

THE PREHISTORY OF NIM GAME

The aim of this paper is to recount the ancestors of the Nim game, taking as a reference the one introduced by Charles Leonard Bouton in his article, 1901.¹ We will see that the Nim already existed in a different form, which is called nowadays the additive version or the one pile game, and our purpose will be to follow its evolution over the centuries.

I. The Origin of the Name “Nim”, Quite a Nimstory!

In a first paragraph, we will see that the word “Nim” appeared with Bouton. Yet, it would be naïve to believe that the Nim was completely invented by Bouton and that it did not exist in other forms or under other names before 1901. Games as simple as Nim, which does not require any board game nor perennial stand, are handed over verbally; therefore this orality is difficult to track and rewriting the story and the evolution of these games is the most often a laborious task.

A. The Germanic Origin

After Bouton’s article was published, a lot has been extensively written about the name “Nim” and its possible origins. A link was pointed out between the Nim and the Fan-Tan,² a game of Chinese origin; but it was wrong, as chance is part of the Fan-Tan. Richard Epstein³ explains the Eastern link by the simplicity of the Nim’s structure and the strategically subtle moves when it comes to a mathematical point of view. Fan-Tan also seemed to have been played by American students as early as the end of the 19th century.⁴

¹ Charles, Bouton, “Nim, A Game with a Complete Mathematical Theory”, *The Annals of Mathematics*, 2nd

² Martin, Gardner, *Hexaflexagons and other mathematical diversions The First Scientific American Book of Puzzles and Games*, 1958, Chicago and London, Revised Edition, The University of Chicago Press, 1988. p. 151.

Raymond, Archibald, “The Binary Scale of Notation, A Russian Pleasant Method of Multiplication, The Game of Nim and Cardan’s Rings”, *The American Mathematical Monthly*, 25.3 (March 1918): 132-142. p.141.

Claude, Berge, “Théorie générale des jeux à n personnes”, *Mémorial des sciences mathématiques*, Fascicule 138, 1957, pp. 1-114, Paris, Gauthier-Villars, 1957. p. 28.

³ Richard, Epstein, *The Theory of Gambling and Statistical Logic*, Elsevier, 2nd Edition, 2009. p. 334.

⁴ James, Glimm, John, Impagliazzo, Isidore, Singer (Editors), *The Legacy of John von Neumann, Proceedings of symposia in pure mathematics*, Vol. 50, American Mathematical Society, 1990. p. 283.

It was not before May 1953 that Alan Ross⁵ denied the Chinese origin of Nim in a short note published in the *Mathematical Gazette*, in which he explained that Chinese names never end with a –m. Nevertheless, this note did not prevent from the confusion being made between the two games. According to Ross, the most likely explanation would be that the word Nim comes from the imperative of the German verb *nehmen*, which means to take (the imperative being *nimm*) and that Bouton would have chosen this word remembering his academic years spent in Leipzig where he received his PhD.⁶

This short note of May 1953 was not to go unnoticed, and in December of the same year,⁷ Joseph Leonard Walsh, a student then a colleague of Bouton at Harvard University, sent a message to the editor of the *Mathematical Gazette* in order to confirm the German origin of the word Nim, explaining that *nimm* was frequently used during a game. And there was more to come. Eight years later, still in the *Mathematical Gazette*, in October 1961, N. L. Haddock⁸ published a note about Alan Ross’s note and suggested a link between the Nim game and the Mancala,⁹ taking as a basis Harold James Ruthven Murray’s work, *A history of board-games other than chess* (1952). Haddock stated that according to Murray, the Mancala game would be of Egyptian or Arab origin, and Nim would be a derived form¹⁰ for the European and American continents. In short, there were many various speculations about the origins of the Nim game and its name:

Did [Bouton] have in mind the German *nimm* (the imperative of *nehmen*, “to take”) or the archaic English “nim”¹¹ (“take”), which became a slang word for “steal”? A letter to *The New Scientist* pointed out that John Gay’s *Beggar’s Opera* of 1727 speaks of a snuffbox “nimm’d by Filch”, and that Shakespeare probably had “nim” in mind when he named one of Falstaff’s thieving attendants Corporal Nym. Others have noticed that NIM becomes WIN when it is inverted.¹²

⁵ Alan, Ross, “Mathematical Note No. 2334 : The Name of the Game of Nim”, *The Mathematical Gazette*, 37.322 (May 1953): 119-120.

⁶ Martin Gardner also considered this possibility in a work in 1983 in which he devoted a chapter to the Nim game and the Hackenbush. See Martin, Gardner, *Wheels, Life and Other Mathematical Amusements*, New York, W. H. Freeman Company, 1983. p. 143.

⁷ Joseph, Walsh, “Correspondence: The Name of the Game of Nim”, *The Mathematical Gazette*, 37.322 (December 1953): 290.

⁸ N. L., Haddock “Mathematical Note No. 2973 : A note on the game of Nim”, *The Mathematical Gazette*, 45.353 (October 1961): 245-246.

⁹ Mancala comes from the Arabic *mākala* and refers to the same game than the *Awélé* (with the exceptions of some variations) or the *Wuri*. Actually, the name depends on the geographical place we are. The game board consists of two parallel rows of six holes, each hole containing the same number of seeds. The aim of the game is “to sow” the seeds, in turns, redistributing them in other holes and taking them when they move from one row.

¹⁰ Indeed, conversely to Mancala, the Nim game does not need any board with holes and its aim is not to win as many points as possible.

¹¹ Richard Epstein uses the word *niman* (and not *nim*) as the verb from the archaic English meaning “to take” or “to steal”. Richard, Epstein, *The Theory of Gambling and Statistical Logic*. p. 335.

¹² Martin, Gardner, *Wheels, Life and Other Mathematical Amusements*. p. 143.

The true story still remains nebulous and will remain so, unless new information about what Bouton had in mind when he wrote his article emerged.

B. The Game “la lulette”, “les luettes”, or “l’alulette”¹³

The game “l’alulette” or “jeu de la vache” is a French card game mainly played in rural and coastal areas between Gironde and the Loire estuary, in a geographical area that is under the influence of the dialects of the Poitou and of Saintongeais. It is a trick-taking game played by four players (two teams of two) with forty-eight Spanish-looking cards, one of which displaying a cow, hence the name of the game “jeu de la vache”. This game was still played in cafés around 1960 but is given up today. It seems it was introduced in France during the 16th century. Indeed we can find the game “les luettes” in François Rabelais’s *Pantagruel* (1532), *Gargantua* (1534) and the *Cinquième livre* (1564). Yet, the rules of the game are not precisely defined and consequently, nothing enables us to state that the game “les luettes” in Rabelais’s works corresponds with the game “l’alulette” as it is known today. But in one of the first French-English dictionaries, published in 1611 by Randle Cotgrave (born 16th, dead in 1634), *A Dictionarie of the French and English Tongues*,¹⁴ we find at the headword “luettes” the following definition: “Luettes. Little bundles of peeces of Ivoirie cast loose upon a table; the play is to take up one without shaking the rest, or else the taker looseth.”¹⁵ Studies were conducted in order to highlight the influence of Rabelais on the interpreters, readers and emulators of the French language, more especially on Cotgrave. In a work dated 1930, Lazare Sainéan explained: “At the beginning of the 17th century, Randle Cotgrave, an English lexicographer, undertook to explain all the specificities of Rabelais’ language, and this significant work (1611) remains nowadays one of the main sources for the comprehension of Rabelais’ works.”¹⁶ “Above all, Cotgrave is Rabelais’ glossary, his first and sole interpreter in the lexicography area.”¹⁷ This suggests that the game “les luettes” in Rabelais’ work consisted of chip stacks and that the initial idea was to take away some chips without moving the others.

¹³ <http://fr.wikipedia.org/w/index.php?title=Alulette&oldid=89760744> [12.11.2013]

¹⁴ Randle, Cotgrave, *A Dictionarie of the French and English Tongues*, London, 1611.

¹⁵ Idem. <http://gallica.bnf.fr/ark:/12148/bpt6k50509g/f588.image.r=cotgrave%20randle.langFR>

¹⁶ Lazare, Sainéan, *L’influence et la réputation de Rabelais*, Paris, Librairie Universitaire J. Gamber, 1930. p. 34: “Au commencement du 17^{ème} siècle, un lexicographe anglais, Randle Cotgrave, s’attacha à expliquer toutes les particularités de la langue de Rabelais, et ce travail considérable (1611) reste encore aujourd’hui une des principales ressources pour l’intelligence de l’œuvre.” My translation.

¹⁷ Idem. p. 82 : “Cotgrave est avant tout le glossateur de Rabelais, son premier et unique interprète dans le domaine de la lexicographie.” My translation.

These rules are closer of those of a game like the Mikado but Bouton may have drawn his inspiration from this ancient game and abandoned the *physical skill* side in favour of a more strategic angle. This kind of transformation can also be observed in the game of Kayles by the famous puzzlist Henry Dudeney: originally, it was a simple bowling game of skill, then it turned into a more mathematical parlour version.

II. Mathematical Recreations in Europe

A. The Birth of Mathematical Recreations

Mathematical recreations can be considered as a new genre, halfway between pure recreation, the full educational tool, and the launch of challenges between scientists. These “marginal”¹⁸ mathematics have not had the same aims over time:

The first recreational works date back to the 1620s, with *Les problèmes plaisants et délectables sur les nombres* by Claude Gaspard Bachet de Méziriac and *Les Récréations mathématiques composées de plusieurs problèmes plaisants et facétieux* by Henry von Etten, while the famous Jacques Ozanam’s *Récréations mathématiques et physiques* were published in 1692. The aim of the ancient recreations was above all to “pique one’s curiosity”, whereas those that appeared at the end of the 19th century and the beginning of the 20th had three other purposes.

The first was to teach mathematics [...] The second was to circulate new mathematics [...] The third aim was to educate with sharing recent historical researches [...] Moreover, these recreations sometimes explicitly compensate for the weaknesses that, at the end of this century [19th], France often acknowledged in the field of its mathematic research as well as in public education.¹⁹

We will see that this new genre significantly developed thanks to Bachet’s and Ozanam’s works during the 17th century, but yet that Luca Pacioli introduced it at the end of the 15th century. Our study mainly deals with the first books of mathematical recreations, those that

¹⁸ “à la marge” : Evelyne, Barbin, “Les Récréations : des mathématiques à la marge”, *Pour la Science* (February-March 2007): 22-25. p. 22. My translation.

¹⁹ Idem. “Les premiers ouvrages de récréations datent des années 1620, avec *Les problèmes plaisants et délectables sur les nombres* de Claude Gaspard Bachet de Méziriac et les *Récréations mathématiques composées de plusieurs problèmes plaisants et facétieux* d’Henry von Etten, tandis que les fameuses *Récréations mathématiques et physiques* de Jacques Ozanam sont publiées en 1692. Ces récréations anciennes ont surtout pour but de « piquer la curiosité », tandis que celles qui paraissent au tournant des XIX^e et XX^e siècles ont trois autres motifs.

Le premier motif est d’instruire aux mathématiques [...]. Le deuxième est de diffuser des mathématiques nouvelles [...]. Le troisième est de cultiver en faisant connaître les recherches historiques récentes. [...] De plus, ces récréations suppléent parfois explicitement aux déficiences que la France de cette fin de siècle [le 19^{ème}] se reconnaît souvent, aussi bien dans sa recherche mathématique que dans son instruction publique.” My translation.

were aimed to “pique one’s curiosity”. We will focus on recreational puzzles presented as a game opposing two players who take turn trying to reach a given number, n , with adding up numbers in the range from 1 to k . This recreational problem is a former version of Bouton’s Nim and the player who knew the solution could easily impress his opponent by displaying his mastery during every game; it was a way to be a social success, even to have a hold over the people who did not have the key. This tendency can be observed too, as we will see further on, in the practice of Tiouk-Tiouk in West Africa.

B. Luca Pacioli (Italy, 1508)

A simplified variation of the Nim game appeared in Europe for the first time during the Renaissance with the Italian mathematician Fra Luca Bartolomeo de Pacioli (1445-1517) and his treatise *De Viribus Quantitatis* written between 1496 and 1508. Pacioli was one of the most famous mathematicians of his time.²⁰ According to David Singmaster,²¹ the *De Viribus Quantitatis* can be considered as one of the first works entirely devoted to mathematical recreations. Pacioli began to write this manuscript in Milan where he was teaching between 1496 and 1499.²² He was then at the height of his career, a prominent member in the intellectual circle of the Duke of Milan, Ludovico il Moro (1452-1500).²³ The opisthographic manuscript, kept at the Bologna University Library, is written in Italian²⁴ and consists in 309 sheets, 24 by 16,5 cm (9.4”x6.5) organised into three parts. The first includes 120 arithmetical recreations (*Delle forze naturali cioè de Arithmetica*, only 81 problems are indexed), the second part consists of 139 problems dealing with geometry and topology (*Delle forze naturali cioè de Arithmetica*, 134 problems are indexed) and the third part contains a few hundred proverbs, poems, riddles and puzzles (*De documenti morali utilissimi*, 85 are indexed). One of Pacioli’s key aims was to reveal the power of numbers and to demonstrate that they could be understood in a concrete way with the card games, dice, Tarot and board games he proposed. One problem of the first part (*XXXVIII*, see Fig. 1), considered as the first

²⁰ In 1496, Pacioli published the most important mathematics work after Fibonacci (1202), the *Summa de Arithmetica, Geometrica, Proportioni et Proportionalità*; it was his major work. See David, Singmaster, “*De Viribus Quantitatis* by Luca Pacioli: The First Recreational Mathematics Book”, in Erik Demaine and Martin Demaine and RODGERS Tom, (Editors), *A Lifetime of Puzzles*, A K Peters, Ltd, Wellesley, 2008. pp. 77-122.

²¹ *Ibid.* p. 77.

²² *Ibid.* pp. 81-82. Pacioli left Milan in 1499 and went to Florence following Sforza’s overthrow by a French invasion. He taught at the universities of Florence and Pisa between 1499 and 1507.

²³ *Idem.*

²⁴ Pacioli preferred Italian to Latin.

one pile game,²⁵ is the following one: “finish any number before the opponent, without taking more than a certain finite number”.²⁶

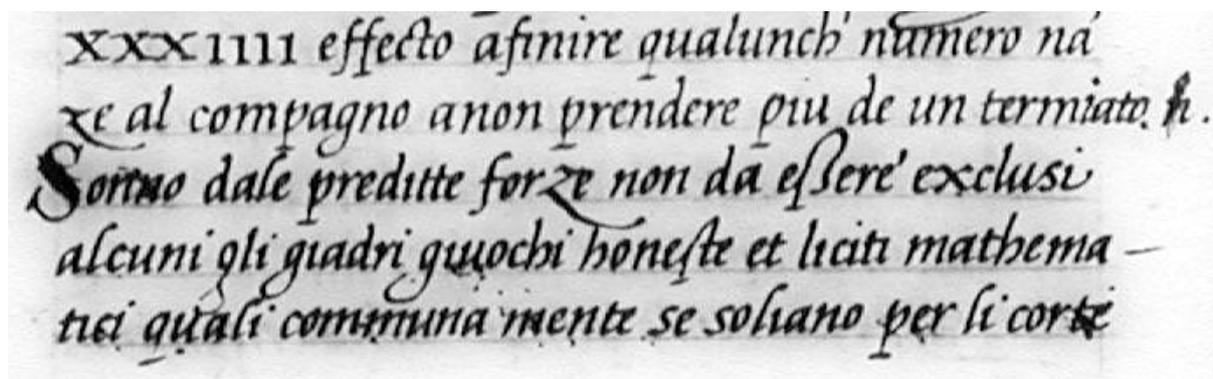


Fig. 1: XXXVIII proposed by Pacioli in the first arithmetical part of *De Viribus Quantitatis*.

Source: www.uriland.it/matematica/DeViribus/Pagine/175.JPG²⁷, page 073v

This rather unclear wording makes sense when Pacioli gives the solution: he suggests that two players reach the number 30 with adding in turns numbers ranging from 1 to 6. Pacioli justifies the number 6 as the higher number of points that can be reached with a dice:

It is customary to say among two [persons], that when you take points from a dice *alternatim* [in turn], you can take any number that you wish as long as it does not exceed 6, because in dice there is no greater [number of] points than 6; and the first person takes it on himself to get to 30 before his companion.²⁸

In fact, it is a simplified version of Bouton’s Nim game, the one that is played with only one pile consisting of 30 objects and in which each removal is limited to a maximum of 6 objects. This version is called “additive” because one adds numbers instead of reducing piles, which was often the case in the first versions of Nim. Yet, the solution remains based on the same principle as Bouton’s Nim: there exist “safe combinations” that secure the win, provided that one plays correctly. Before revealing the method to find the safe combinations, Pacioli

²⁵ *Ibid.* pp. 91-92.

²⁶ Luca Pacioli, *De Viribus Quantitatis*, 1508. p. 073v: “effecto a finire qualunch’ numero na’ze al compagno anon prendere piu de un termiato .n.” My translation.

²⁷ The whole manuscript was digitized and can be found on the following website: www.uriland.it/de-viribus-quantitatis-pages

²⁸ This translation of Pacioli’s Effect XXXVIII was kindly given to me by David Singmaster during the Board Game Studies Colloquium XVI, held in April, 3rd-6th 2013. It is part of David Singmaster’s translation (draft) of *De Viribus Quantitatis*. I warmly thank him for this help.

explains that this game is part of recreational, honest and legitimate mathematical games, which are fully entitled to be part of mathematics lessons and which everybody can go in as a mathematical recreation. Then he wonders if there is any advantage for the first or the second player to begin the game; finally, he quickly gives the winning strategy to be applied: the four safety steps, 2, 9, 16 and 23 have to be reached. At this stage, Pacioli does not explain how he has determined these safety steps, but it seems obvious that he succeeded in doing so using the following backward induction: if I don't want my opponent to win, I must put forward the highest number such as if my opponent added 1, 2, 3, 4, 5 or 6, he cannot reach 30. This number is 23; indeed, whatever number is added to 23, the resulting number is higher than or equal to 24, but remains less than or equal to 29. So, I will be able to complete up to 30 at the following turn. Reasoning in the same way with 23, the safety step before 23 is 16, then 9 and finally 2. So, I must manage to be the first to reach one of these safety steps, then the others up to 30. Pacioli does not explain safety steps by this method but he suggest a general method to find the safety steps of any game: "always divide the number that you wish to arrive at by one more than has been taken and the remainder of the said division will always be the first [step of the] progression [...]"²⁹ If the division comes out exactly and the remainder is zero, Pacioli considers that this case is more difficult and he clearly explains the backward induction he applies in a precise example. He takes the case when the number to be reached is 35 by adding numbers between 1 and 6:

[...] For 35, take away 7, and 28 remains for the [step]; the other [step] takes away 7, and 21 remains; the other [step] 14; the other [step] 7. Therefore, he takes whatever he wants up to 6, and you will take, or actually you will make 7 the first degree, and then 14, 21, 28, and 35, and so on [...]"³⁰

For Pacioli, the two situations are to be differentiated, whereas they actually need the same reasoning; they could come down to the following result: if the remainder of the division of n by $k+1$ (n being the number to reach and k the maximal number that can be added) is equal to zero, then we subtract $k+1$ to find the last safe combination, and so on until the first safety step is reached. If the remainder of the division of n by $k+1$ is different from 0, therefore it is the first safe combination and the others are determined by adding this remainder to the former safe combination. It must be noticed that Pacioli never gives this explanation by generalizing numbers and by using n and k . Indeed, it was not before 1901,

²⁹ Idem.

³⁰ Idem.

with Bouton's article, that the solution of a combinatorial game was to be formalized and studied as a general case with any number of piles and any number of chips in each pile. This is due to the belated development of algebra and the lack of an appropriate symbolism, which is required to represent unknown quantities and write equations. For centuries, men used clever arithmetical methods in order to solve problems that we would tackle nowadays with algebra. Consequently, it was then impossible to obtain general conclusions; each case was studied independently, and no formula, for which specific data to each example had only to be replaced, was developed.³¹ René Taton confirms that fact:

Consequently, the Renaissance algebra never presents *formulas*, but gives *rules* and offers *examples*. It is exactly what grammar does too, giving us *rules* that we have to follow, and *examples* that we have to comply with by declining names and conjugating verbs.³²

Actually, this aspect can be found in all the cases of additive Nim game we analyse in this work.

The *De Viribus Quantitatis* manuscript is a collection of *mathematici ludi*, i.e. recreational mathematics including games or problems through which the author wished to teach mathematics avoiding the boredom due to the repetition of exercises frequently asked.³³ Before Pacioli, other authors had the same idea, such as Fibonacci, from whom Pacioli recognized frequent borrowings, Francesco and Pier Maria Calandri, but in the other arithmetic treatises (*trattati d'abbaco*), recreational problems are simply placed here and there in the text, in order to provide a rest to the reader who is learning; consequently, Pacioli's manuscripts can be considered as the first true treatise on the subject.³⁴ Vanni Bossi points out a *magical* nature, easily intelligible, in most of Pacioli's assertions.³⁵ It would seem that Pacioli did want the secret of the method to be well kept, in order to surprise the audience; indeed secret is the fundamental principle of any magic trick. Keeping this secret is the

³¹ Vera, Sanford, *The History and Significance of Certain Standard Problems in Algebra*, Teachers College, Columbia University, New York City, 1927. p. 17.

³² René, Taton, *La science moderne de 1450 à 1800*, Presses universitaires de France, Paris, 1958 (1^{ère} édition Quadrige, avril 1995) p. 52: "En conséquence, l'algèbre de la Renaissance ne nous présente jamais de *formules*, mais nous donne des *règles* et nous offre des *exemples*. Exactement comme le fait la grammaire qui, elle aussi, nous donne des *règles* que nous devons suivre, et des *exemples* auxquels nous devons nous conformer en déclinant les noms et en conjuguant les verbes." My translation.

³³ Vanni, Bossi, "Magic and Card Tricks in Luca Pacioli's *De Viribus Quantitatis*", in Erik, Demaine and Martin, Demaine and Tom, Rodgers (Editors), *A Lifetime of Puzzles*, A K Peters, Ltd, Wellesley, 2008. pp. 123-130.

³⁴ *Ibid.* p. 123.

³⁵ *Ibid.* pp. 123-130.

essential condition for being able to amaze one's friends, especially women, "maxime donne", and those who do not know mathematic principles because they had no access to arithmetic knowledge.³⁶ This dimension of intellectual domination thanks to the control of the game is also to be found in the West African game Tiouk-Tiouk that we will talk about further.

Pacioli did not claim his originality loud and clear; some of the problems he suggested came from more ancient works or were studied or talked about in public schools at that time and handed out orally.³⁷ Some problems were even invented by his students and Pacioli encouraged them to do so. For instance, in one chapter, he mentions his disciple Carlo de Sansone from Perugia and in another one he names Catano de Aniballe Catani from Borgo, who would have played one of the problems in Naples, in 1486.³⁸ This date allows us to suppose that most of the problems in Pacioli's manuscript were in fact invented during the last quarter of the 15th century.

At the same time as *De Viribus Quantitatis*, the *Triparty en la science des nombres* by Nicolas Chuquet,³⁹ a Parisian doctor, came out in France without yet being published. This work "achieves a much higher level than the former works, even than Luca Pacioli's *Summa*."⁴⁰ Very few information is available on Chuquet, except that he was from Lyon and that he had a deep knowledge in arithmetic and algebra. Recently, it has been proved that Chuquet maintained contacts with the Italian tradition through provincial intermediaries.⁴¹ The notebook of Francesco Bartoli, an Italian entrepreneur who regularly travelled between Italy and the South of France, gives some rare proofs of the hand-over of recreational problems through Europe:

In addition to arithmetic tools such as exchange and multiplication tables, it contains a collection of thirty problems of the recreational sort. We can assume that Bartoli was only one of the many links in the trade routes by which the tradition of recreational mathematics passed from Italy to France and the Low Countries.⁴²

The *Triparty* consists of two distinct parts: the first contains the *Triparty en la science des nombres* and the second deals with the *Applications des Règles du Triparty* with some

³⁶ *Ibid.* p. 124.

³⁷ Albrecht, Heeffter, *Récréations Mathématiques* (1624), *A Study on its Authorship, Sources and Influences*, October 2004. pp. 1-37. p. 18. Available on: <http://logica.ugent.be/albrecht/thesis/Etten-intro.pdf> [20.12.2011]

³⁸ Vanni, Bossi, *Magic and card tricks...* p. 125.

³⁹ (born between 1445-55- dead 1487-88)

⁴⁰ René, Taton, *La science moderne...* p. 19.

⁴¹ Albrecht, Heeffter, p. 16.

⁴² *Idem.*

sheets devoted to *Jeux et esbatements qui par la science des nombres se font*.⁴³ Unfortunately, we have not found games that could resemble the problem suggested by Pacioli, “yet, if we cannot assert that one of the works had directly influenced the others, their similarities prove that they are all part of a same tradition.”⁴⁴ René Taton regrets that the *Triparty* had not the influence it should have had upon the development of algebra, “and it was Pacioli’s *Summa* that, during the following century, was used as a starting point, and as a secondary source, for the theoretical and practical mathematical knowledge.”⁴⁵

Determining the (hi)story of a recreational game is extremely difficult because authors did not precise whether they borrowed the idea from someone else or not. This is particularly true in the case of the French Claude-Gaspard Bachet de Méziriac.

C. Claude-Gaspard Bachet (France, 1612)

Luca Pacioli’s manuscript was kept in the archives of the University of Bologna for nearly five hundred years without being published. Maria Garlaschi Peirani⁴⁶ transcribed it into modern Italian in 1997, and in 2007, according to *The Guardian*, a translation into English had been undertaken.⁴⁷ The original book had rarely been consulted since the Middle Ages but it was undoubtedly a reference for later works. Actually, the additive version of Nim crossed centuries and frontiers and it was to appear again in various books of recreational mathematics as early as the 17th century. It is the case of *Problemes plaisans et delectables, qui se font par les nombres*,⁴⁸ the famous book by Claude-Gaspard Bachet known as de Méziriac (1581-1638), which is often considered as the first work on mathematical recreations (Fig. 2).

⁴³ Aristide, Marre, *Le triparty en la science des nombres par maistre Nicolas Chuquet Parisien publié d’après le manuscrit Fonds Français N°. 1346 de la Bibliothèque Nationale de Paris et précédé d’une notice par M. Aristide Marre*, Rome, 1881. p.32: “games and recreations, which are played thanks to the science of numbers.” My translation.

⁴⁴ René, Taton, *La science moderne...* p. 19: “toutefois, si l’on ne peut affirmer que l’un de ces ouvrages ait influencé directement sur les autres, leurs similitudes démontrent qu’ils appartiennent à une même tradition.” My translation.

⁴⁵ *Ibid.* pp. 21-22: “et ce fut la *Summa* de Pacioli qui, pour le siècle à venir, servit de point de départ, et de source secondaire, du savoir mathématique théorique et pratique.” My translation.

⁴⁶ Luca, Pacioli, *De viribus quantitatis*, trascrizione di Maria Garlaschi Peirani dal Codice n. 250 della Biblioteca universitaria di Bologna ; prefazione e direzione di Augusto Marinoni, 1997.

⁴⁷ <http://www.theguardian.com/world/2007/apr/10/italy.books>

Lucy, McDonald, “And that’s Renaissance magic...” *The Guardian*, 10 April, 2007.

As far as I know, no official translation has been published yet, conversely to what was announced in the article.

⁴⁸ Claude-Gaspard, Bachet, *Problemes plaisans et delectables, qui se font par les nombres*, Lyon, 1st edition, 1612.

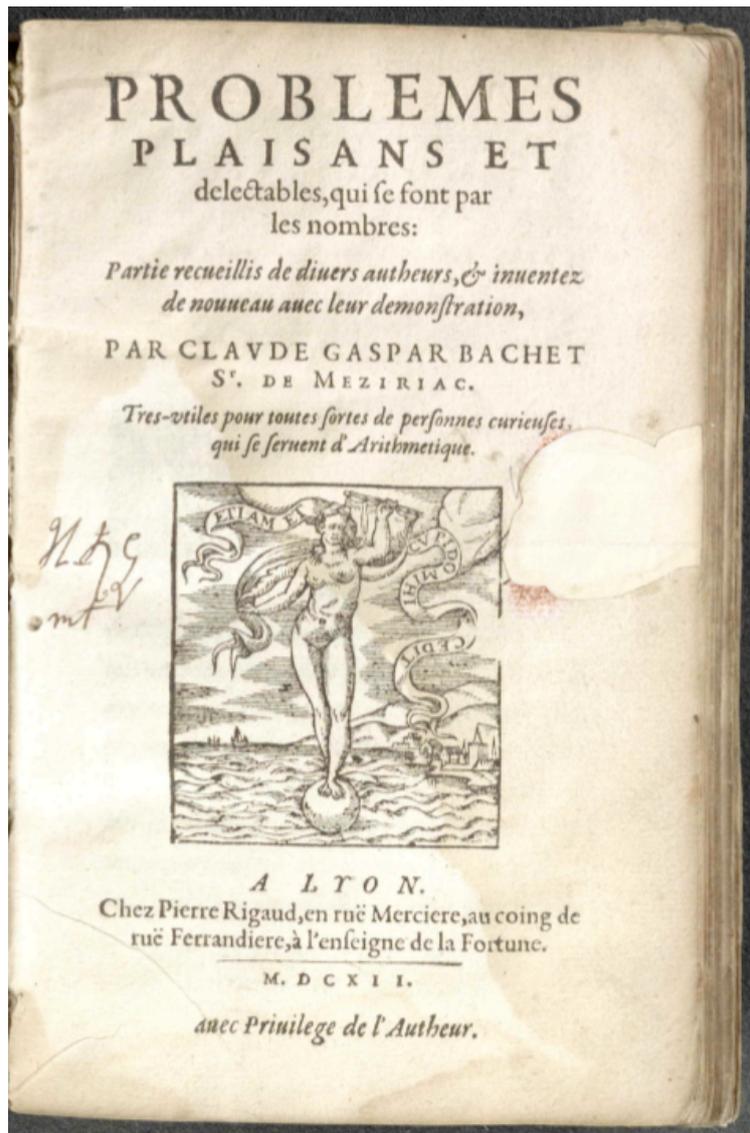


Fig. 2: Frontispiece of *Problemes plaisans et delectables, qui se font par les nombres* de Claude-Gaspard Bachet, first edition, 1612.

Source: Claude-Gaspard, Bachet (1612)

It was indeed the first book on the subject to be published, but the original idea of a collection of recreational mathematic problems was due to Pacioli.⁴⁹ Bachet was a mathematician and a French translator (Fig. 3), known for publishing the Greek text of Diophante's *Arithmétique* (3rd century) with a Latin translation added.

⁴⁹ Vanni, Bossi, *Magic and Card Tricks...* p. 124.



Fig. 3: Claude-Gaspard Bachet de Méziriac.

Source: Wikipedia,

[http://commons.wikimedia.org/wiki/File:WP_Claude_Gaspard_Bachet_de_M%C3%A9ziriac.j
pg](http://commons.wikimedia.org/wiki/File:WP_Claude_Gaspard_Bachet_de_M%C3%A9ziriac.jpg)

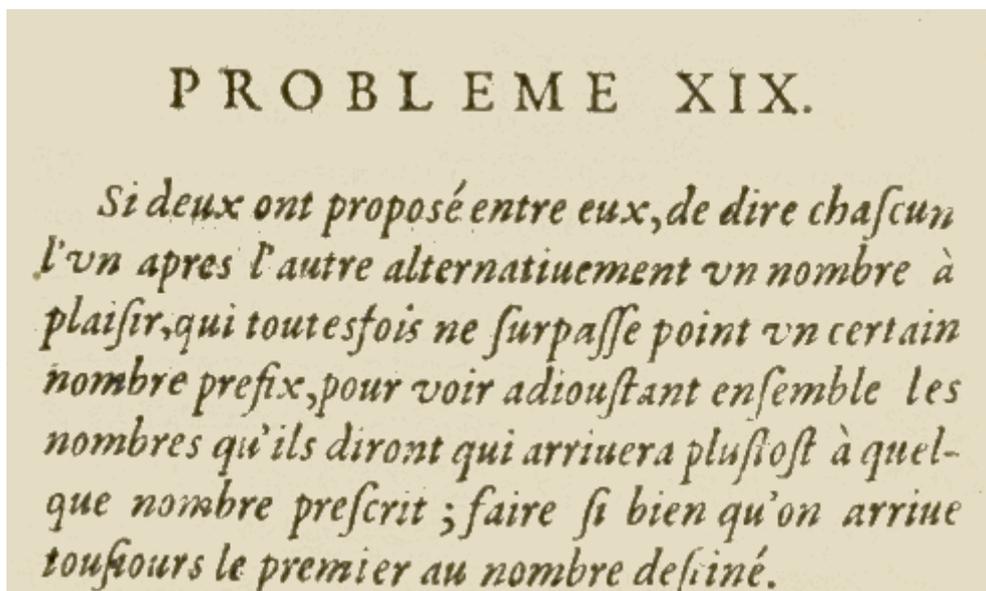


Fig. 4: Problem XIX presented in Bachet's 1612 work

Source: Claude-Gaspard, Bachet, p. 99

Problem XIX (Fig. 4) in the first edition of Bachet's 1612 book about mathematic recreations proposes a version of Nim that is similar to the one formulated by Pacioli.⁵⁰ Bachet studies the case when two players must reach 100 by adding numbers ranging from 1 to 10, "or any smaller number [...] such as the player who will say the number that achieves 100 is recognized the winner."⁵¹ Then he explains the strategy that ensures the win: "But to win unerringly, add 1 to the number that cannot be exceeded, here 10; you obtain 11 and then, always remove 11 of the number to be reached, 100; you will obtain these numbers 89, 78, 67, 56, 45, 34, 23, 12, 1."⁵² These are the equivalents of Pacioli's safety steps, applied to a different configuration of numbers. It can be noticed that Bachet gives the solution, without proving it, of the given example; yet he makes a start on a generalisation when he advises the reader to add 1 to the number that must not be exceeded. Indeed, in any case, when the number to be reached is n and the added number not to be exceeded is k ($k \leq n$), the steps to reach are the numbers m such as $m \equiv 1 \pmod{k}$, that is to say 1, $k+1$, $2k+1$, $3k+1$... Bachet rightly points out: "if the two players know the trick, the one who begins will inevitably win."⁵³ Then he suggests a demonstration to "formulate the general rule",⁵⁴ which is actually not that general, because the author's proof is limited to one example with 100 and 10 and Bachet only explains why it is clever to reach the steps 89, 78, etc. Backward induction, which is essential to this kind of games, is used here: Bachet starts at the end of the game, demonstrating that the opponent cannot reach 100 from 89 and goes back to the beginning of the game to determine the starting situation. Yet, he concludes: "the rule is infallible and perfectly demonstrated."⁵⁵ A notice meant to "bring diversity in the game practice"⁵⁶ proposes to play with other numbers – reach 120 without adding numbers higher than 10, or reach 100 with numbers not exceeding 8 or 9. Next, Bachet advises to choose an opponent who does not know anything about the strategy, and nevertheless to remain clever:

So, if your opponent ignores the subtlety of the game, you must not take the same remarkable numbers necessary to win unfaillingly, because doing so, you will highlight

⁵⁰ The fact that Pacioli was a major source of inspiration for Bachet has been studied in detail: about a third of the problems suggested by Bachet are directly linked to Pacioli. See Albrecht Heeffer, p. 18.

⁵¹ Claude-Gaspard, Bachet, 1612. p.100: "ou tout nombre moindre. [...] et que celui qui dira le nombre accomplissant 100, soit réputé pour vainqueur." My translation.

⁵² Idem. "Or pour vaincre infailliblement, ajoute 1 au nombre qu'on ne peut passer, qu'est ici 10, tu auras 11, et ôte continuellement 11, du nombre destiné 100, tu auras ces nombres 89, 78, 67, 56, 45, 34, 23, 12, 1." My translation.

⁵³ Idem. "si les deux qui jouent à ce jeu savent tous deux la finesse infailliblement celui qui commence remporte la victoire." My translation.

⁵⁴ *Ibid.* p.101: "pour former la règle générale." My translation.

⁵⁵ Idem. "la règle est infaillible et parfaitement démontrée." My translation.

⁵⁶ Idem. "pour apporter de la diversité dans la pratique de ce jeu." My translation.

the trick, and if he is a man of good intelligence, he will immediately notice these numbers, as he will see you are always choosing the same: but at the beginning, you can say other numbers on the fly, until you came nearer the wanted number, because then you will be able to add some of the necessary numbers for fear of being surprised.⁵⁷

Better being safe than sorry...

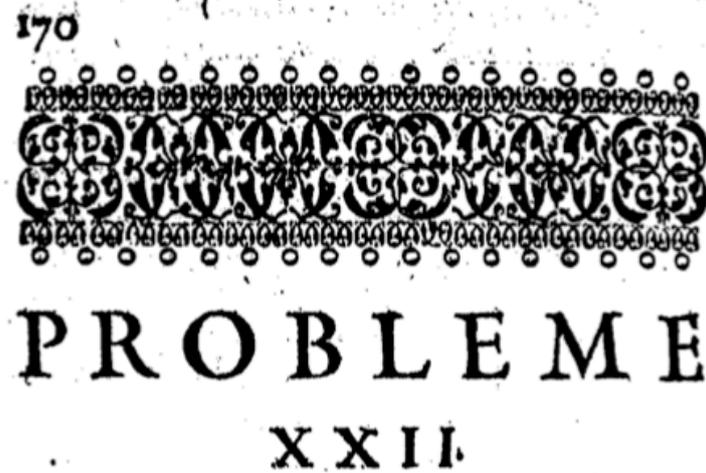
This problem and its resolution appear in the second edition in 1624 (Fig. 5), under the same statement (Fig. 6); only the numbering is changing.



Fig. 5: Frontispiece of *Problemes plaisans et delectables, qui se font par les nombres* by Claude-Gaspard Bachet, second edition 1624.

Source: Claude-Gaspard, Bachet (1624)

⁵⁷ *Ibid.* p.102: "Partant, si ton adversaire ne sait pas la finesse du jeu, tu ne dois pas prendre toujours les nombres remarquables et nécessaires, pour gagner infailliblement, car faisant ainsi, tu découvriras trop l'artifice, et s'il est homme de bon esprit il remarquera tout incontinent ces nombres là, voyant que tu choisis toujours les mêmes : mais au commencement tu peux dire à la volée des autres nombres, jusqu'à ce que tu approches du nombre destiné, car alors tu pourras facilement accrocher quelque un des nombres nécessaires de peur d'être surpris." My translation.



*Si deux ont proposé entre eux, de dire
chascun l'un apres l'autre alternatiue-
ment un nombre à plaisir, qui toutes-
fois ne surpasse point un certain nom-
bre prefix, pour voir adioustant en-
semble les nombres qu'ils diront, qui ar-
riuera plustost à quelque nombre pres-
crit ; faire si bien qu'on arrive tousiours
le premier au nombre destiné.*

Fig. 6: Problem XXII as stated in Bachet's book 1624.

Source: Claude-Gaspard, Bachet, p. 170

In the same year, at the French university of Pont-à-Mousson, an octavo volume entitled *Récréation mathématique, composée de plusieurs problèmes plaisants et facétieux, En fait d'Arithmétique, Géométrie, Mécanique, Optique, et d'autres parties de ces belles sciences* was published. It was the first appearance of the words “mathematical recreations” in the title of a book.⁵⁸ The numerous revised and corrected editions that followed this publication do not make it possible to confidently state the paternity of this work, as the frontispiece does not mention any author. Opinions differ on this point and three names are brought forward: Henry

⁵⁸ Albrecht, Heffer.

Van Etten, who signed the dedication, Jean Leurechon (1591-1670), a Jesuit whose name is used in almost every library to register the book, and Jean Appier Hanzelet (1596-1647), an engraver and printer at the university of Pont-à-Mousson, who published the book. Albrecht Heeffer led a survey to discover the true author of *Recréations mathématiques*. This survey also lists the different sources of the mathematical problems found in the volume. It would seem that the thirty-one arithmetical and combinatorial problems, about one third of the collection, directly come from *Problèmes plaisans* by Bachet who is quoted in the preface and in some of the notes.⁵⁹ It is amusing to notice that the authors of mathematical recreation books often referred to Bachet,⁶⁰ simply because this latter forgot to give the origin of his recreations, whereas he collected them from Pacioli, Alcuin, Tartaglia, Cardan,⁶¹ probably Chuquet,⁶² and certainly from oral tradition too. Nevertheless, it must be admitted that as Bachet was the author of the most important seventeenth-century translation of Diophante's *Arithmetica*, it is therefore not surprising if one of his main concerns was related to arithmetical problems. Bachet hardly gave references, but a long tradition of recreational mathematics throughout the Middle-Ages and the Renaissance has made it possible to find some sources, among them Luca Pacioli.⁶³ Let us now consider another source of Nim, in additive version, which appeared in Germany shortly after the publishing of Bachet's work.

D. Daniel Schwenter (Germany, 1636)

One of the first German occurrences of the one-pile Nim can be found in *Deliciae Physico-Mathematicae* by Daniel Schwenter (1585-1636), a mathematician, inventor, poet and bookseller. Problem XLV is stated as follows: “you both must count to 30. The winner is the one who first reaches 30. But it is not allowed to add more than 6 at each turn.”⁶⁴

⁵⁹ Albrecht, Heeffer, p. 13.

⁶⁰ For example, it is the case of Jacques Ozanam (1640-1718), a French mathematician, author of *Récréations mathématiques et physiques, Qui contiennent les Problèmes et les Questions les plus remarquables, et les plus propres à piquer la curiosité, tant des Mathématiques que de la Physique ; le tout traité d'une manière à la portée des Lecteurs qui ont seulement quelques connaissances légères de ces Sciences*, Paris, 1778. His work will be detailed later.

⁶¹ Walter, Rouse Ball, rev. by Harold Coxeter, *Mathematical Recreations and Essays*, New York, The Macmillan Company, American Edition, 1947. p. 2.

⁶² Albrecht, Heeffer, p. 17.

⁶³ *Ibid.* pp. 15-22.

⁶⁴ Daniel, Schwenter, *Deliciae Physico-Mathematicae*, Nuremberg, 1636. p. 78: “So ihr zwei sollt miteinander bis 30 zählen. Wer als erstes auf 30 kommt, hat gewonnen. Es darf aber keiner auf einmal über 6 zählen” My translation with help from a German friend...

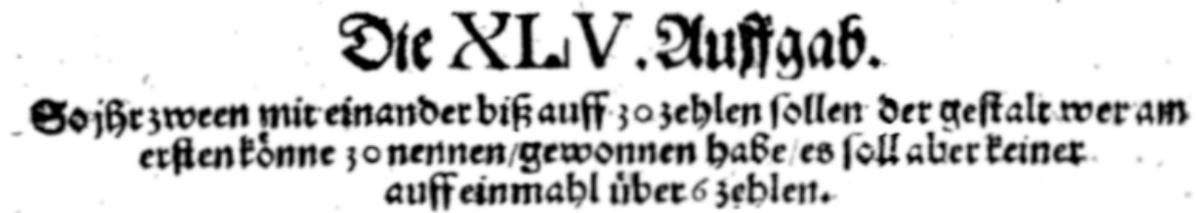


Fig. 7: Problem XLV presented in Schwenter's book, ed.1636.

Source: Daniel, Schwenter, p. 78

It is not clear whether Schwenter knew Pacioli's *De Viribus Quantitatis*, because he mentions Professor Gustavus Selenus⁶⁵ and his work about cryptography⁶⁶ in the very first sentence. According to Schwenter, Selenus explains that the winner will be the one who will choose numbers 9, 16 and 23. Unfortunately, because of the complexity of the Latin text, we have not found this passage in Selenus' book yet...

Schwenter is an interesting example because it shows how difficult it is to trace the links between the authors of books containing arithmetical problems. Former sources are sometimes quoted explicitly but it is not sure that these former sources did not take inspiration from other works that would not be mentioned.

E. Jacques Ozanam (France, 1694)

Jacques Ozanam (1640–1718) was a French mathematician more particularly known for his writings about trigonometric and logarithmic tables. The first edition of his *Récréations mathématiques et physiques*⁶⁷ dates back to 1694; many republications, added with revisions and additions, were to follow such as Jean-Etienne Montucla's edition in 1778.⁶⁸ These numerous republications make William Schaaf say: "Ozanam may be regarded as the forerunner of modern books on mathematical recreations."⁶⁹ Nevertheless, Schaaf

⁶⁵ Gustave Selenus was the pseudo used by August II von Brunswick-Wolfenbüttel (1579-1666).

⁶⁶ Selenus wrote two books : one about Chess, *Das Schach – öder Königsspiel* (1616) and the second about cryptography, *Cryptomenytices et Cryptographiae libri IX* (1624).

⁶⁷ Jacques, Ozanam, *Récréations mathématiques et physiques, Qui contiennent les Problèmes et les Questions les plus remarquables, et les plus propres à piquer la curiosité, tant des Mathématiques que de la Physique ; le tout traité d'une manière à la portée des Lecteurs qui ont seulement quelques connaissances légères de ces Sciences*, Paris, 1778.

⁶⁸ Jean-Etienne Montucla (1725-1799) was a French mathematician, the author of *Histoire des Mathématiques* (1758).

⁶⁹ William, Schaaf, *Recreational Mathematics A Guide to the Literature*, The National Council of Teachers of Mathematics, INC., 1963. p. 1.

admits that Ozanam drew inspiration from Bachet's, Mydorge's and Leurechon's works and that "[...] his own contributions were somewhat less significant".⁷⁰

We have based our study on the posthumous edition of *Récréations mathématiques et physiques*, dated 1778 and published by Claude-Antoine Jombert (173?-1788). In the first tome, *Contenant l'Arithmétique et la Géométrie*, the additive version of Bachet can be found under its simplest formulation:

Problem XIV: two players agree to take in turns numbers smaller than a given number, for example 11, and to add them until one of the two persons can reach, for instance, 100; how should we proceed to be the first without fail?⁷¹

It is worth noticing that Ozanam openly asks the question: which is the strategy to apply in order to win, whereas his predecessors put down the problem without any questioning. Ozanam gives an explanation of the solution that is not so different from the one found in Bachet's book. He only completes the strategy to use if, instead of adding numbers ranging from 1 to 10, it is decided to choose numbers between 1 and 9. But he does not bring any changes in the statement of the problem nor in the explanation of the solution. This "inertia" in the evolution of the content in recreational books, numerous problems are actually similar in various books, is linked to the inertia in the solving methods of the given problems. Changes are generally to be found in the way problems are stated more than in the way they are solved. This "stasis" is also due to the fact that authors tended to copy the problems they had found in other sources:

In certain instances, authors have been careful to state the immediate origin of their questions, but it must be confessed that this is done when one writer wishes to correct the work of another rather than when he merely wishes to acknowledge his use of the other's book.⁷²

We will see later that copying a problem and including a slight variation without noticing that this variation completely changes the solution can sometimes prove to be tricky for the authors!

⁷⁰ Idem.

⁷¹ Jacques, Ozanam, pp. 162-163. "PROBLÈME XIV : Deux personnes conviennent de prendre alternativement des nombres moindres qu'un nombre donné, par exemple 11, et de les ajouter ensemble jusqu'à ce l'un des deux puisse atteindre, par exemple, 100 ; comment doit-on faire pour y arriver infailliblement le premier ?" My translation.

⁷² Vera, Sanford, *The History and Significance...* pp. 79-80.

F. André-Joseph Panckoucke (France, 1749)

In France, it is not before the end of the eighteenth century that an additive version of Nim was to be found in *Les amusemens mathématiques*⁷³ (Fig. 8) by André J. Panckoucke (1703-1753), a writer, bookseller and editor in Lille (1703-1753).⁷⁴ Panckoucke owned a bookshop situated Place Rihour⁷⁵ between 1728 and 1733 and he published a large number of works.⁷⁶ Among other books, he wrote the *Dictionnaire des proverbes françois et des façons de parler comiques, burlesques et familières...*(1748), *L'art de se désopiler la rate* (1754) and the *Amusemens mathématiques* (1749).⁷⁷ It seems that this editor from the North of France appreciated enjoyable pastime! But Panckoucke was also well read and an erudite who paid attention to scientific developments and to their practical applications.⁷⁸ His bookshop provided the intellectual elite of Lille with a large choice of books.

⁷³ André-Joseph, Panckoucke, *Les amusemens mathématiques, précédés Des Eléments d'Arithmétique, d'Algèbre & de Géométrie nécessaires pour l'intelligence des Problèmes*, Lille, 1749.

⁷⁴ His son, Charles-Joseph Panckoucke (1736 – 1798), was more famous than his father as he became the official editor and bookseller of the Imprimerie royale and of the Académie royale des sciences. He was a leading figure in the world of edition and diffusion of the encyclopaedic knowledge of the Enlightenment. He also corresponded with Voltaire and Rousseau.

⁷⁵ Place Rihour is a tourist place in Lille where the remains of a fifteenth-century castle can be seen. Palais Rihour was built by the Dukes of Burgundy of the Valois dynasty.

⁷⁶ See: André Joseph Panckoucke. (2013, November 2nd). *Wikipedia*.

http://fr.wikipedia.org/w/index.php?title=Andr%C3%A9_Joseph_Panckoucke&oldid=97949812. [2.12.2013]

⁷⁷ For a better understanding, I have translated these three titles as follows:

- Dictionary of French Proverbs and of the Different Comical, Farcical and Colloquial Ways of Speaking.
- The Art of Killing Oneself Laughing.
- Mathematical Pastimes.

⁷⁸ Gilbert, Dalmasso, *Présence de la "chymie" dans la France du Nord, de la deuxième moitié du XVIIIe siècle au premier tiers du XIXe : sa diffusion et son enseignement public et privé, son application aux Arts*, PhD thesis submitted at the University of Lille 3 in 2005.

LES
AMUSEMENS
MATHÉMATIQUES

PRÉCÉDÉS

Des Elémens d'Arithmétique, d'Algèbre &
de Géométrie nécessaires pour l'intelli-
gence des Problèmes.

*Sapientem decet interdum Remittere aciem rebus
agendis intentam. Aug. de Mus.*



A LILLE,

Chez ANDRÉ-JOSEPH PANCKOUCKE,

ET SE VEND A PARIS,

Chez TILLIARD, Libraire, Quai des
Augustins, près le Pont Saint-Michel,
à S. Paul.

M. DCC. XLIX.

Avec Approbation & Privilège du Roi.



Fig. 8: Frontispiece of *Amusemens mathématiques* by André-Joseph Panckoucke, 1749.

Source: André-Joseph, Panckoucke (1749)

As the title indicates it, the book displays general results of arithmetic, algebra and geometry that are useful for resolving the 239 problems, which are proposed later with their solutions. Problem 10 is called “Le Piquet des Cavaliers”⁷⁹ and its wording is more fictionalized than in Pacioli’s, Bachet’s, Ozanam’s or Schwenter’s books. Below is a copy of this problem (Fig. 9):

⁷⁹ André-Joseph, Panckoucke, p.130.

PROBLEME 10.
Le Piquet des Cavaliers.
Deux amis voyagent à cheval, l'un propose à l'autre un cent de Piquet sans carte.
Ils conviennent, 1°. Que celui qui arrivera le premier à 100 sera défrayé du souper, 2°. qu'ils ne pourront prendre alternativement plus de 10 à la fois.

Fig. 9: *Le Piquet des Cavaliers*, pastime proposed by Panckoucke.

Source: André-Joseph, Panckoucke, p. 130⁸⁰

Originally, *Piquet*⁸¹ was a card game, which up to the middle of the nineteenth century, was one of the three games considered as the most dignified with Chess and Backgammon.⁸² During the seventeenth century, it was played with 36 cards (the lowest being the 6). This game is described in the French comédie-ballet *Les Fâcheux* by Molière, performed in 1661. *Piquet* was also mentioned, spelled *Picquet*, in *Gargantua* by Rabelais (1534). The two players must take, in turns, a card from the pack and add its value to the sum already obtained with the former draws. The riders of our problem do not have any cards, which would not be very useful for riding, and they play orally, which is equivalent to one pile Nim. The given solution is very short; below is a copy (Fig. 10).

⁸⁰ My translation: “Two friends are riding; one of them suggests to play one “cent (one hundred) de Piquet” without cards.

Both agree that 1°. The first who will reach 100 will not pay the diner, 2°. They will not be allowed to take in turns a number higher than 10.”

⁸¹ Formerly, *Piquet* was called *Cent* (one hundred) because it was the number to reach in order to win a match.

⁸² See: <http://academiedesjeux.jeuxsoc.fr/piquet.htm>

SOLUTION.

Le premier qui commencera à compter doit toujours se saisir de ces époques.

1, 12, 23, 34, 45, 56, 67, 78, 89, &c.

D'où l'on peut conclure que celui qui commenceroit par 1, & qui avec chaque mise de son compagnon formeroit toujours 11, arriveroit le premier à 89, & par conséquent son adversaire ne pouvant prendre tout au plus que 10 ne formeroit que 99, reste au premier à dire 100.

Il n'est pas nécessaire de s'assurer des époques, quand on a affaire à un homme qui ignore la finesse du jeu, il suffira de se saisir à propos des dernières.

Fig. 10: The solution of problem 10, *Le Piquet des Cavaliers* by Panckoucke.

Source: André-Joseph, Panckoucke, p. 130⁸³

The steps to reach to ensure a win are called “époques”, epochs, which is different from the words used by Bachet and Ozanam; this term was to be reused later by Guyot. The last sentence makes us think that Panckoucke read *Problèmes plaisants et délectables*, for he encourages the reader not to insist in reaching all the steps, “époques”, but only those that are close to the number to be reached, just like Bachet did; a rather simple strategy, which is nevertheless efficient enough to eat out cheaply! Some forty years later, Henry Decremps (1746 – 1826) used this phrase: “*Principes mathématiques sur le piquet à cheval, ou l’art de gagner son dîner en se promenant*”⁸⁴ in his *Codicile de Jérôme Sharp*.⁸⁵ This assertion reinforces the idea that the player who knows the winning strategy can take advantage of his knowledge to obtain a favour from his fellow player. In this additive version of Nim, Panckoucke stresses the control you can exert over your opponent if you know the trick of the

⁸³ My translation:

“The first who will begin to count must always reach these steps: 1, 12, 23, 34, 45, 56, 67, 78, 89, etc... Therefore, it can be concluded that the first who would begin with 1 and who would always make 11 when added with his friend’s stake, would reach first 89; as his opponent cannot add more than 10, he could not reach but 99; consequently, it only remains to the first player to announce 100.

When playing with a man who is not aware of the finer points of the game, it is not necessary to ensure the first steps; it will be sufficient to ensure the last ones.”

⁸⁴ For a better understanding, I give my translation of the title: “Mathematical Principles about the “Piquet à cheval”, or the Art of Earning One’s Diner while Riding”.

⁸⁵ David, Singmaster, *Sources in Recreational Mathematics An Annotated Bibliography*, 9th preliminary edition, March 2004, available for consultation on: www.gotham-corp.com/sources.htm

game. It is no longer a matter of simply displaying an intellectual superiority but also a question of using it for material purposes. This reference highlights several new aspects of the game: first, the wording of the problem is formulated in a more fictionalized way, even if the given solution is the same as far as the solving method remains arithmetically identical; and secondly, the aim of the game and the interest of knowing the winning strategy are clearly emphasized.

G. Edmé-Gilles Guyot (France, 1769)

Edmé-Gilles Guyot (1706 – 1786) was a French physician and inventor, an author in the area of mathematics and physics, which he used to perform magic tricks such as optical illusions, projection of figures into smoke. Guyot worked on the development of magic lanterns used in phantasmagoria in order to show his experiments before a live audience and to popularize his discoveries. During the eighteenth century, this was indeed a common practice for teaching and disseminating sciences in France: “[...] mathematical exercises in public, which multiplied at that time with educational purposes [...] were meant to stress on applied mathematics, which were easier to understand by the common people who attended the meetings [...]”⁸⁶ Guyot’s works were translated into English and German and were largely circulated in Europe.⁸⁷ In 1769, Guyot tackled the French edition of *Récréations Mathématiques*, which had been reissued more than twenty-five times between 1629 and 1680 by Claude Mydorge, Jacques Ozanam and Jean-Etienne Montucla who published the work in four volumes. Guyot titled the second volume *Nouvelles récréations physiques et mathématiques, Contenant, Toutes celles qui ont été découvertes et imaginées dans ces derniers temps, sur l’Aimant, les Nombres, l’Optique, la Chymie, etc. et quantité d’autres qui n’ont jamais été rendues publiques. Où l’on a joint leurs causes, leurs effets, la manière de les construire, et l’amusement qu’on peut en tirer pour étonner agréablement*.⁸⁸ This volume is devoted to recreations with numbers. Panckoucke’s version of *Piquet à cheval*⁸⁹ can be found,

⁸⁶ René, Taton, *La science moderne...*, p.55: “[...] les exercices publics sur les mathématiques, qui se multiplient à l’époque avec des intentions pédagogiques [...], sont portés à mettre fort l’accent sur les mathématiques appliquées, plus accessibles aux honnêtes gens qui y assistaient [...]” My translation.

⁸⁷ See the English site of Wikipedia: http://en.wikipedia.org/wiki/Edm%C3%A9-Gilles_Guyot

⁸⁸ My translation: *New Physical and Mathematical Recreations including All of Those that have been Discovered or Created in Recent Times; Upon Magnet, Numbers, Optics, Chemistry, etc... and Many Other Things Never Made Public. To Which are Attached their Causes, their Effects, the Way to Construct Them and the Entertainment that can be Drawn in Order to Surprise Pleasantly.*

⁸⁹ Edmé-Gille, Guyot, *Nouvelles récréations physiques et mathématiques, Contenant, Toutes celles qui ont été découvertes et imaginées dans ces derniers temps, sur l’Aimant, les Nombres, l’Optique, la Chymie, etc. et*

in which “two riders who travel together, are bored when thinking of the distance that is still to be covered; they create a game that could help them to pass the time more pleasantly and agree to play a “Cent de Piquet” [...]”⁹⁰ What is at stake here is no longer the winning of a diner; it is simply a verbal recreation that helps passing the time. We can point out the use of the word “recreation” instead of “problem”, which was used until then when referring to the object of the statement. The solution Guyot gives is similar to those given by the authors we studied previously; yet, the way the author presents it is rather different. Indeed, Guyot states:

In order that the player who gives the first number can reach 100, and that his opponent cannot, he must remember the numbers 11, 22, 33 etc... of the problem mentioned above, and count such as there is always one unit more than these numbers; furthermore, he should give first the number 1, and because his opponent cannot take a number higher than 10, this latter will not be able to reach 12 that the first player will take, and then consequently, the numbers or “époques” (steps) 23, 34, 45, 56, 67, 78 and 89; reaching this last step, his opponent cannot prevent him from reaching 100 at the following turn, whatever number he himself could choose.⁹¹

Guyot does not clearly explain the backward induction necessary to find the “époques” and suggests right away that the first player should choose 1 so that his opponent could not reach the step 12. The solution is quite close to Panckoucke’s in its wording as well as in its terminology, using “époque”. Guyot also advises the player, “if his opponent does not know the trick and in order to better disguise this Recreation, to give indistinctly any numbers in the first turns, as far as around the end of the game, he takes the two or three last numbers necessary to win.”⁹² He adds that this recreation has no interest and is not pleasant if played by two people aware of the tricks, as far as “the first who gives the first number always wins.”⁹³ Once again, it is better to play with someone who does not know the strategy. Guyot

quantité d'autres qui n'ont jamais été rendues publiques. Où l'on a joint leurs causes, leurs effets, la manière de les construire, et l'amusement qu'on peut en tirer pour étonner agréablement, Paris, 1769. pp. 27-29.

⁹⁰ *Ibid.* p. 27: “deux cavaliers qui voyagent ensemble, ennuyés du chemin qu’il leur reste encore à faire, inventent un jeu qui puisse leur faire passer le temps plus agréablement, et conviennent ensemble de jouer un Cent de Piquet.” My translation.

⁹¹ *Ibid.* pp. 27-28: “Afin que le premier qui nomme le nombre puisse arriver à 100, et que son adversaire n’y puisse y parvenir, il doit se souvenir des nombres 11, 22, 33, etc. du problème ci-dessus, et compter de façon qu’il se trouve toujours d’une unité au-dessus de ces nombres ; ayant en outre attention de nommer d’abord 1, attendu que son adversaire ne pouvant prendre un nombre plus grand que 10, ne pourra arriver au nombre 12, qu’il prendra alors lui-même et conséquemment ensuite les nombres ou époques 23, 34, 45, 56, 67, 78, et 89, à laquelle étant arrivé, quelques nombres que puisse choisir son adversaire, il ne peut l’empêcher de parvenir, le coup suivant, à 100.” My translation.

⁹² *Ibid.* p. 28: “ne connaît l’artifice de ce coup peut (pour mieux déguiser cette Récréation) prendre indistinctement toutes sortes de nombres dans les premiers coups, pourvu que vers la fin de Partie, il s’empare des deux ou trois derniers nombres qu’il faut avoir pour gagner.” My translation.

⁹³ *Idem.* “attendu que celui qui nomme le premier a toujours gagné.” My translation.

differentiates himself from his predecessors by insisting on the fact that the game is completely uninteresting if the two players know the solution. Furthermore, the aim of the game stated by Guyot differs from Panckoucke's; it is no longer a matter of using one's knowledge to achieve one's purpose. The author concludes with this interesting point:

It (the recreation) can be played with any other numbers; and then, if the first wants to win, the number to be reached must not be equal to the one he can stake, because he could loose; but it is necessary to divide the higher number by the lower, and the remainder will be the number that the first player must say first to make sure he will win.⁹⁴

Thus, the recreation is generalized for any numbers; Guyot gives an example that helps us to understand what he is trying to get at:

If the number that we agree to reach is 30, and if the number we are allowed to say must be lower than 7, as 4 times 7 make 28, it remains 2 to reach 30 and this number is the one that the first player must pronounce in the first place; and so, whatever number his opponent will say, he must choose the number that will make 7 when added to the other, and he will necessary reach first the number 30.⁹⁵

Guyot uses the division of n by $k+1$ and more particularly the remainder of this division, which sets the first number to say to ensure the win. This notion of division does not appear in Bachet's, Ozanam's and Panckoucke's works. Besides, Guyot gives an example, taking $n = 30$ and $k + 1 = 7$ to illustrate this generalization. This leads us to think that he got acquainted not only with Panckoucke's *Amusemens mathématiques* as far as the wording, the fictionalization and the terminology are concerned, but also with Pacioli's *De Viribus Quantitatis* or Schwenter's *Deliciae Physico-Mathematicae*; indeed, these two authors were the only ones who used the numbers 30 and 6 before 1769 in this additive version of one-pile Nim.

⁹⁴ Idem. "Elle [la récréation] peut se faire aussi avec tous autres nombres ; et alors si le premier veut gagner, il ne faut pas que le nombre où l'on doit arriver, mesure exactement celui jusqu'ou on peut atteindre pour gagner, car alors on pourrait perdre ; mais il faut diviser le plus grand par le plus petit, et le reste de la division sera le nombre que le premier doit nommer d'abord pour être assurer du gain de la Partie." My translation.

⁹⁵ *Ibid.* pp. 28-29: "Si le nombre auquel on se propose d'atteindre est 30, et le nombre au-dessous duquel on doit nommer 7, 4 fois 7 faisant 28, il reste 2 pour aller à 30, et ce nombre est celui que le premier doit nommer d'abord ; alors quelque nombre que nomme l'adversaire, s'il y ajoute celui qui convient pour former avec lui celui de 7, il parviendra de nécessité le premier au nombre 30." My translation.

H. William Hooper (England, 1774)

The additive version of Nim crossed the Channel and landed on William's Hooper book entitled *Rational Recreations* in which "[...] the author has selected the principal part of the experiments from the writers on recreative philosophy of the last and present centuries".⁹⁶ The first edition of Hooper's book dates back to 1774 and the second, which we have worked on, dates back to 1783. Four volumes make up this work; the first mainly deals with "arithmetical and mechanical experiments"⁹⁷ and the others are related to optics, chromatic and acoustics (vol.2), pneumatic, hydrology and pyrotechnics (vol.4) and finally electrical and magnetical experiments in volume 3. The presentation of the eighth recreation of the first volume, in which we can find the additive version of Nim, is slightly different from the "problems", "recreation" or "effect" that were present in the former works. First, the recreation is entitled *The Magical Century*⁹⁸ and no longer *Cent de Piquet*. Then Hooper starts with an arithmetical reminder concerning the multiplication of the first nine digits by 11: "If the number 11 be multiplied by any one of the nine digits, the two figures of the product will always be familiar."⁹⁹ He illustrates this reminder with the following eleven-time table (Fig. 11):

II								
I	2	3	4	5	6	7	8	9
II	22	33	44	55	66	77	88	99

Fig. 11: The multiplication table of the nine first non-zero natural numbers by 11.

Source: William, Hooper, p. 31

Next, Hooper chooses to add in turns counters piled up on a table, until he obtains 100, yet without adding more than 10 counters at the same time. It is worth noticing that Hooper opts for a more visual layout of the recreation, choosing the possibility to manipulate counters

⁹⁶ William, Hooper, *Rational Recreations. Volume the first. Containing arithmetical and mechanical experiments*, 2nd edition, London, 1783. p. i.

⁹⁷ *Ibid.* Frontispiece.

⁹⁸ *Ibid.* p. 31.

⁹⁹ *Idem.*

instead of abstract values that must be kept in mind. This recreation is really closer to a game including the handling of objects than an exercise of mental arithmetic. We can also point out a more direct link with the Nim of Bouton who considered piles of various-sized counters, which had to be manipulated too. Hooper states that, before starting to play, you must modestly (!) tell your opponent, “moreover, that if you stake first he shall never make the even century, but you will.”¹⁰⁰ In order to achieve this, you must start with staking only one counter and then:

[...] Remembering the order of the above series, 11, 22, 33, etc. you constantly add, to what he stakes, as many as will make one more than the numbers of that series, that is, as will make 12, 23, 34, etc. till you come 89, after which the other party cannot make the century himself, or prevent you from making it.¹⁰¹

In this last statement, we can easily understand the deadlock in which the opponent is, as he cannot win, nor make us lose. These properties can be directly compared to the ones of Bouton’s safe combinations. Just like his predecessors did, Hooper suggests that “if the other party has no knowledge of numbers [...]”,¹⁰² you should choose any number in the first turns and then secure your win around the last steps such as 56, 67, 78 and 89. He specifies that “this Recreation may be performed with other numbers [...]”¹⁰³ and that, in order to win, “[...] you must divide the number to be attained, by a number that has one digit more than you can stake each time, and the remainder will be the number you must first stake.”¹⁰⁴ Solving the problem by using the division reinforces the idea that Hooper had knowledge of Guyot’s *Nouvelles récréations physiques et mathématiques*. Yet, Hooper adds that in order to win, there must always be a remainder, which is true if we confine ourselves to playing the first stake and with an opponent who has the *knowledge of numbers*!

I. John Badcock (England, 1820)

The way John Badcock presents *A curious Recreation with a Hundred Numbers, usually called the Magical Century*¹⁰⁵ is exactly the same than Hooper’s, at least in the first

¹⁰⁰ Idem.

¹⁰¹ *Ibid.* pp. 31-32.

¹⁰² *Ibid.* p. 32.

¹⁰³ Idem.

¹⁰⁴ *Ibid.* p. 31.

¹⁰⁵ John, Badcock, *Philosophical Recreations, or Winter Amusements: A Collection of Entertaining and Surprising Experiments in Mechanics, Arithmetic, Optics, Hydrostatics, Hydraulics, Pneumatics, Electricity,*

lines... Badcock echoes the explanation using the multiplication table of the nine first non-zero natural numbers by 11; he also assumes that players handle counters and have to reach 100 without adding more than 10 at each turn. Yet, at the start, each player has 50 counters. As a solution, Badcock does nothing else but copying Hooper's one, almost word for word, without noticing that the apparently light variation he introduced considerably changes the game and its solving! Indeed, if we start the game playing 1 counter, and if our opponent adds only one counter in each of his turns, we will have to use 10 counters in each turn to reach the safety steps (12, 23, 34, 45 etc.). Within five turns, we will already have taken 41 counters from our stock, compared with 5 used by our opponent, which makes a sum amounting to 46. Therefore, it is absolutely impossible for the first player to reach 100 first if he plays the way described above. Consequently, the game Badcock suggests is totally different from Hooper's *Magic Century*, because even if we agree that the player who has no counter left loses the game, the goal to reach at all costs is no longer the "époques" or the safety steps; we must also make sure that we have enough counters to keep on playing. This fact makes the solution considerably harder. Sometimes, it may be wiser to simply copy one's predecessors' works instead of introducing variations that are not mastered...

This change in the statement proves to be an interesting variation because it changes the solution. Unfortunately, the author did not take the chance to provide a brilliant contribution.

J. John Jackson (England, 1821)

The additive version of Nim can be found once again in an English work, dated 1821 and entitled *Rational amusement for winter evenings ; or, A collection of above 200 curious and interesting puzzles and paradoxes relating to arithmetic, geometry, geography, etc.*¹⁰⁶ John Jackson was a "Private Teacher of the Mathematics";¹⁰⁷ the preface precises that the author came across arithmetical and geometrical puzzles and that he regarded as relevant the idea to compile the most interesting riddles with their solutions into a small volume.¹⁰⁸ The

Chemistry, Magnetism, & Pyrotechny, Or Art of Making Fire Works, Together with the Wonders of the Air Pump, Magic Lanthorn, Camera Obscura, &c. &c. &c. and A Variety of Tricks with Cards, London, Thomas Hughes ed., 1820. p. 33.

¹⁰⁶ John, Jackson, *Rational amusement for winter evenings ; or, A collection of above 200 curious and interesting puzzles and paradoxes relating to arithmetic, geometry, geography, etc. with their solutions, and four plates*, London, 1821.

¹⁰⁷ *Ibid.* Frontispiece.

¹⁰⁸ *Ibid.* p. i.

problem we are interested in is the forty-seventh of the chapter devoted to *arithmetical puzzles*¹⁰⁹ and is stated quite briefly:

Two persons agree to take, alternately, numbers less than a given number; suppose less than 11, and add them together till one of them has reached a certain sum; suppose 100. By what means can one of them infallibly attain to that number before the other?¹¹⁰

The solution Jackson gives is as succinct as the statement; it suggests to choose the numbers 1, 12, 23, 34 etc. “[...] a series in Arithmetical Progression, the first term of which is 1, the common difference 11, and the last term 100”,¹¹¹ in order to reach 89 and hence to win unfaillingly. What is new here is the use of the words “Arithmetical Progression”, written in capital letters in the book; this expression had never been used before. The idea of “progression”, whether geometrical or arithmetical, goes back to the most ancient mathematical recreations: “This fact may be due to an innate fondness for rhythm and repetition, a trait that seems to be universal, or it may be due to the mystery of a series of numbers whose values increase so rapidly,”¹¹² [concerning the geometrical progressions].¹¹³ Vera Sanford gives examples of riddles that appeal to geometrical and arithmetical progressions in Rhind Mathematical Papyrus, in Fibonacci, Tartaglia or Cardan.¹¹⁴ Yet, none of these authors realized that these progressions, which they did not name so, were to be more than simple mathematical curiosities, “the reason may be found in the lack of symbolism and lack of scientific knowledge which operated previously [...]”.¹¹⁵ They talked about problems in a purely arithmetical way, because they had no other mathematical tools, such as logarithms in some cases, to solve them. It was no longer the case in the nineteenth century and John Jackson, as a mathematics teacher, had undoubtedly knowledge of arithmetical and geometrical series, as well as their properties.

At this point, we stop our study on the ancestors of Nim related to the one-pile additive version. From the beginning of the nineteenth century, this version could be found in

¹⁰⁹ *Ibid.* p. 1.

¹¹⁰ *Ibid.* p. 11.

¹¹¹ *Ibid.* p. 64.

¹¹² Vera, Sanford, *The History and Significance...* p. 55.

¹¹³ My clarification.

¹¹⁴ *Ibid.* pp. 55-57.

¹¹⁵ *Ibid.* pp. 57-58.

numerous works of mathematical riddles and recreations in France, England and Germany,¹¹⁶ sometimes presented as “Piquet sans cartes” or simply as a two-player game. Some authors set out the game under a subtractive version,¹¹⁷ and a version “misère” appeared, in which the one who is the last to play is the loser. The range of works we have analysed represents the very first sources in which the ancestral version of Nim can be found and we have seen how difficult, sometimes impossible it is to create strong links between the different authors. We have also understood that it is rather simple to win when playing these additive versions, as far as we know addition and multiplication tables.

On the other hand, things become more laborious when it comes to the Bouton’s Nim of 1901: it is still possible to do some mental arithmetic, but a more important mathematical knowledge, abstract thinking and a longer time to reflect are required to reach the win. This is also the case for combinatorial games that came after Nim, such as Wythoff’s Nim or Moore’s Nim_k , the solutions of which appeal to even deeper mathematics knowledge. The initial aim of the first Nim games that were displayed as mathematical recreations has disappeared through the ages; it is no longer a matter of creating puzzles in order to impress the fairer sex during high-society evenings, but a matter of discovering interesting mathematical properties, even discovering new ones that could lead to theories still undeveloped.

The next part of our study will be devoted to Tiouk-Tiouk, an African game, which could seem far from Bouton’s Nim as it requires a board and offers the possibility to block a piece, but yet is similar to Bouton’s Nim when we consider its solving. Additionally, Tiouk-Tiouk might be an ancestor of Nim game, but once again tracing its sources and its first appearances has proved to be difficult, especially because boards have been made of sand...

III. Tiouk-Tiouk in Western Africa

In 1955, Charles Béart, a school principal in tropical Africa published a work in two volumes, in which he made an inventory of games and toys in Western Africa.¹¹⁸ One chapter is devoted to two-player combinatorial games without chance, such as board games (Chess,

¹¹⁶ David, Singmaster, *Sources in Recreational Mathematics An Annotated Bibliography*, www.gotham-corp.com/sources.htm

¹¹⁷ We start with number n and we can take off at the most k in each turn.

¹¹⁸ Charles, Béart, *Jeux et jouets de l’Ouest africain*, Mémoires de l’Institut Français d’Afrique Noire n°42, IFAN DAKAR, Tome I et II, 1955.

The player who succeeds in blocking all the counters of his partner will win.”¹²⁰ If we compare this version with Bouton’s Nim, we have in this case 6, 8, 10 or 12 piles and each contains the same number of objects; the gap between the seeds and the sticks is the same for each row. Transcribed into binary system, the starting position is a safe combination and the winner is the one who plays in second turn.¹²¹ We do find this notion of safety step in the explanation given by Béart who names it “balance of distances”:¹²²

At the start, the opposite counters are equidistant. The one who will play first and who, therefore, will break this balance, will loose. The second player will only have to restore the balance of distances with moving his counter forwards in the second column in such a way that intervals are equal in the two columns. He will keep this strategy until the end of the game, with always balancing the smallest interval proposed by the first player who will be finally stuck and who will loose.¹²³

For example, in Fig. 13, the player S who has sticks, only needs to move his piece in A3 to rebalance the distances. On the other hand, in the configuration seen on picture Fig. 14, the player who must play will loose because he will inevitably break the balance of distances.

¹²⁰ *Ibid.* Tome II, p.471: “chaque pion peut se déplacer en avant ou en arrière à volonté, et d’autant de cases qu’il lui plaît, mais ne peut pas sauter par-dessus le pion du partenaire. A gagné qui arrive à bloquer tous les pions du partenaire.” My translation.

¹²¹ Providing that both players know the optimal strategy.

¹²² *Ibid.* Tome II, p.471: “équilibre de distances.” My translation.

¹²³ *Idem.* “Au départ, les pièces opposées sont à égale distance. Celui qui joue le premier et qui, par conséquent rompt cet équilibre, perdra. Il suffira au second de rétablir l’égalité des distances en avançant sa pièce dans la deuxième colonne de telle sorte que les intervalles soient les mêmes dans les deux colonnes. Le second joueur continuera cette tactique jusqu’à la fin de la partie en égalisant chaque fois sur le plus petit intervalle proposé par le premier joueur, qui, finalement, sera bloqué et perdra.” My translation.

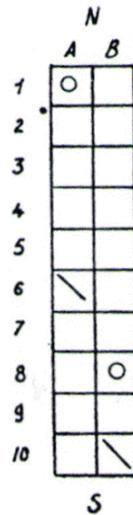


Fig. 13: Forward position of a Tiouk-Tiouk game: the player S has to move forward his stick in A3 to rebalance the distances.

Source: Charles, Béart, Tome II, p. 470

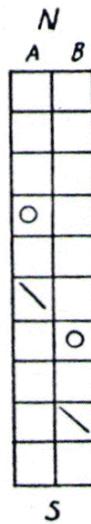


Fig. 14: Forward position of a Tiouk-Tiouk game: the player whose turn it is will loose for he will inevitably disturb the balance of distances.

Source: Charles, Béart, Tome II, p. 470

Béart specifies that Tiouk-Tiouk is generally proposed to a shepherd by a griot¹²⁴ who “generously offers him to draw the board and take the first-move advantage.”¹²⁵ But we know now how disadvantageous it is to start the game! Yet, the griot does not cheat because “it is

¹²⁴ A griot is a traditional storyteller in West Africa. His origins go back to a time when writing did not exist. The griot is the keeper of oral tradition. The status of griot is passed within a cast. Griot families are specialises in divers areas: history, genealogy, storytelling and music.

¹²⁵ *Ibid.* p. 470: “généreusement il lui accorde le trait” My translation.

unnecessary, he is sure to win, when he wants if he does not start, and quite sure to win if he starts.”¹²⁶ However there is a lot at stake with the outcome of this game.

Tiouk-Tiouk was listed in 1955 but it is impossible to date this game precisely and even to find its geographical origin. Béart classifies it within the grid games that have a peculiar status: “these are serious game par excellence, adult games; they are or were mainly the privilege of adults, of men, even of leaders; women and children could only imitate these games [...]”¹²⁷ This elitist aspect of Tiouk-Tiouk and other grid pattern games could be compared to the sixteenth-seventeenth-century mathematical recreations that were listed in works intended for educated classes who could afford to buy these books. “The most simple explanation for this singular situation lies in the fact that these games were introduced by a dominant society setting among a subjugated population.”¹²⁸ Indeed, we can understand the intellectual pressure one can exert on somebody else who would not know the strategy and who would loose each game.

In 1988, Harry Eiss, author of *Dictionary of Mathematical Games, Puzzles and Amusements*,¹²⁹ listed another African version, and an Asian one, of Nim game and noticed that “whatever its origin, the game seems to have a universal appeal.”¹³⁰ He added:

A form of it known as Pebbles or Odds has been played in Africa and Asia for centuries. In this version, an odd number of pebbles, seeds, or whatever is placed in a pile, and players take turns selecting one, two, or three until all have been drawn. The player with an odd number in possession wins.¹³¹

Unfortunately, Eiss gives no other references and it is therefore difficult to trace a game that only requires pebbles or seeds and that does not leave lasting prints.

Games and mathematical riddles have been puzzling people for centuries, “[...] the human nature has changed but little, and problems that whet the imagination prove more fascinating than do the prosaic ones, whether a person lives in the sixteenth century or in the

¹²⁶ *Ibid.* pp. 470-471: “ce n’est pas nécessaire, il est sûr de gagner, quand il voudra, s’il n’a pas le trait, et à peu près sûr de gagner s’il l’a.” My translation.

¹²⁷ *Ibid.* p. 451: “Ce sont là par excellence des jeux sérieux, des jeux d’adultes, et ils sont, ou ils furent, assez réservés aux adultes, et aux hommes, souvent même aux chefs ; les femmes et les enfants ne pouvaient qu’imiter ces jeux [...]” My translation.

¹²⁸ *Idem.* : “L’explication de cette position singulière, la plus simple, est que ces jeux ont été introduits par une société dominante s’installant au sein d’une population subjuguée.” My translation.

¹²⁹ Harry, Eiss, *Dictionary of Mathematical Games, Puzzles and Amusements*, Westport, Greenwood Press, 1988.

¹³⁰ *Ibid.* p. 188.

¹³¹ *Idem.*

twentieth.”¹³² By the way, mathematical recreations were far from being outfashioned at the beginning of the twentieth century. Famous puzzlists Samuel Loyd’s or Henry Dudeney’s contributions are among the most popular; human infatuation for mysterious problems is timeless. This interest for puzzles and riddles is also marked by the growing complexity of the solutions, for example Piet Hein’s *Superellipse*, Solomon Golomb’s *Polyominoes*, Penrose’s *Tilings* or Conway’s *Surreal Numbers*. This complexity has enabled some mathematical theories to develop. Recreations have been considered as challenges to be taken up, yet within an entertaining frame. This tendency can be found as early as the first versions of Nim, of which solutions are rather easy to find out, provided that we take some time to do so. On the other hand, some no-trivial solutions were to appear later with Bouton’s Nim and its variations. From that time on, the real mathematical history of Nim and its theorization have begun, when a sufficient keenness, even a genuine mathematical skill have been necessary to discover winning strategies. But this is another story...

¹³² Vera, Sanford, *The History and Significance...* p. 62.

