COLOR ADDITION ACROSS THE SPECTRUM OF MATHEMATICS

RON TAYLOR

Abstract. In this paper we introduce two sequential games whose rules are mathematical in nature, though no explicit mathematics is necessary during game play. Both games are based on color mixing rules which can admit a variety of mathematical interpretations. We discuss several of these realizations with an eye toward the novelty of the interpretation and from the perspective of using game play as a pedagogical strategy.

1. Introduction

> "*A feeling of adventure is an element of games. We compete against the uncertainty of fate, and experience how we grab hold of it through our own efforts.*" – Alex Randolph, game author

There are many games which can be analyzed using mathematics. Some of these involve some notion of chance like poker or games played with dice. Games which do not involve a notion of chance include checkers, tic-tac-toe and the ancient Chinese game of Go. While all of these games can be studied using mathematics, no explicit mathematics is necessary during game play. The commonalities between all such games, as with other parlour games, are the following:

(1) the *game* starts at a given state
(2) players play in succession according to a prescribed order
(3) each player has a choice of moves during his or her turn.

In between moves, or as a part of each move, there may be other elements to the game like the roll of a die or shuffling of cards.

These elements are characteristics of so-called sequential combinatorial games. The sequential nature arises from the fact that each player makes a

Date: October 16, 2013.

move before other players are allowed to make a move. This requirement, really a combination of (2) and (3), allows the remaining players to have this information available to them to incorporate into their own choices. The combinatorial nature is due to the the complexity of a game that is due to a – potentially large – number of possible moves for each player at each stage of the game in (3). The potential for a large number of legal moves may make it difficult for a player to nail down a winning strategy. According to Bewersdorff in [4]:

> "*There is no unified theory for the combinatorial elements in games. Nonetheless, a variety of mathematical methods can be used for answering general questions as well as solving particular problems.*"

While many games have analogs to serious real world problems which have a meaningful payoff at the end, and much of game theory is focused on such concerns, we will consider the payoff of these games here to be the amusement mentioned in the following definition of **game** found in [13]:

> *Game (n) – a competitive activity involving skill, chance or endurance on the part of two or more persons who play according to a set of rules, usually for their own amusement or that of spectators.*

There may be additional benefits in a pedagogical situation, and we do make a case for the pedagogical benefit of game play, but we purposefully exclude the notion of an outcome with any monetary value or increase in prestige. Meanwhile there are aspects of mathematical games that not only assist in preserving the enjoyment derived from playing a game over and over, but also enable the games to yield mathematical interpretations. These include the aforementioned chance element and the number of legal moves that each player can choose between at each stage of the game. An additional mitigating factor is the potential for different states of information among the individual players. That is, at each state of the game, when a player makes a move he may know more or less about the progress of the game. In chess, for example, each player is aware of every move that has been made so far. This also provides information about the possible remaining moves because of the positions of the pieces on the board. This is an example of a *perfect information* game. On the other hand, in a card game like bridge, the players may know what moves have been made so far, but players ordinarily do not know the cards being held by the other players. This is an example of an *imperfect information game*.

In the following section we introduce a pair of sequential games that involve some aspect of chance, but not to the same degree as poker or dice games. These games also have a strategic component like checkers or Go, but the chance element mitigates the pure strategy. One game, played with colored stones similar to those used in Mancala, is a perfect information game where the set of available moves and the current position of each player is known to all players at every stage of the game after the initial draw. The other game, played with special dominoes, is an imperfect information game where the set of available moves is known to all players, but each player's set of potential moves is hidden from all other players. After a brief introduction to the game play of both games, we turn to the structure of the common rules underlying both games and see that they can be interpreted in a variety of ways as different mathematical structures. Our goal is not to present a body of research about paths to an endgame, but rather we consider the structure of the rules of the games with an eye toward the pedagogical implementation of certain related mathematical concepts via a relatively simple method for playing with colors.

2. The games

The first game, called *Al-Jabar*, is played with a collection of colored stones. This game was initially developed by a father to be played with his son.[10] The game, as it currently exists, has evolved from this humble beginning to a sequential game with a robust mathematical structure. The game pieces are comprised of 10 stones each of the colors red, orange, yellow, green, blue, purple, and white along with 30 black stones.[1] Throughout the subsequent sections we denote black with the symbol K when we begin thinking about the mathematical structure of the games, and reserve B to denote the color blue.

Game play of *Al-Jabar* goes as follows:

(1) The black stones are placed in the middle of the playing surface.
(2) From the bag of the remaining stones, each player is dealt 13 stones at random.
(3) One colored stone is placed in the middle along with the black pieces as a starter.
(4) Players take turns exchanging pieces from their hands for pieces in the middle – up to 3 at a time – with the goal of reducing the number

[1]Note that the official rules describe these pieces as "clear or black" rather than just black. Here we will use the color black to describe the pieces, in part because this choice translates more easily to the discussion of the second game and the visual structure of the rules.

of stones in their hand. These exchanges are made using the rules of color addition.

(5) The final round of the game is indicated when a player ends his turn with exactly one piece in his hand or begins his turn with 2 or 3 pieces and no move that will reduce the number of pieces in his hand.

For a full description of the rules, along with information about rules variations, please see [2].

The second game, called *Spectrominoes*, is played with special dominoes according to the rules of the familiar domino game *Mexican Train*, see [9], as follows:

(1) Each player is dealt a number of dominoes, depending on the number of players.
(2) Players take turns playing their dominoes in a row by matching the colors along the ends or sides of the pieces according to the rules of color addition.
(3) A round ends when a player has run out of tiles to play.

The distinctive characteristic of this game is that the dominoes, instead of having a number of pips on each end, have one block of color on each end where the colors are chosen from the collection: red, orange, yellow, green, blue, purple, white and black. Sample moves are shown below.

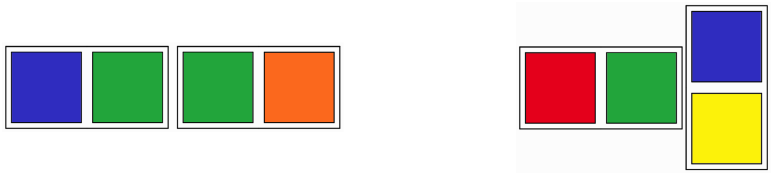


FIGURE 1. Sample moves in the game of *Spectrominoes*.

In each case, the question remains: *What are the rules of color addition?*

3. Stipulation for pigmentation combination

The two games briefly described in the previous section are based on a set of straightforward color addition rules that are a mixture of additive and subtractive color mixing. Subtractive color mixing is based on a classical RYB color model that can be thought of as mixing paint colors. This is

something that anyone who has played with finger paints would intuitively understand. An example of this is shown in Figure 2.

FIGURE 2. A fingerpainted masterpiece.

Additive color mixing is a more modern interpretation of color mixing where new colors are created by mixing different colors of light. One of the canonical models for this is the RGB standard for color on a computer monitor or television screen. Another example is the overlapping of colored lights in settings such as theaters as shown in Figure 3.

FIGURE 3. Color mixing via colored lights.

In the color addition rules for *Al-Jabar* and *Spectrominoes*, we mix the color mixing metaphors and default to having red, yellow and blue as the primary colors in both models. Additionally, using the metaphor of color mixing via colors of light, we include a notion of flipping a switch to turn on the light. This final idea gives the color addition rules a binary flavor which leads to a variety of mathematical interpretations.

There are five different color addition rules, with one of them being the same for each of the different colors. We describe them below in terms of their reliance on subtractive and additive color mixing fundamentals beginning with three rules based on subtractive color mixing. These are the more or less obvious rules that follow directly from the idea of mixing different colors of paint. They are:

$$\begin{aligned} \text{RED} + \text{YELLOW} &= \text{ORANGE} \\ \text{RED} + \text{BLUE} &= \text{PURPLE} \\ \text{YELLOW} + \text{BLUE} &= \text{GREEN} \end{aligned}$$

The other two rules are based on additive color mixing, with the extra notion of flipping a switch to turn on a specific light. The first of these is:

$$\text{RED} + \text{YELLOW} + \text{BLUE} = \text{WHITE}$$

We can interpret this as getting white light by mixing all of the colors of the spectrum which we can, in turn, create from the three primary colors red, yellow and blue. We will call this the **spectrum rule**. The final rule is a binary notion that arises when we think of getting a color by flipping a switch.

$$\text{RED} + \text{RED} = \text{BLACK}$$

In this case we can imagine that if we toggle the red light switch and then toggle it again, we get dark – symbolized by the color black – instead of a red light as a combination of red and red. There are, of course, eight versions of this rule, one for each color. These rules give rise to a variety of other unusual rules for color addition, which we will discuss as we consider the various ways we can interpret the rules using mathematical structures.

Now that we have the rules in place, we can have the means to experience abstract mathematical concepts somewhat tangibly through visual perception. This also allows us to investigate connections between these concepts and see the richness of the mathematical landscape. In [14] Wells gives us a nice motivation for making these connections.

> *"Exploration leads – as it does in natural history and geography – to important structures and features being identified, named and classified, so that the game develops its own language. These structures make abstract games playable and mathematics manageable."*

One of the pedagogical benefits of using the games as an approach to teaching mathematical concepts is that the rules are essentially simple, albeit somewhat counterintuitive, but this is where part of the benefit lies. Based on very straightforward ideas, surprising things can happen. That these things can illustrate a variety of mathematical principles can be a useful pedagogical tool since recreational mathematics can be though of as

a mechanism for making serious mathematics more approachable to those in the process of developing expertise. We begin with the interpretation that uses, perhaps, the most obvious mathematical structure and continue to other less apparent connections.

4. The game is called Al-Jabar for a reason

The initial interpretation of the color addition rules, once the robust structure was in place, was as a group. In particular, we can think of the rules as a colored version of the structure of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ where we have the colors corresponding to the elements of the group as follows.

RED	$\longleftrightarrow$	$(1,0,0)$	ORANGE	$\longleftrightarrow$	$(1,1,0)$
YELLOW	$\longleftrightarrow$	$(0,1,0)$	GREEN	$\longleftrightarrow$	$(0,1,1)$
BLUE	$\longleftrightarrow$	$(0,0,1)$	PURPLE	$\longleftrightarrow$	$(1,0,1)$
WHITE	$\longleftrightarrow$	$(1,1,1)$	BLACK	$\longleftrightarrow$	$(0,0,0)$

This allows us to rewrite the color addition rules from the previous section in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as follows:

$$\begin{aligned} R+Y &= (1,0,0)+(0,1,0) = (1,1,0) = O \\ R+B &= (1,0,0)+(0,0,1) = (1,0,1) = P \\ Y+B &= (0,1,0)+(0,0,1) = (0,1,1) = G \end{aligned}$$

$$R+Y+B = (1,0,0)+(0,1,0)+(0,0,1) = (1,1,1) = W$$

$$R+R = (1,0,0)+(1,0,0) = (0,0,0) = K$$

This also allows us to establish the somewhat counterintuitive color addition result of $GREEN + PURPLE = ORANGE$ as follows:

$$G+P = (0,1,1)+(1,0,1) = (1,1,0) = O.$$

For the sake of completeness, we have the following group table for color addition on the set of eight colors.

$\star$	B	R	O	Y	G	B	P	W
K	K	R	O	Y	G	B	P	W
R	R	K	Y	O	W	P	B	G
O	O	Y	K	R	P	W	G	B
Y	Y	O	R	K	B	G	W	P
G	G	W	P	B	K	Y	O	R
B	B	P	W	G	Y	K	R	O
P	P	B	G	W	O	R	K	Y
W	W	G	B	P	R	O	Y	K

FIGURE 4. The group table for color addition.

In a classroom setting, game play of *Al-Jabar* would give students the experience of interacting with the group structure in a visceral way. Assembling pieces to make an exchange amounts to calculating a sum in the group. In order to determine combinations that create a BLACK sum, we would look for the inverse of a collection of pieces with a particular sum.

If, instead of thinking of the colors as ordered triples, we think of them as vectors in 3-space, then we can realize a geometric structure of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as the vertices of a cube with the action of vector addition taken under mod 2 addition.

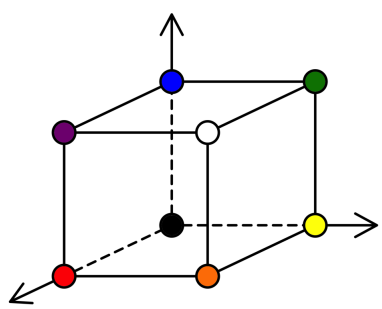


FIGURE 5. The rules of color addition as vectors in 3-space.

This may have applications in linear algebra or vector calculus as students begin to think about geometric and algebraic structures outside of the plane.

5. FINITA GEOMETRICUM

While the first geometric interpretation is closely related to the algebraic interpretation, being essentially the same thing with a slight change of notation, the second one is a bit more surprising. One caveat here is that we

do not have an exact representation since one essential element is lacking in practice. However, we can overcome this with a bit of cleverness. In the meantime, this second interpretation gives a very nice visual representation of the rules at the same time as an introduction to a non-Euclidean geometry. The corresponding picture can also act as a visual aid during game play. This new geometric interpretation is as a coloring of the Fano plane. The Fano plane is a finite geometry with 7 points and 7 lines. The general structure looks like the picture below where the black dots are the seven points and the line segments, rather than being actual lines in the geometry them selves, indicate which points are on the lines. Note that the *circle*

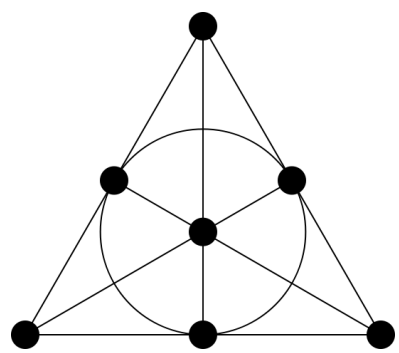


FIGURE 6. The Fano plane.

connecting the three midpoints indicates that these three points form a line in this geometry.

According to the Pólya Enumeration Theorem, there are

$$\frac{1}{168}\left(n^7 + 21n^5 + 98n^3 + 48n\right)$$

colorings of the Fano plane with n colors. For $n = 7$, we can choose one of the 7205 colorings as shown below to create a visual guide to the addition rules – with the exception of the binary structure given by black being the sum of any color with itself.[2] Notice that the midpoint of each side of the triangle is the secondary color corresponding to the sum of the colors of the two corresponding vertices, as in the metaphor of subtractive color mixing. But since the geometry does not have an inherent notion of betweenness, we can consider any point on any line to be the one *in the middle* so that we get RED=PURPLE+BLUE as well. To recover the spectrum rule, we can

[2]We make the following convention: we call such a coloring, using the 7 colors $\{R, O, Y, G, B, P, W, K\}$, a *spectral coloring* – or refer to the figure as being *spectrally colored* – as a nod to the colors of the spectrum.

perform the following calculation,

$$\begin{aligned} RED + YELLOW + BLUE &= RED + (YELLOW + BLUE) \\ &= RED + GREEN \\ &= WHITE \end{aligned}$$

by calculating the sums along individual lines. But this essentially forces the sum of the colors on a line to be black so that we recover, in some sense, the notion of the binary structure.

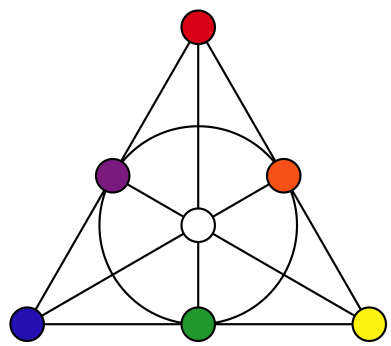


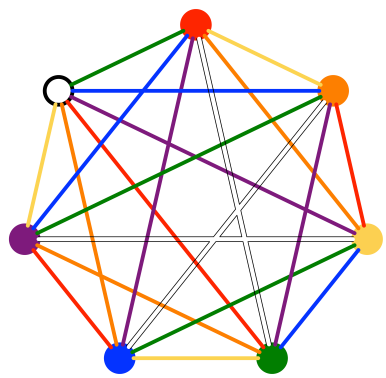
FIGURE 7. A *spectral* coloring of the Fano plane.

As an extension of this idea, we see from the work of Baez that we can recover the structure of the octonians from our coloring of the Fano plane.[4] In order to mitigate the appearance of the additive inverses, we can restrict the field of coefficients to $\mathbb{Z}_2$. We leave it as an exercise for the reader to determine the correspondence between the colors and the non-identity elements of the octonians.

This representation of finite geometries and division algebras through color addition seems to be a means by which students can be introduced gently to concepts that are far outside of their experiences from their secondary mathematics classes.

6. Totally colored complete graphs

In this section we consider a spectral coloring of K_7 in order to see the effects of various moves and the interdependencies between the colors. This graph is undirected and the color of the edge indicates the color that would be added to an individual vertex to create the color of the adjacent vertex. Similarly, the sum of the colors of the vertices at the ends of a given edge would be the corresponding color of the edge.



Notice that this is a proper coloring of the vertex set of K_7. It is also a proper coloring of the edge set of K_7. This makes the coloring a proper total coloring of K_7. This final feature is a variation on graph coloring that is both a proper vertex coloring and a proper edge coloring and also a coloring such that no edge shares a color with either of its incident vertices.

Moreover, this particular coloring is a minimal example of such a coloring since one fundamental property of the total coloring number $\chi''(G)$ of a graph G is that $\chi''(G) \geq \Delta G + 1$ where $\Delta(G)$ is the maximum degree of G. In this case we get equality, which is a well known result for complete graphs on an odd number of vertices.

With this particular coloring in place we have the following two results.

Proposition. *The edge sum of a cycle in a spectrally colored K_7 is black.*

Proof. Let S be a spectrally colored K_7 and let $C = e_1 e_2 \cdots e_k$ be a cycle in S. Then we can write the edge sum as

$$\begin{aligned} e_1 + e_2 + \cdots + e_k &= (v_1 + v_2) + (v_2 + v_3) + \cdots (v_k + v_1) \\ &= (v_1 + v_1) + (v_2 + v_2) + \cdots + (v_k + v_k) \\ &= K. \end{aligned}$$

Therefore the edge sum of a cycle is black. □

Proposition. *The vertex sum of a cycle in a spectrally colored K_7 is equal to the vertex sum of the remaining vertices.*

Proof. Let S be a spectrally colored K_7 and let $C = v_1 v_2 \cdots v_k$ be a cycle in S. Then the vertex sum in S is

$$\begin{aligned} c_{v_1} + c_{v_2} + \cdots + c_{v_k} &= [c_{v_1} + c_{v_2} + \cdots + c_{v_k}] + \sum_{v \in S} c_v \\ &= [c_{v_1} + c_{v_2} + \cdots + c_{v_k}] + (c_{v_1} + c_{v_2} + \cdots + c_{v_k}) + \sum_{v \notin C} c_v \\ &= (c_{v_1} + c_{v_1}) + \cdots + (c_{v_k} + c_{v_k}) + \sum v \notin C c_v \\ &= K + \cdots K + \sum_{v \notin C} c_v \\ &= \sum_{v \notin C} c_v \end{aligned}$$

as desired. □

The second proposition allows us to think about the exchanges made in the games in terms of different sets of pieces. It also provides an alternate reason for the unusual color addition rule

$$\mathsf{O} + \mathsf{G} + \mathsf{P} = \mathsf{K}$$

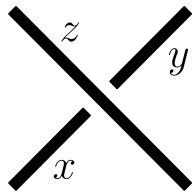
since the colors not in the cycle O–G–P are the colors in the spectrum move, whose sum is black.

Graph theory is an interesting visual branch of mathematics which is easily accessible to undergraduate students without requiring them to have an extensive background in order to understand interesting questions and even to begin doing research. Graph coloring is an active area in this discipline and the picture above is a nice entry point into this domain.

7. A knotty interpretation

The basic study of knot theory can be thought of as the search for ways to distinguish between knots. When this search is successful, the methods for distinguishing between knots are measurements of the complexity of a knot – so-called **knot invariants**. That is, the measurement given does not depend on the particular picture of the knot, or the way the knot is sitting in space, but rather the essential structure of the knot. One particular invariant is related to colorings of knot projections, those pictures of knots where a break in the string indicates that a strand is passing behind another strand. For a given prime number p, a knot is said to be p-colorable if there exists a projection of the knot which can be colored using the *colors* $\{0, 1, 2, \ldots, p-1\}$ so that at each crossing the sum of the two understands

is equal to twice the over strand, mod p. That is, at a given crossing with the indicated coloring we have $x + y \equiv 2z \pmod{p}$.



$x + y \equiv 2z \pmod{p}$

FIGURE 8. The p-colorability relation.

We can make a connection to knot theory by considering the following spectrally colored picture of the 7_1 knot:

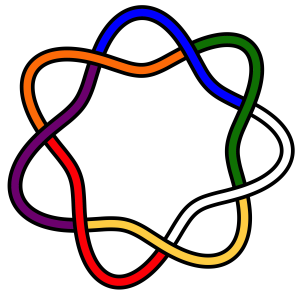


FIGURE 9. A colored projection of the 7_1 knot.

If we define a mapping $\bowtie$: $\{R, O, Y, G, B, P, W\} \to \{0, 1, 2, 3, 4, 5, 6\}$ between the usual colors in the picture and the mod 7 natural numbers as shown below, then we have a colored picture that satisfies the modular relationship described above, and which also follows the rules of color addition.

$$\bowtie = \{(\mathsf{W}, 0), (\mathsf{Y}, 1)(\mathsf{R}, 2), (\mathsf{P}, 3), (\mathsf{O}, 4), (\mathsf{B}, 5), (\mathsf{G}, 6)\}$$

Then we see that at the PURPLE-ORANGE-BLUE crossing the understrand sum would be $P + B \bowtie 3 + 5 \equiv 1 \pmod{7}$ while $2O \bowtie 2(4) \equiv 1 \pmod{7}$. Moreover, the sum of the three colors, as we have considered before is given by $P + O + B = Y \bowtie 1$. So we have a relationship that resembles the knot theory calculation for p-colorability and also retains aspects of color addition. We note here that we have not used the color BLACK because this notion of colorability uses a prime number of colors. So this correspondence would not account for a crossing in a knot in which all of the strands were

the same color, which is an allowable coloring since $x + x \equiv 2x \pmod{p}$ for any p and for any $0 \leq x < p$.

Knot theory is another very visual entry point into higher mathematics for students, and colorability is one of the nice properties that is straightforward to check while also having deep connections to other knot theory invariants.

8. The difference is it

We have saved what is perhaps the most interesting interpretation for last, and it is in some ways the most abstract. As usual we consider the set of colors used in the game.

$$S = \{\mathsf{R}, \mathsf{O}, \mathsf{Y}, \mathsf{G}, \mathsf{B}, \mathsf{P}, \mathsf{W}, \mathsf{K}\}$$

From this set we can create the power set $\mathcal{P}(S)$ which will have $2^8 = 256$ elements. Then we can define an equivalence relation $\triangle$ on the power set by saying that sets X and Y are equivalent, or $X \triangle Y$ if the sum of the colors in the sets are equal. For example if we have

$$C_1 = \{\mathsf{O}, \mathsf{G}\} \quad \text{and} \quad C_2 = \{\mathsf{R}, \mathsf{B}\}$$

then we know $C_1 \triangle C_2$ since the sums of the colors in both sets is purple. There is still the question of how to define the *color* of the empty set. For various reasons, it is logical to define the color of the empty set to be black. We see that if we define the color of the empty set to be clear, that is $\varnothing \triangle \{\mathsf{K}\}$, then

Using this equivalence relation we can consider the function

$$\triangle : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$$

where $\triangle$ is the familiar symmetric difference operator. That is

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$

Under this binary relation we can recover the structure of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by the operation $[A] \triangle [B] = [C]$ where $[X]$ is the singleton set which we choose as the representative of the associated equivalence class.

If we define the set $\mathcal{A}$ to be the elements of the power set of S with at most three elements along with the set $\{R, Y, B, W\}$ then we have a list of the possible moves in the game *Al-Jabar*. For example, there would be eight subsets in $\mathcal{A}$ which are equivalent to RED:

$$[\mathsf{RED}] = \left\{ \{\mathsf{R}\}, \{\mathsf{Y}, \mathsf{O}\}, \{\mathsf{G}, \mathsf{W}\}, \{\mathsf{B}, \mathsf{P}\}, \{\mathsf{O}, \mathsf{G}, \mathsf{B}\}, \{\mathsf{O}, \mathsf{P}, \mathsf{W}\}, \{\mathsf{Y}, \mathsf{G}, \mathsf{P}\}, \{\mathsf{Y}, \mathsf{B}, \mathsf{W}\} \right\}$$

Finally, if we allow for multisets with multiplicity two, then we can recover the idea of the sum of two colors being equal to black.

References

[1] C. Adams, *The knot book: An elementary introduction to the mathematical theory of knots*, Amer. Math. Soc., Providence, RI, 2004.

[2] *Al-Jabar: A Mathematical Game of Strategy*, `http://al-jabargames.com`, last accessed October 16, 2013.

[3] J. Baez, *The octonians*, Bull. Amer. Math. Soc. **39** (2002), 145-205.

[4] J. Bewersdorff, *Luck, logic and white lies: The mathematics of games*, translated by D. Kramer, AK Peters, Wellesley, MA, 2005.

[5] G. Chartrand & L. Lesniak, *Graphs and digraphs*, Chapman & Hall, London, 1996.

[6] M. Davis, *Game theory: A nontechnical introduction*, Basic Books, New York, 1983.

[7] T. Huffman, *Total colorings of graphs*, Master's Thesis, San Jose State University, 1989.

[8] G. Owen, *Game theory*, W.B. Saunders, Philadelphia, 1968.

[9] Rules of *Mexican Train*, `http://en.wikipedia.org/wiki/Mexican_train`, last accessed October 16, 2013.

[10] R. Schneider, personal communication.

[11] *Spectrominoes*, `https://www.facebook.com/Spectrominoes`, last accessed October 16, 2013.

[12] S. Vajda, *Mathematical games and how to play them*, Ellis Horwood, West Sussex, 1992.

[13] *Websters Encyclopedic Unabridged Dictionary of the English Language*, Dilithium Press, Ltd. New Jersey, 1994.

[14] D. Wells, *Games and mathematics: Subtle connections*, Cambridge University Press, 2012.