fall heads up.
A more striking way to demonstrate the tendency of new pennies to fall heads up is to balance 10 of them on their edge. Give the table a strong blow to make the coins topple. You'll find that a majority of the pennies, at times all 10, will fall heads side up.


I'LL BET YOU DIDN'T KNOW
1997 and 2011
In both books, he added a new claim that involved toppling a penny balanced on its edge.
Here he also said the bias was toward heads.
He seemed to assume that the physical mechanism would be the same for both toppling and spinning — namely, a bias in the edge.
The Biased Penny


SUMMARY
1992 - PRESENT
1992: Martin Gardner says 80% tails (physics magazine).
1996: He says 80% tails (magic magazine).
1997: He says 80% heads (games and puzzles book).
2011: He says 80% heads (games and puzzles reprinted).
Sometime along the way, "the Internet" happened, and now it seems that everyone says that spinning pennies fall tails up 80% of the time.
Many, MANY websites say this is because the heads side is heavier.
They rarely mention a bias in the edges.

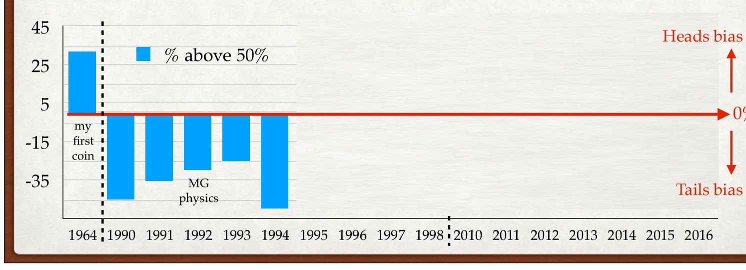
I'LL BET YOU DIDN'T KNOW

MY DATA

- The first time I read Martin's book, I pulled a penny out of my pocket, and it contradicted what Martin had said.
  - But it was an old, brown penny from 1964.
- Once I started teaching statistics, I found that spinning pennies in class was easy to turn into a class project.
- Statistics-class pennies showed no bias, or Martin's bias of 20/80, or the opposite bias of 80/20.
- And I found that Martin's change of mind could possibly have been based on actual data!

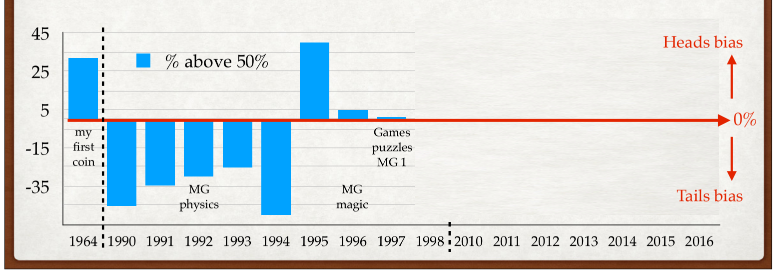
IT DEPENDS ON THE YEAR

- I obtained pennies from various years, and started spinning.
- And spinning. ☹ AND SPINNING. ☹☹☹☹
- It turns out that there was indeed a tails bias in the 1990s that continued up through 1994.
- Martin was correct in his first column (physics magazine).



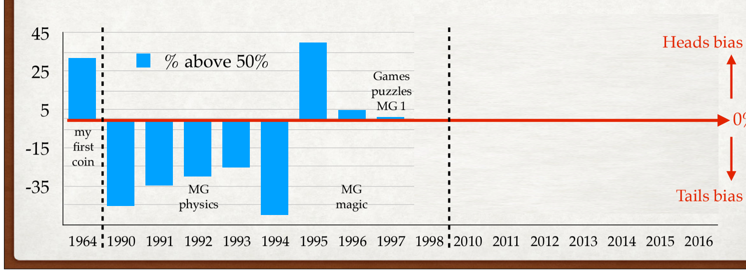
IT DEPENDS ON THE YEAR

- But something changed in 1995's pennies.
- About then, Martin was presumably writing the book that stated a heads bias (published 1996), not a tails bias.
- And my question to Jim Gardner is:
  You and your brother were too old to be spinning pennies for your dad in the 1990s; do you know how he got the data?



IT DEPENDS ON THE YEAR

- And what happened thereafter? It evened out!
- Except: in 2012 it restored a tails bias, and...
  - In 2013 it went to a heads bias, and...
  - After that it evened out again.



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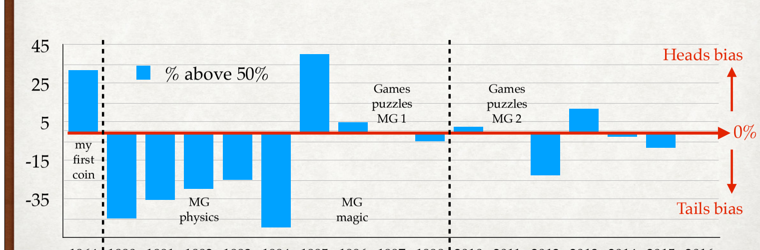


AND FINALLY, THE PHYSICS

- In closing, let me note that the physics of spinning coins is deceptive.
- Do spinning coins usually fall with their heavier side down?
- NO!

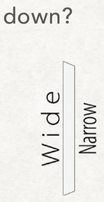
AND FINALLY, THE PHYSICS

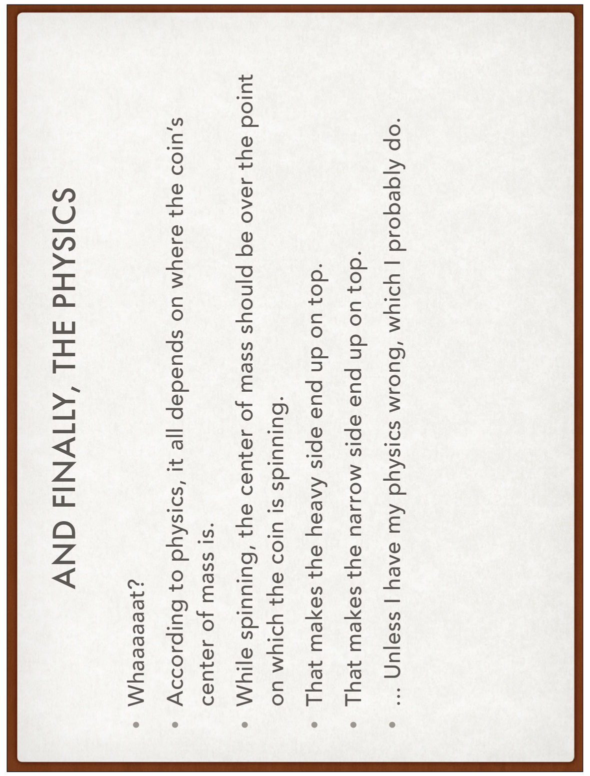
- In closing, let me note that the physics of spinning coins is deceptive.
- Do spinning coins usually fall with their heavier side down?
- NO!

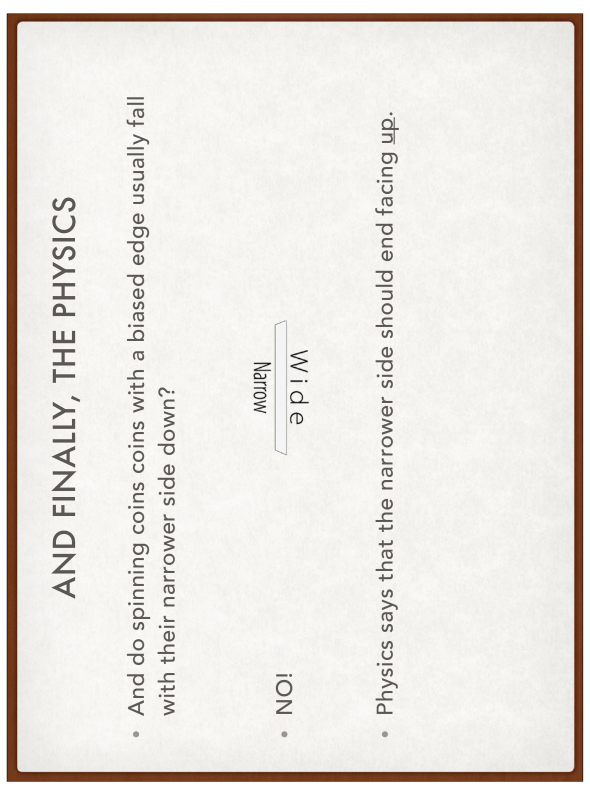
- Physics says that the heavier side should end up.
- Moreover, the heads side of a penny is probably the lighter side — so "the Internet" is doubly wrong.

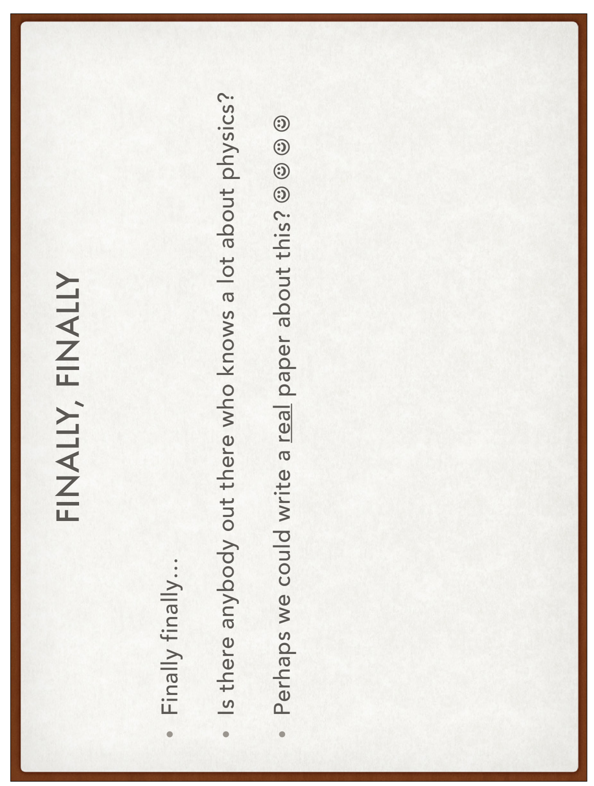
AND FINALLY, THE PHYSICS

- And do spinning coins coins with a biased edge usually fall with their narrower side down?
- NO!









SCIENCE

Introducing the Mesmoid | Kenneth Brecher | Page 340

Introducing the MESMOID™

Kenneth Brecher
Departments of Astronomy and Physics
Boston University
Boston, MA 02215

brecher@bu.edu

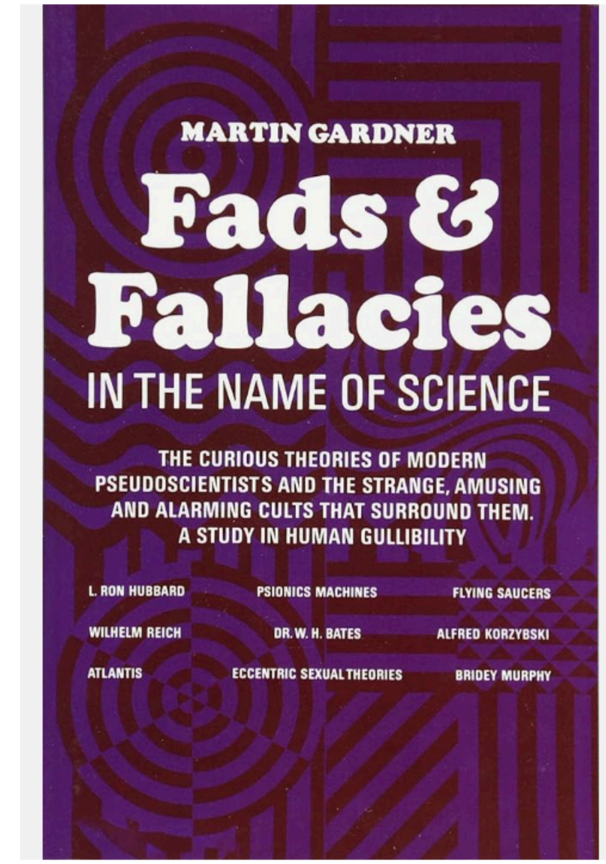
https://www.siriusenigmas.com/

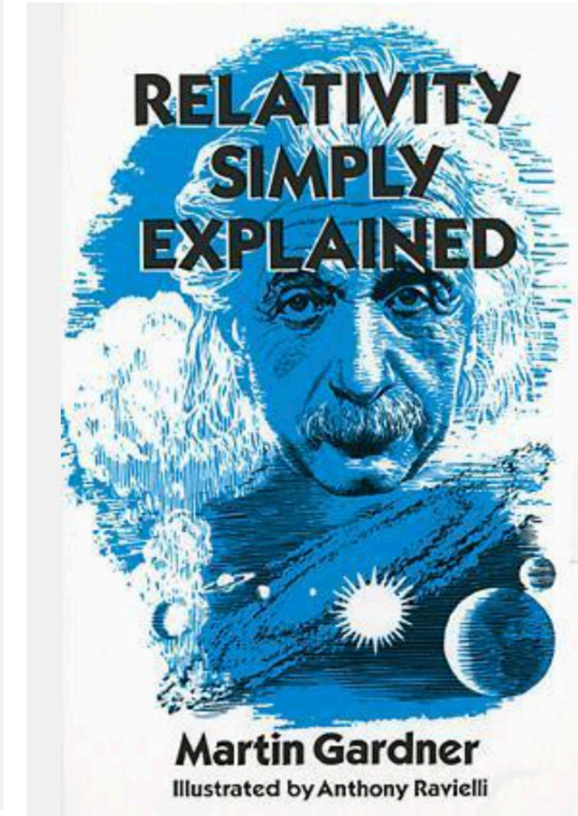
Introduction

This is a talk about a new kind of pendulum: a “rolling pendulum”. Why a talk about a pendulum at G4G15? Martin Gardner wrote several pieces about pendulums in physics and about dowsing rods in pseudoscience:

CHAPTER 9

Dowsing Rods and Doodlebugs





What is the MESMOID™?

It is a right circular cylinder with an eccentric circle cut out of it. Because of its design, it can act like a rolling pendulum. Unlike a coin or a wheel, when rolled it moves back and forth instead of in one direction only.

https://photos.google.com/share/AF1QipOViADNr9JeMAOK2VCexdsKuy5NrWgSGQBw0Ul5-a8IDieUYURyngOs4HLwM_QeqA/photo/AF1QipO-1s0xTmZGirYx0nOgKOfTvKvcV3cpOaUjr0O_?key=cEE2QnA0a2g4OFh0V2RCdTl0SDdNUW9XX1FsaVp3

patent pending

It is a miniature work of kinetic art.
It is an executive fidget device.
It is a tactile stress reliever.
It is an elegant paperweight.
It is a companion piece to the
ϕTOP®, πTOP®, *e*TOP, *i*TOP & δCELT®.

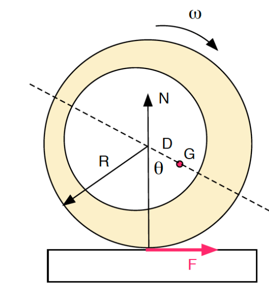
Roll the MESMOID™ on a hard, smooth surface and its back-and-forth motion is mesmerizing. Notice that it sometimes moves in a slightly curved path. Why???

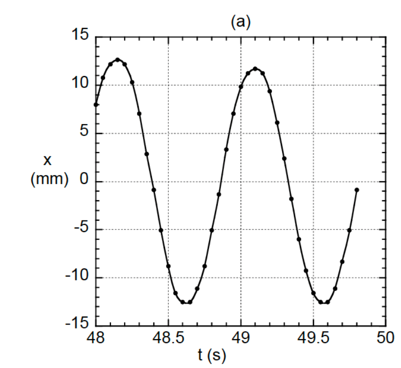
A short video showing the MESMOID™ rolling on a black marble surface in front of a version of "Newton's Cradle" pendulums can be seen here:

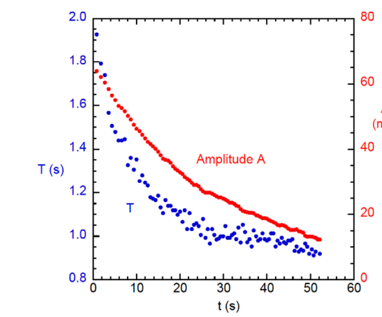
https://www.youtube.com/watch?v=qCjtNJk0Ndk

MESMOID Physics

***from* "Pendulum Motion of an Eccentric Disk" by R. Cross and K. Brecher, TPT, in press, 2024**







Measured period, T, and amplitude, A, vs time.

Acknowledgements

Many thanks to Heitor Mourato of the BU Scientific Instrument Facility, Rod Cross of the University of Sydney, and especially to Kaz Brecher

More information about the MESMOID™ can be found at:

www.siriusenigmas.com

If you want one, you can get it here:

https://www.etsy.com/shop/SiriusEnigmas

G4G COLLEGIALITY STATEMENT

G4G is a gathering of people of all ages from many disciplines with different interests. However, we all have one thing in common and that is our love of Martin Gardner and his legacy which we are here to celebrate. We are here to share our ideas and learn about yours. We are here to see old friends and make new ones and to have a great time!

Some of us have been to all of the Gatherings and some of us are here for the first time. So, here are a few guidelines to ensure that we all have a great time.

Please welcome new attendees, especially the young attendees who might not find it easy to engage a more senior attendee in conversation.

We welcome and encourage diversity and different opinions and ideas, but G4G is not a political platform.

We encourage mutual respect and ask that you refrain from promoting bias and stereotypes about race, nationality, religion, gender and sexual orientation, or other personal characteristics.

If you feel uncomfortable at any time during the meeting because of unbecoming behavior, please notify a G4G Board Member who will address the issues confidentially.

Thank you and enjoy the gathering.

www.Gathering4Gardner.org

Presented by G4G Foundation