GATHERING FOR

12

G4G12 Exchange Book

VOLUME 1

Art, Games, Magic, & Math

Atlanta, Georgia

MARCH 30 - APRIL 3, 2016

G4G12 Exchange Book

VOLUME 1

The Gift Exchange is an integral part of the Gathering 4 Gardner biennial conferences. Gathering participants exchange gifts, papers, puzzles and other interesting artifacts. This book contains gift exchange papers from the conference held in Atlanta, Georgia from Wednesday, March 30th through Sunday, April 3rd, 2016. It combines all of the papers offered as exchange gifts in two volumes.

Acknowledgments

Gathering 4 Gardner would like to offer thanks to the following individuals:

Freddy Bendekgey – for editing and laying out the pages of this book
Katie LaSeur – for project management and art direction
John Miller – for helping manage the papers and following up with authors
Nancy Blachman – for getting this book started

There are many things you can buy or make yourself, but there are some things that can only be created by a group. This book is such an item.

In Memoriam

With sadness, we note the passing of the following Gathering 4 Gardner attendees:

Raymond Smullyan (5/25/1919 - 2/6/2017) – Puzzle-creating Logician
Solomon Golomb (5/30/1932 - 5/1/2016) – Known for work on Mathematical games & pentominoes
Laurie Brokenshire (10/20/1952 - 8/4/2017) – Magician and world-class puzzle solver
Marc Pelletier (8/4/1958 - 11/6/2017) – Co-founder of Zometool

Preface

It's not often that a conference is so culturally diverse that its presenters and patrons include mathematicians, physicists, philosophers, logicians, jugglers, puzzle designers, artists, card players, and knitters. Yet it happens every two years at the Gatherings for Gardner, held in honor of Martin Gardner, author of Scientific American's mathematical recreation column for nearly a quarter-century. The 82 papers of these two volumes are write-ups of presentations at the twelfth conference in this series: G4G12.

These contributions range from technical to playful, instructive to entertaining. They include serious mathematical papers and a paper on how to knit strange topologies. Others papers refer not to presentations but to exchange gifts, such as one that gives a literal twist to rotating tetrahedral rings. Papers are included on puzzles, fractals, tilings, hexaflexagons, and surprising applications of the number 12. If you had to choose a single descriptive title for these two volumes, it might be: Exhibiting mathematics in novel, surprising, and tangible ways.

These papers represent but one thread of G4G12's full creative richness. Many presentations were not written up here: John Conway's on "Chemical Pi," for example, a way to memorize over a thousand places of Pi by associating each chemical element with 10 places; Andrea Hawksley's on making Fibonacci lemonade, whose intensity of flavors increases exponentially as you drink down; or Eric Demaine's on juggling patterns where the balls' trajectories resemble letters and therefore create the possibility of communicating secret messages.

Furthermore, these presentations were but fuel to prime the real creative energy of G4G12, which took place in its interstities – in the conversations one had before and after the presentations, or chance encounters during the excursions or evening events. Toy collector Tim Rowett might pull some recent acquisition out of his pocket to show you, Gwen Fisher might explain how to knit a Moebius strip, magician Mark Mitton demonstrate a new trick, or other people insist that you let them teach you to trisect an angle or cheat at dice.

As you enjoy these write-ups, think of them as assists to getting you into the Gardner-zone of playful mathematical thinking, enacted again and again at these conferences.

Robert P. Crease

Chair, Department of Philosophy
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Stony Brook, NY

G4G12 video presentations can be found on our website:
www.Gathering4Gardner.org/G4G12-Videos

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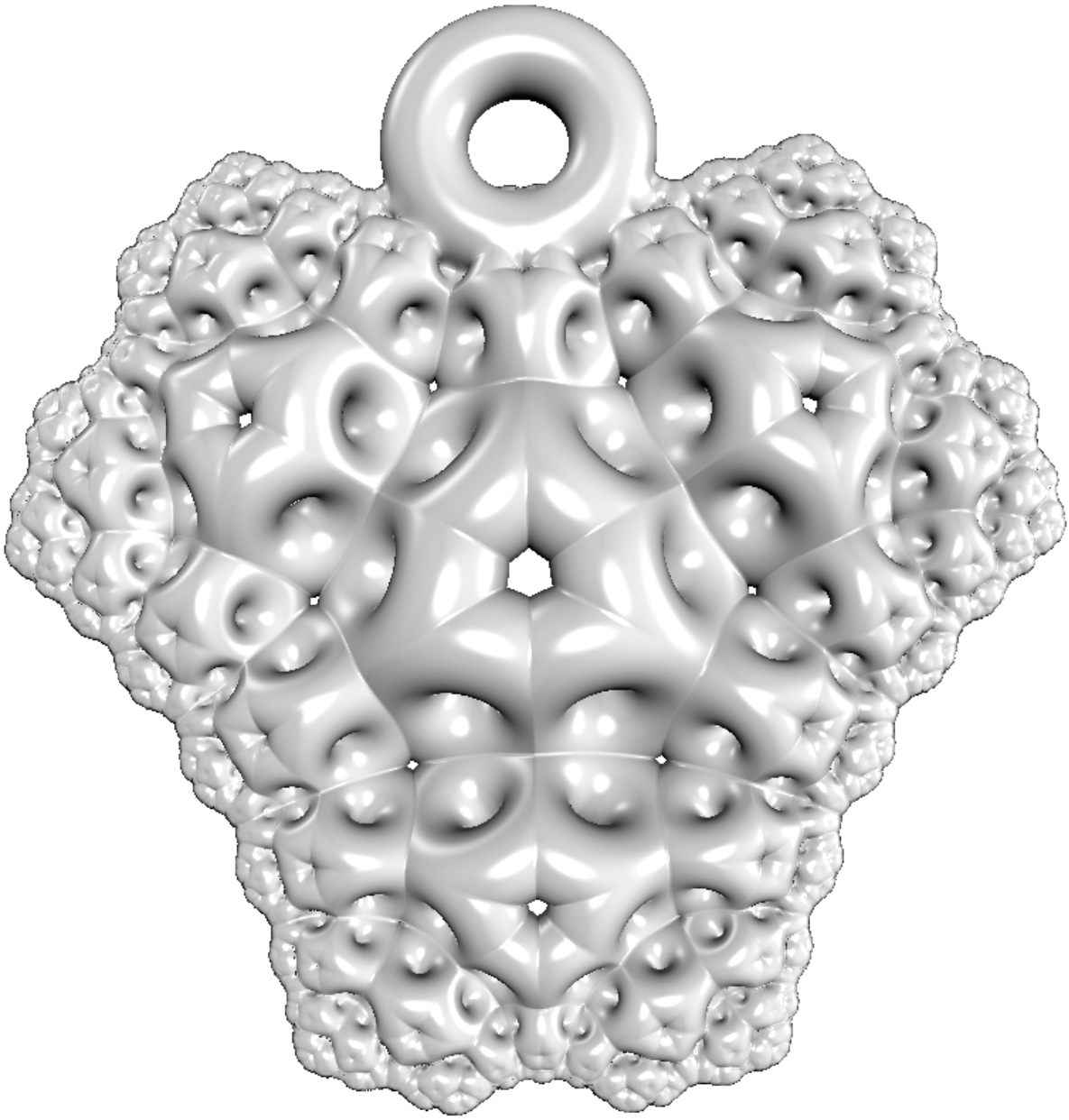
3D-Printed Fractal Pendant

by Vladimir Bulatov



Almada Negreiros and the Geometric Canon

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January 30, 2015

Abstract

This paper presents a mathematical analysis of a series of geometrical abstract artworks by Portuguese author Almada Negreiros (1893-1970), understood in the context of the author's search for a *canon*.

After a brief description of Almada's work in the frame of 20th century visual arts, we examine the mathematical elements in three of his works: illustrations for a newspaper interview, two drawings from a collection called *Language of the Square* and his last visual work, the mural *Começar*.

The analysis revealed that some of the author's geometrical constructions were mathematically exact whereas others were approximations. We used computer based drawings, along with mathematical deductions to examine the constructions presented in the aforementioned works, which we believe to be representative of Almada's geometric statements. Our findings show that, albeit limited by the self-taught nature of his endeavor, the mathematical content of these artworks is surprisingly rich.

The paper is meant to be an introduction to Almada's work from a mathematical point of view, showing the importance of a comprehensive study of the mathematical elements in the author's body of work.

1 Introduction

José de Almada Negreiros (São Tomé and Príncipe, 1893 – Lisbon, 1970) was a key figure of 20th century Portuguese culture, in both visual arts and literature. See Figure 1 for a photo of Almada.[1]

As a member of the group *Orpheu* (composed by the new literary generation of the beginning of the century) and by proclaiming himself a futurist, he clearly demonstrates from a young age his desire to break with the academic traditions of his time. In the beginning of his artistic and literary career he made himself noticed by his active and openly critical stance on the Portuguese art scene of the time, with drawings, exhibitions, collaborating in literary journals, or his famous Anti-Dantas Manifest (1915). In this text he reacts to reactionary positions in the Portuguese art scene. Even if his visual artwork is not easily associated with futurism, being closer to cubism or geometric abstractionism, Almada remains a crucial figure of Portuguese modernism.

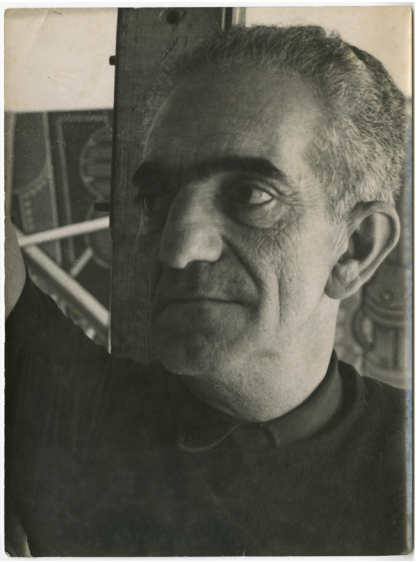

Figure 1: Almada Negreiros in the late 1940s

In his early years, Almada was close to Amadeo de Souza-Cardoso[2] and Santa-Rita Pintor[3], which were the main figures in Portuguese modern art. Although Amadeo and Santa-Rita died in 1918, Almada continued to work intensely for many decades. Over the years, his initial restless attitude gave way to a more lyrical approach in his works, both in visual art and literature.

[1] The artist himself chose to use just the name "Almada". His signature can be seen in some of the artworks presented in this paper.

[2] Amadeo de Souza-Cardoso (1887-1918): Probably the most internationally recognized Portuguese modernist painter, he left an extensive body of work that articulates Cubism, Futurism and Expressionism.

[3] Santa-Rita Pintor (1889-1918): A member of the first generation of Portuguese modernist painters, he died prematurely leaving orders to have his work destroyed. Few examples of his artwork remain.

After a short stay in Paris (1919-1920) he resided for some time in Madrid (1927-1932) where he began articulating his work with architecture, something he would continue to do for many years with different media. Returning to Portugal, where he would spend the rest of his life, he marries the painter Sarah Afonso[4] and carries out a long list of public works in collaboration with architects, namely Pardal Monteiro[5]. Among these the most remarkable are the murals of the buildings at the maritime stations of Rocha Conde de Óbidos and Alcântara, in Lisbon. Overall, Almada leaves a vast legacy, which includes poetry, plays, lectures, painting and public art such as murals, frescoes, stained glass windows and tapestries.

Over the years, his work came to incorporate more geometric abstraction—from a very early stage geometry was one of the author's admitted passions (notably in the understanding and analysis of Portuguese renaissance painting, as we will see in the next section). In the 1950s his artwork begins to have a strong geometric tone bearing some relations, in form and artistic intention, to Mondrian ou Le Corbusier. In the 1960s he publishes several interviews (cf. [Va60]) on the importance of geometry in art, proposing a universalistic theory on the matter, which he called a canon. Motivated by the search for this canon, to be found in various artistic manifestations throughout time, Almada devoted himself, for decades, to research on geometry. His last piece, *Começar*, which will be studied in this paper, is a remarkable conclusion of this search.

Almada Negreiros' estate is currently undergoing inventory and analysis by a multidisciplinary team that includes researchers from several areas, such as literature, fine arts, history and mathematics. This research made possible the analysis of artworks that were, up to now, unknown to the public.

To our knowledge, only the books [Fr77] and [Ne82], the article [Co94], and the thesis [Va13] presented some mathematical analysis of Almada's work, some of the material about *Começar* is already studied in some of these works, but not with the detail we present here. We have published an analysis of some drawings in [CF14], which is a bilingual publication (Portuguese and English). The material about the *Language of the Square*

[4]Sarah Afonso (1899-1983): From the second generation of Portuguese modernist painters, the wife of Almada Negreiros studied painting in Paris during the 1920s but decided to stop pursuing her career in 1940.

[5]Pardal Monteiro (1897-1957): One of the most important Portuguese architects of the first half of the twentieth century. He was responsible, along with a small group of other architects, for the implementation of modernist architecture in Portugal.

we present is original.

Other than these, we know of no other published studies of the mathematical elements in Almada's visual work.

2 Almada and Renaissance Art

Almada's interest in geometry stemmed from his studies of Portuguese renaissance visual art. He was particularly interested in two masterpieces: a Portuguese renaissance polyptych (composed of six panels), the *Painéis de S. Vicente*, by Nuno Gonçalves, and *Ecce Homo*, originally also attributed to Nuno Gonçalves, now known to be a later work. Both of these are still on display at the Portuguese *Museu de Arte Antiga*.

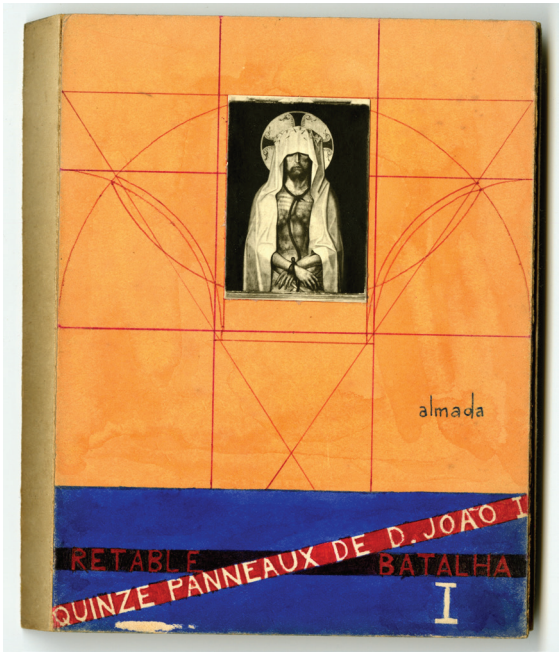

Figure 2: Cover of a notebook with a photograph of *Ecce Homo*

Figure 2 is the cover of one of Almada's notebooks, with a photo of the *Ecce Homo* and some geometric studies.

This fascination with *Ecce Homo* and the *Painéis de S. Vicente* led Almada to a compositional analysis of these pieces, using circles, squares, rectangles, and other geometrical constructions, superimposed on the paintings (as can be hinted from the geometric lines on the notebook cover).

Some of these studies on perspective actually led to a redefinition of the way the polyptych was presented: nowadays, the six panels are presented as an ensemble, instead of being divided in two triptychs.

This analysis was deeply rooted in geometric considerations. It started probably in the 1920s and lasted until the end of his life, in 1970. As his investigations developed, Almada postulated the existence of a *canon*, present in all art throughout history, manifesting itself in specific rules in each artistic period. He used several expressions to refer to this canon, the most famous one being "relation nine/ten".

It is not clear what were the contents of this canon, one can only guess what they were from the art works produced and some brief (and often obscure) explanations by Almada. For instance, here's how he described the elements of the canon in an interview, [Va60]:

> The simultaneous division of the circle in equal and proportional parts is the simultaneous origin of the constants of the relation nine/ten, degree, mean and extreme ratio and casting out nines.

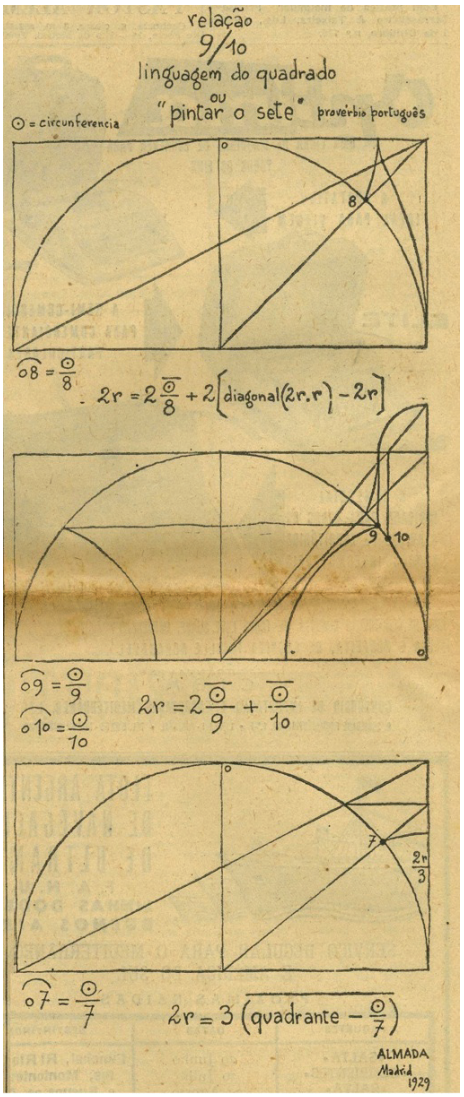

Figure 3: Drawings from 1929, published in the newspaper *Diário de Notícias* in 1960 illustrating the interviews [Va60]

As Almada became more interested in this canon, he

started to produce more geometrically based artworks, some of them meant to reveal this canon, directly and without explanation. As a first example, Figure 3 presents three drawings of canonic elements. The drawings are from 1929 and are included in the interviews [Va60].

The title reads

relation
9/10
language of the square
or
"paint the seven"[6]

In these drawings Almada presents three relations between some measurements in the semicircle and the circumscribed rectangle.

The first one states that the chord of the 8th part of the circle, added to the difference between the diagonal and the side of the rectangle is equal to the radius of the circle.

The second one presents constructions for the 9th and 10th parts of the circle, stating that the diameter is equal to the chord of the 10th part plus two times the chord of the 9th part.

The third one starts by constructing the 7th part of the circle, stating afterwards that the chord of difference between the quarter circle and the 7th part of the circle is two-thirds of the radius.

As we will see, all these three statements are approximations. Even without any computations, one can see that the two last ones can only be approximations because of the following theorem.

Gauss-Wantzel's Theorem. *The division of the circle in n equal parts with straightedge and compass is possible if and only if*

$$n = 2^k p_1 \dots p_t$$

where $p_1, \dots, p_t$ are distinct Fermat primes.

A *Fermat prime* is a prime of the form $2^{2^m} + 1$. Presently, the only known Fermat primes are 3, 5, 17, 257 and 65537.

This result implies that the 9th and the 7th parts of the circle cannot be determined with straightedge and compass, since 7 is not a Fremat prime

[6]This is a Portuguese expression, with a meaning similar to "Paint the town". Almada's own interpretation was "to produce wonders".

and $9 = 3 \times 3$, the Fermat prime 3 appears twice in the factorization of 9. Therefore, the last two constructions cannot be exact, as they include the chords of the 7th and 9th parts of the circle.

As for the first construction, one can also conclude that it is approximate, using the following formula for the chord of the n-th part of the circle of radius r:

$$\text{chord}(n) = 2r \sin(\pi/n). \tag{1}$$

The value of $\sin(\pi/8)$ can be found using elementary complex analysis, and it equals $\frac{1}{2}\sqrt{2-\sqrt{2}}$, so $\text{chord}(8) = \sqrt{2-\sqrt{2}}r$. The difference between the diagonal and the long side of the rectangle is $\sqrt{5}r - 2r = (\sqrt{5}-2)r$, so

$$(\text{chord}(8) + \sqrt{5} - 2)r = (\sqrt{2-\sqrt{2}} + \sqrt{5} - 2)r = 1.001434r.$$

The approximations in the other constructions are of the same order of magnitude and can be easily calculated with any dynamic geometry application, such as geogebra.

So, in the present case, all three constructions are approximations, but in the next section we will present some exact ones. It is not clear whether Almada was aware of this, as he presents them all without explanations, and without distinguishing the exact ones from the approximate ones. There is another quote from Almada, also in [Va60], which may refer to this: "Perfection contains and corrects exactness".

The presence of mathematics in Almada's work is thus justified by the nature of the author's program: to find and reveal the canon underlying all art. One that, in his own words "is not the work of Man, but his possible capturing of immanence. It is the initial advent of epistemological light" (interviews [Va60]). Mathematics is particularly appropriate for this project, as its constructions are abstract and general, not directly connected to any particular art period.

In this work we do not wish do dwell on the philosophical and artistic issues that would develop from Almada's postulate of the existence of this canon. We are primarily interested in analysing the geometric artworks from a mathematical viewpoint (as we have just done).

3 Language of the Square

Almada expanded this type of simple geometric drawings in a collection, dating probably from the 1960s, which he called *Language of the Square*.

It consists of a series of fifty-two drawings on paper (all with 63 × 45 cm) done with pencil, ink, marker and ballpoint pen. It also includes sketches and quotes (from Aristotle and Vitruvius among others).

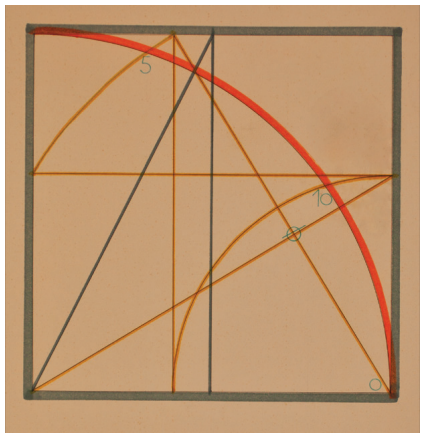

Figure 4: One of the drawings of the collection *Language of the Square*

The composition of the drawings generally starts from a square with a quarter-circle inscribed in it — a quadrant — or with a rectangle, formed by two of these figures. From this starting point it is possible to obtain several geometric elements considered by Almada as canonical, such as the circle's division in equal parts or the golden ratio.

Figure 4 is one of these drawings, a rather simple and elegant construction which gives us an exact determination of the golden rectangle and the 5th and 10th parts of the circle.

It is not too difficult to prove the exactness of the construction. We now refer to Figure 5, where we reproduce the original drawing, with some of the original notation for points (O, 5 and 10), along with some other notations we needed to add.

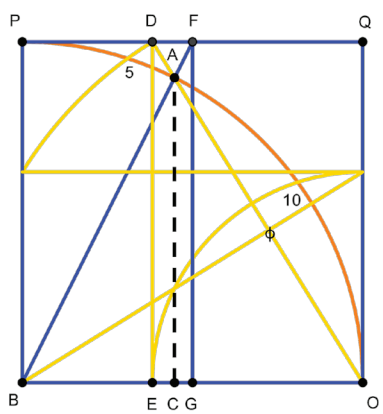

Figure 5: A reproduction of the drawing in Figure 4

We first prove that $DEOQ$ is a golden rectangle. For this, we prove that

$EO/DE = 1/\phi$. Since $\phi^2 - \phi - 1 = 0$, we have

$$\frac{1}{\phi} = \phi - 1 = \frac{1+\sqrt{5}}{2} - 1 = \frac{\sqrt{5}-1}{2},$$

so we need $EO/DE = (\sqrt{5}-1)/2$.

Since the triangles DEO and ACO are similar, we have

$$\frac{EO}{DE} = \frac{OC}{AC} = \frac{OB - BC}{AC} = \frac{OB}{AC} - \frac{BC}{AC} = \frac{AB}{AC} - \frac{BC}{AC}.$$

Now we use the similarity of triangles ACB and FGB, noting that in both of them the larger leg measures twice the smaller, since the rectangle $BPFG$ is a half square. This means that $BC = (1/2)AC$ and

$$AC^2 + \left(\frac{1}{2}AC\right)^2 = AB^2 \Leftrightarrow \frac{AB^2}{AC^2} = \frac{5}{4}.$$

Therefore

$$\frac{EO}{DE} = \frac{AB}{AC} - \frac{BC}{AC} = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{1}{\phi}.$$

For the 5th and 10th parts of the circle, we will use the formula for the chord, which we have already presented. We need the sines of $\pi/5$ and $\pi/10$, which can be determined exactly:

$$\sin(\pi/5) = \frac{\sqrt{2(5-\sqrt{5})}}{4} \qquad \sin(\pi/10) = \frac{\sqrt{5}-1}{4} = \frac{1}{2\phi}.$$

According to formula (1), page 7,

$$\text{chord}(5) = \frac{\sqrt{2(5-\sqrt{5})}}{2}r \qquad \text{chord}(10) = \frac{r}{\phi}.$$

Therefore, all that remains to be proved is that

$$OD = \frac{\sqrt{2(5-\sqrt{5})}}{2}r \ \text{ and } \ OE = \frac{r}{\phi}.$$

Both assertions are easy consequences of the fact that $DEOQ$ is a golden rectangle.

Figure 6 sketches the standard constructions for the 5th part of the circle and the golden rectangle.

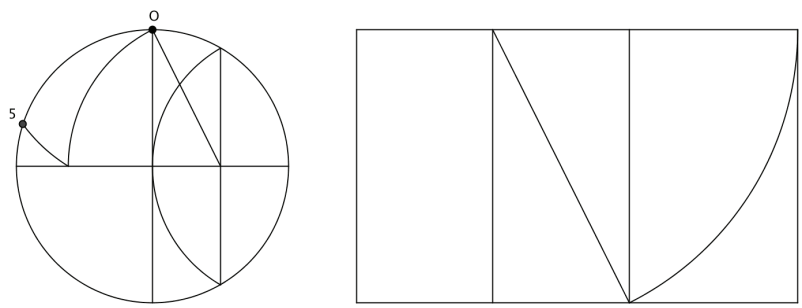

Figure 6: Standard constructions for the 5th part of the circle and the golden rectangle

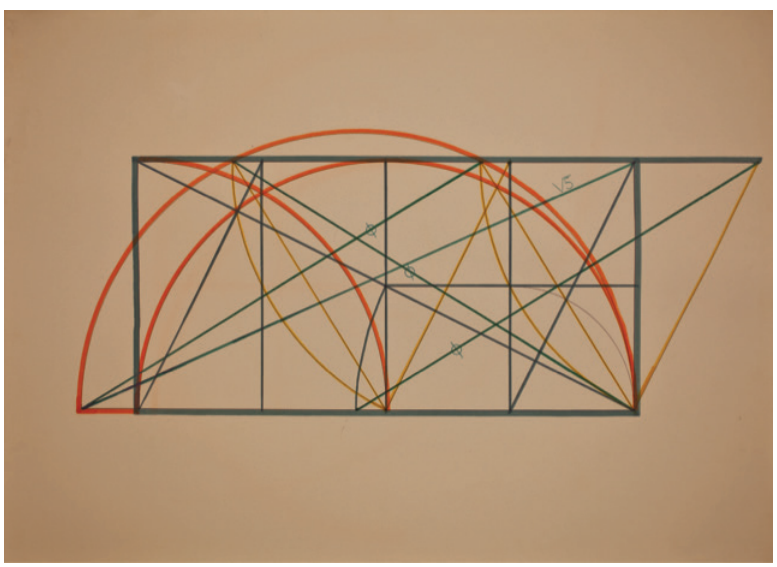

Figure 7: Another drawing from the collection *Language of the Square*

To finish this section, we present, in Figure 7, another drawing from the same collection in which the previous construction appears.

All lines marked ϕ are diagonals of golden rectangles, and the line marked $\sqrt{5}$ is the diagonal of a rectangle in which the length of the larger side is $\sqrt{5}$ times that of the smaller one. Also, one can see an intersection of four lines, two straight lines and two arcs of circle, in the top left (we invite the reader to check that the four lines do intersect). The two blue straight lines in this

intersection are called *reciprocal diagonals* in Hambidge's book [Ha67]. These are perpendicular lines that can be used to construct similar rectangles, which have these lines as diagonals. Hambidge also makes an extensive study of rectangles of given proportions (ϕ and $\sqrt{5}$ are among them) applying these to the analysis of art (especially of greek vases). Almada acknowledged Hambidge's influence in his own work, albeit claiming originality in his own constructions.

4 Começar

Finished in 1969, a year before Almada's death, *Começar* (*To begin*) is a colored sunk relief, on marble, found on Calouste Gulbenkian Foundation's main atrium wall, in Lisbon. It assembles, anthologically, on a composition of impressive size (about 2 metres by 13 metres), many of the author's discoveries on this matter, see Figure 8.

Figure 8: *Começar*, mural, 1968/69

Some of the structures and constructions presented in *Language of the square* appear here, woven into a very intricate network of lines. Some of this geometric material appeared in a tapestry from 1958, simply called *Número*

(*Number*). Figure 9 presents a detail of this tapestry, showing some of the geometric elements.

In this fragment of the tapestry we see some of what Almada called "epochal manifestations of the 9/10 relation", elements of art that follow rules inspired by the canon. From top to bottom, we see several elements accompanied by geometric descriptions. The two top ones are a Babylonian vase and an element from a frieze in a greek palace in Knossos. Afterwards we find the Pythagorean *Tetracys* (an arrangement of ten sticks in a triangle) and a right triangle with sides measuring 3, 4 and 5 — Almada calls it the 3.4.5 triangle.

Figure 9: Detail of the tapestry *Número*, 1958.

Right below we find a geometric construction of a square and a triangle inscribed in a circle. This is Almada's interpretation of the so-called *Point of the Bauhütte*. It refers to a four verse stanza (which Almada found in a text by architect Ernest Mössel), going back to a medieval guild of masons called the Bauhütte. The stanza refers to a point, critical for the work of stone masons, which was kept a secret, and is stated as follows in [Gh77, p. 120]:

> A point in the Circle
> And which sits in the Square and in the Triangle.
> Do you know the point? then all is right,
> Don't you know it? Then all is vain.

The stanza does not give a clear construction of the point (probably on purpose). Usually it is understood as referring to a point common to a circle, a square and a triangle, the two latter ones inscribed in the circle. This is the case with Almada's construction, which we will analyse below in more detail.

Finally, we have a 16-pointed star which Almada attributes to Leonardo da Vinci, called the *Figura superflua ex errore*. It is also represented in *Começar*, where only half the figure is visible.

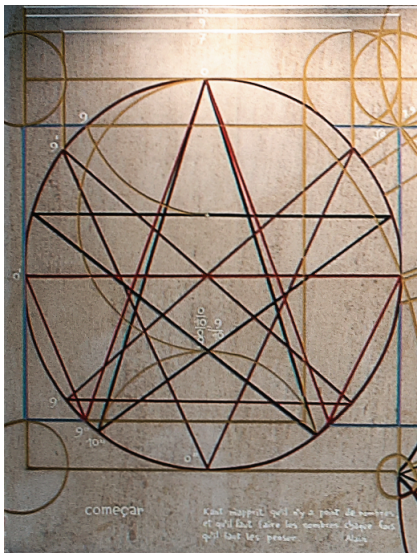

Figure 10: Detail on the left of the mural *Começar*

As we can see these elements are displayed in chronological order, in an atempt to demonstrate the influence of the canon in art throughout history.

We now go back to to *Começar*, focusing on the leftmost and rightmost parts of the mural.

Figure 10 details the left side, Figure 11 reproduces the drawings, preserving the original notation for some points.

Among other elements, we find a regular pentagram, drawn in black, inscribed in a circle. With this figure Almada comes back to one of his canonical relations, already presented in the drawings of 1929, which states that the diameter of the circle is twice the chord of the 9th part plus the chord of the 10th part:

$$2r = 2\frac{\overline{\odot}}{9} + \frac{\overline{\odot}}{10}$$

In the drawing, Almada states that OM has the same length as the chord of 9th part of the circle ($M\frown 9$) and that $OM = MN$, via the half circle $O\frown N$. Finally, arc $N\frown 10''$ states that NO'' is the chord of the 10th part of the circle, thus illustrating his statement.

As we have seen, the 9th part of the circle cannot be exact, in this construction the error is $0.006954r$. All the remaining constructions (the 10th part of the circle and the equality of OM and MN) are exact.

On the rightmost part of the panel we find an area with a very dense

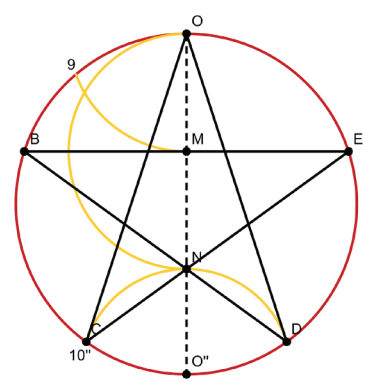

Figure 11: A reproduction of the drawing in Figure 10

mesh of lines, see Figure 12. Again, we will stick to the drawings in black, a triangle and a square, both inscribed in a circle, a representation of the Point of the Bauhütte. Figure 13 reproduces the drawing.

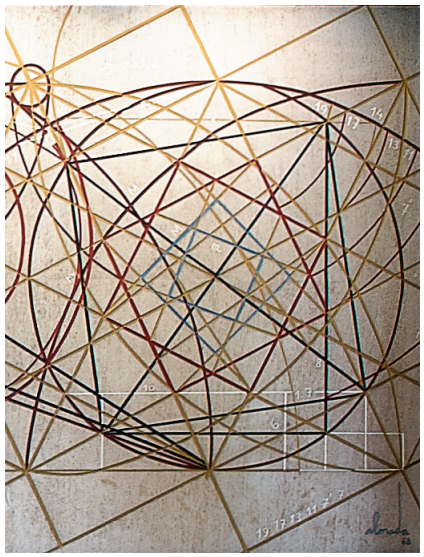

Figure 12: Detail on the right of the mural *Começar*

However, Almada himself shows us a much easier way of proving this, in one of his studies for the construction, using grid paper, see Figure 14. Notice that there is even a reference to this grid in the mural, near the bottom, which can be seen in Figure 12.

In the first of the drawings, Almada states that point E, in our previous figure, is actually a point on the grid lines. Let E' be the point on the grid lines of this first drawing, corresponding to point E.

It is easy to see that E' is the diagonal, by considering the two grid squares to its top left. On the

The diagonal of the rectangle $ABDJ$ determines point E in the circle, from which an inscribed square is drawn. From point H we now draw a vertical line HI, followed by a horizontal line IF, thus constructing a right triangle inscribed in a semicircle.

The remarkable thing is that this is a 3.4.5 triangle. This means, of course, that if we take as measurement unit a third of the shorter leg IF, then the longer leg HI will measure 4 and the hypotenuse FH will measure 5. One can prove that this is the case by computing the tangent of angle $\measuredangle HFI$ and proving it is equal to 4/3 (one has to compare this angle to angles $\angle JCE$, $\angle JAD$ and $\angle ADB$, knowing that $\tan(\measuredangle ADB) = 1/2$).

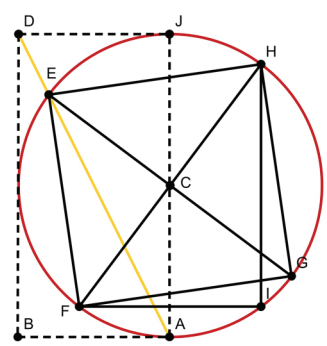

Figure 13: A reproduction of the drawing in Figure 12

other hand, if we imagine two coordinate axes through the center of the circle, with a measurement unit equal to the side of the square of the grid, then E' has coordinates $(-4, 3)$.

Since the circle has radius 5, by the Pythagorean theorem, the point lies in the circle. Therefore, it is the intersection point of the circle and the line.

With similar considerations, one would conclude that point G (in our figure) is also on the grid lines, and therefore the triangle GHE has sides that comprise 8, 4 and 10 squares of the grid. It is therefore a 3.4.5 triangle!

In the third drawing Almada has yet another reference to the number seven, stating that the four sides of the square are diagonals of grid rectangles of sides 1 and 7.

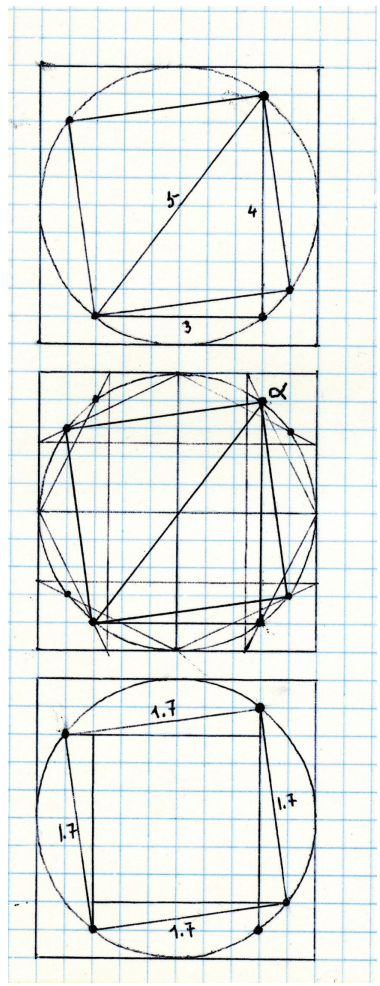

Figure 14: Study for the construction in Figure 12, from one of Almada's notebooks

5 Conclusions

The analysis we have presented here is part of an ongoing inventory process of Almada Negreiros' work. His quest for the canon involved research primarily on geometric principles that the author considered to underlie all art and that could be immediately understood by every person, with no need for explanation. These propositions are interesting from the mathematical point of view because they establish unexpected connections between apparently disparate concepts.

As we have noticed, is not clear if Almada was aware that some of these constructions were approximations,

while some others were exact. As we have said, Almada never produced any mathematical explanation or analysis, which suggests that all his geometrical work was done by extensive, almost obsessive, trial and error. His aim was to exhibit canonical elements and connections between them, without dwelling on mathematical accuracy. Even though he did acknowledge the influence of others (such as Hambidge), he did claim authorship of the constructions he presented.

Beyond the interest of their geometric program, these works are of unquestionable artistic value. They form an unusual (maybe even unique) ensemble of art works with almost exclusively mathematical content, in perfect harmony with an aesthetical intention. For over forty years, Almada experimented on geometry, as a self-taught process, constantly producing works of art, which constitute, on its own, the final format of his theoretical statements.

6 Acknowledgements

This work was made possible by the *Modernismo online* project (responsible for the site `www.modernismo.pt`), of the Faculdade de Ciências Sociais e Humanas of the Universidade Nova de Lisboa, financed by the Fundação para a Ciência e Tecnologia, that gathers and archives in digital format the material heritage of Portuguese modernism. It now includes a vast archive of Almada's work, which can be consulted online.

We would like to thank the *Centro de Arte Moderna* of the Calouste Gulbenkian Foundation (CAM-FCG) and the family of Almada Negreiros for their kind permission to use these images.

This is an Authors Original Manuscript of an article published by Taylor & Francis Group in Journal of Mathematics and the Arts on January 2015, available online:

http://www.tandfonline.com/10.1080/17513472.2015.1012699

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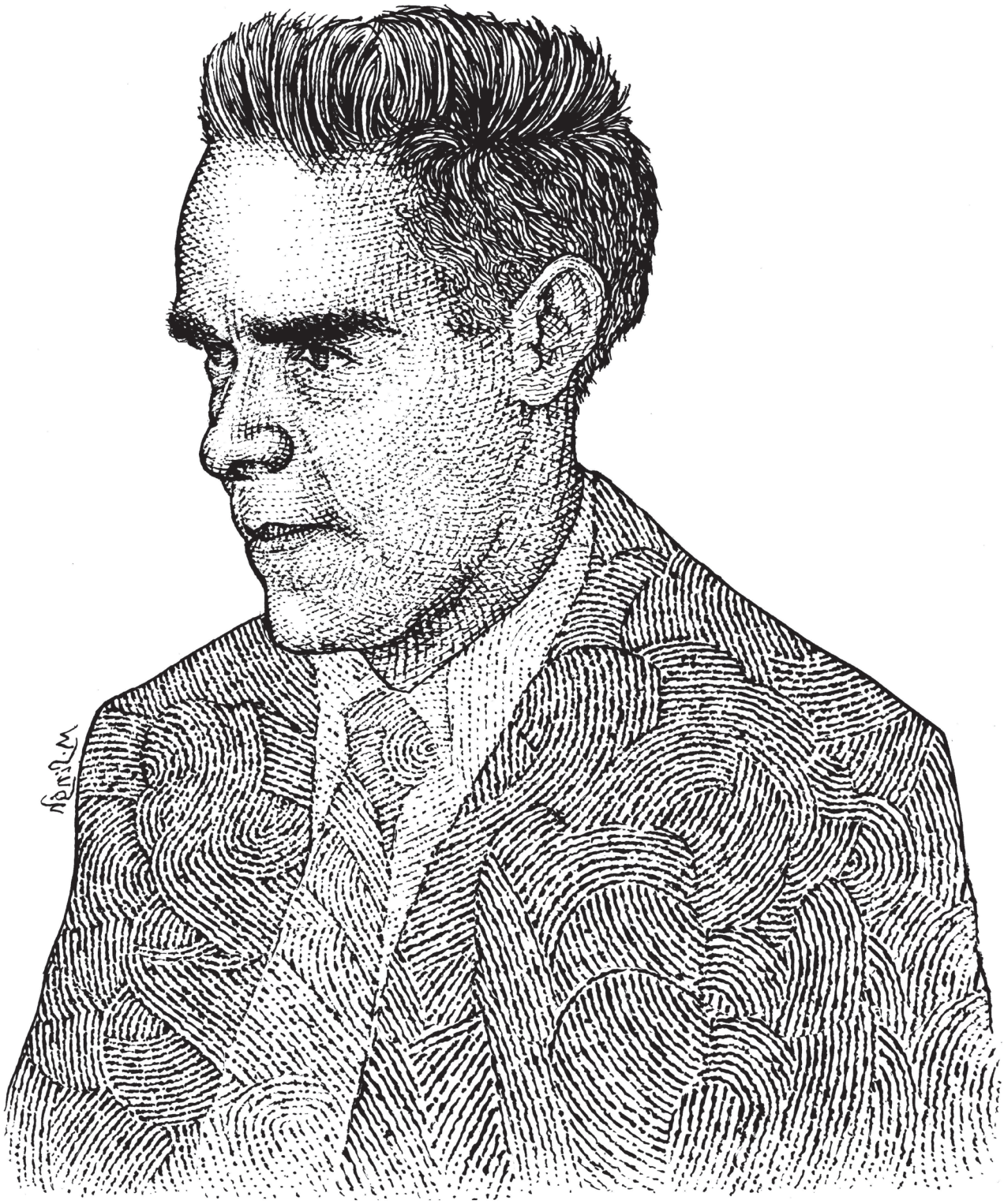
Cover Drawing

by Mark Burstein and Jim Gardner



Double Take

by Alexa Meade

Folding Method for “OSU Triptych No. 2”

Robert Orndorff
June 2016
Seattle

The work itself is a permanent manifestation of an ephemeral artwork, namely, one solution for a specific one-straight-cut problem. Such problems are usually stated as follows: How must one fold a paper rectangle into a flat figure such that one straight cut through all of the layers will produce a given planar straight-line graph? Here the problem has been solved with paper and then represented in acrylic. To a significant degree the work relies on transmitted and reflected light, and so it never looks the same twice. The figure (the letters "OSU") has been divided into three frames. The crease patterns for the left and right letters are pedestrian, but the crease pattern for the central letter is sublime. The short note that I am contributing to the gift exchange includes the crease pattern and a description of the folding process. The Joint Mathematics Meetings Art Exhibit was an international juried event. Three awards were given, and this work received one of them.

Fig. 1. “OSU Triptych No. 2.” 2015. Acrylic.

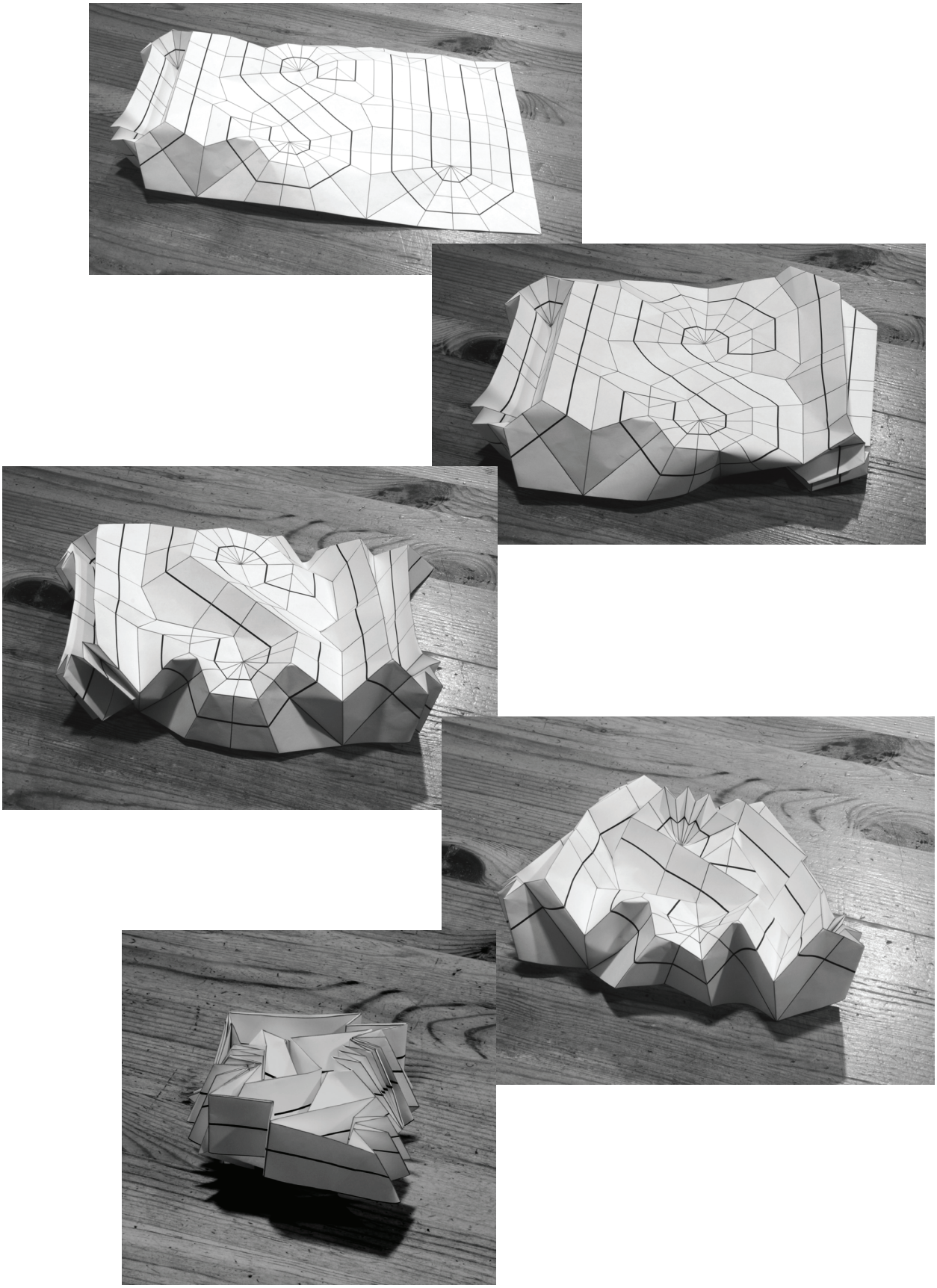


Fig. 2. Folding method for “OSU Triptych No. 2.”

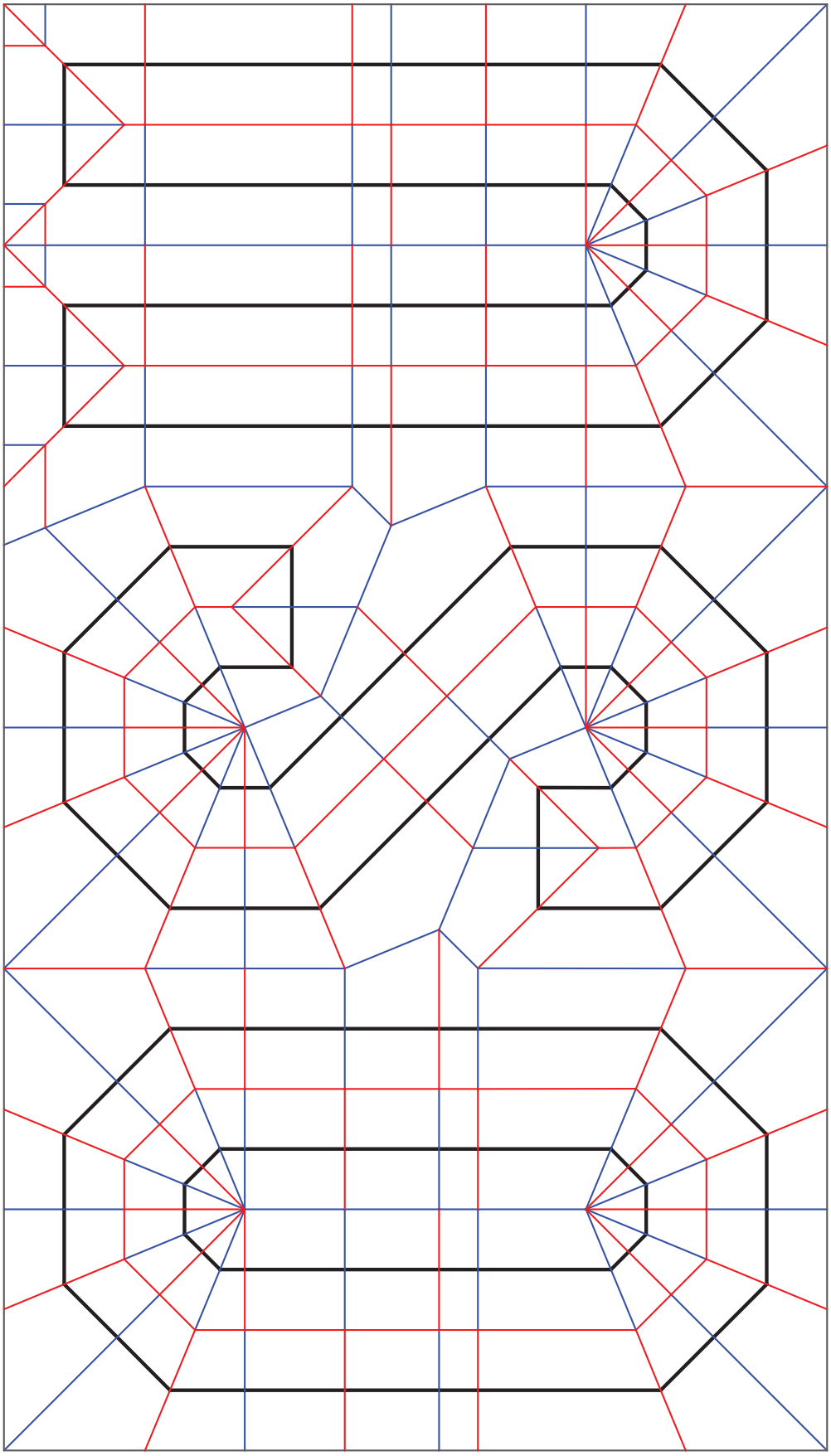
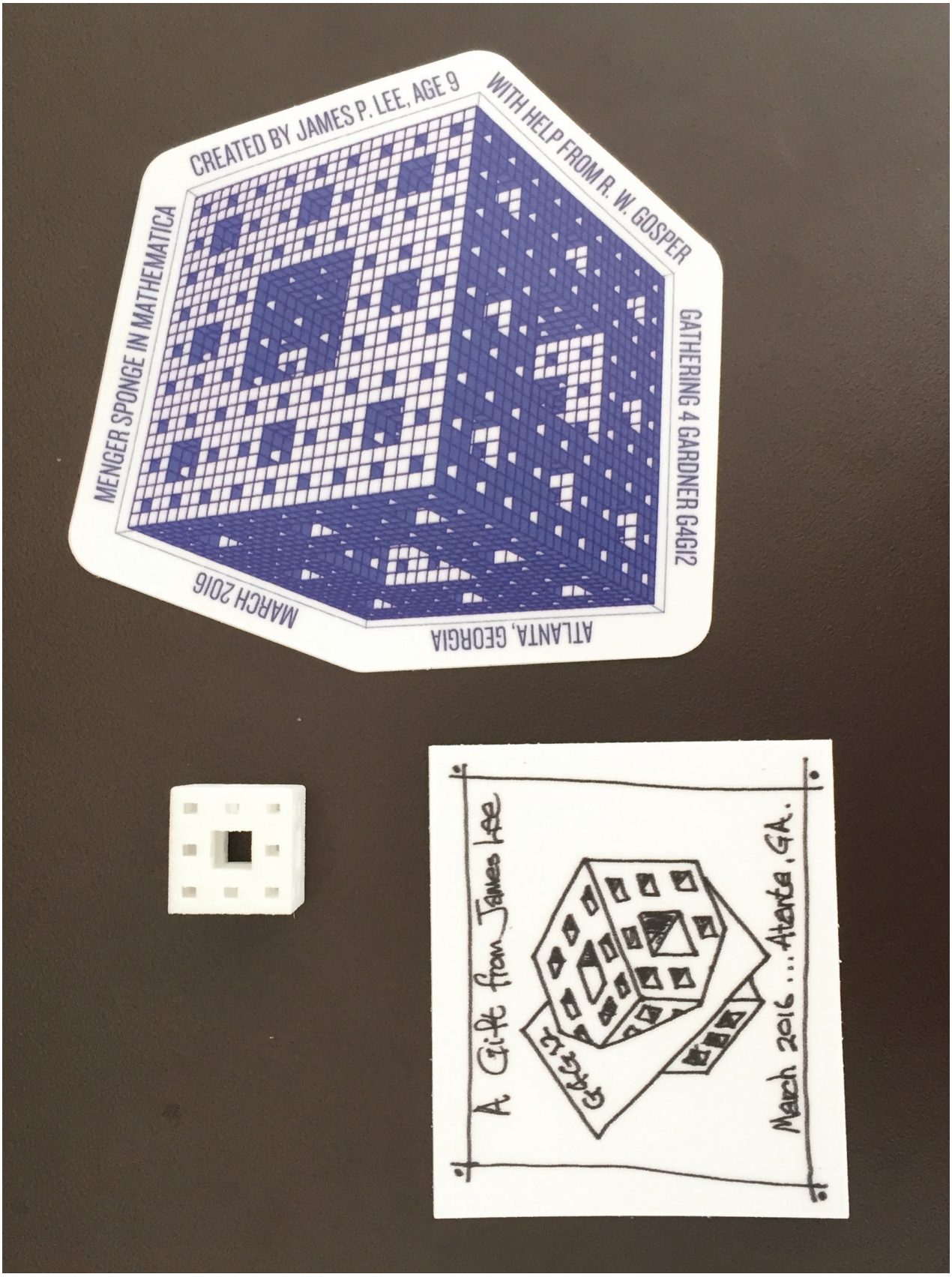


Fig. 3. Crease pattern for “OSU Triptych No. 2.” The red lines indicate mountain folds (convex); the blue lines, valley folds (concave). The bold black lines represent the figure that the single straight cut produces. There is one small error in the artwork (Fig. 1) which has been corrected here.

G4G Gift

by James P. Lee



Greek Cross Puzzle

by Norton Starr

My G4G12 gift is a put-together puzzle with (at least!) three outcomes.

It is inspired by puzzle shown at the bottom, middle of page 29 of

“Creative Puzzles of the World”, by Pieter van Delft & Jack Botermans.

Head Lines

by Lew Goldklang

March 31, 2016

[slide] **Welcome to G4GL!** (L is the 12th letter.)
It's an **honor** and a **pleasure** to be here with you again to celebrate this special event!

Listen Carefully!
There's a long-running narrative here. If you get lost following the threads, it's all right.
Whenever you hear a **self-reference**, an **N-tendre**, a **palindrome**, or a **paradox**,
[slide] **drink!**

Credentials
I'm **Dr. Lew** (that's a self-reference – **drink!**) I'm the **C**hief **D**evelopment **O**fficer of the **Marin Forensic Language Pathology Lab**. As we explained at **G4GK**, I may be
[slide] accused of having **O**bsessive **C**ompulsive **D**isorder, but at least I arranged the letters in their correct alphabetical sequence.

Clarity & Precision
At the Lab, we encourage speaking **numerically**, for clarity and precision. Please visit our headquarters in **94941** at
[slide] GPS (**37.887289**, **−122.532983**, 17z) = (**N** 37° 53' 14.2404", **W** 122° 31' 58.7388").

Mission / Correct Language / Surface Hygiene
I have traveled across this great Republic – from **94941** to **94965**, and here to **30303** every two years – to **correct** everyone's **language** for **logic**, clarity, accuracy, precision,
[slide] **syntax**, grammar, eloquence, erudition, **vocabulary**, usage, idiom, **spelling**, punctuation, **voice**, style, wit, tone, & nuance. Also to correct their **belief systems**, and to improve their **surface hygiene** through the magical therapeutic properties of
[slide] **fig soap**, as I will explain later.

Accounting
Our Lab hired a brilliant Orthodox accountant, specializing in **double entry bookkeeping**, to keep track of our finances, and prevent us from drowning in data. His name is
[slide] **Noah Counting** (that's almost a triple entendre – **drink!**) **Bookkeeper** & **bookkeeping** are the only two English words containing a triple set of consecutive double letters.

Nutrition
Our Lab manufactures highly concentrated, energy-packed, **wheat**-based snacks, extruded into flexible **crystal** filaments, shaped into **handguns**.

The prototype product name was **Crystal Pistols**, but our genius Marketing Department changed it to
[slide] **Rootin' Tootin' Gluten!** The dried-cherry trigger adds a nice touch.

We signed up a very influential health enthusiast as our international distributor.
[slide] **Putin's Rootin' Tootin' Gluten!** That's Vlad in our Quality Control room.

Headlines

[slide] I saw this newspaper ad with an **ambiguous headline**, for some sort of **language** contest run by an Indian casino up in 94928. It validates our earlier product's concept. (If you missed G4GK, I will explain later.) I immediately realized this **headline** has at least **11** different meanings. As **Safe Cracker** experts, we zipped up there in our
[slide] **Race car** (with **palindromatic** transmission – **drink!**), and won these rare, valuable portraits of Treasury Secretaries
[slides] **Alexander Hamilton** and **Salmon P. Chase**.

At the Lab, we write clear, unambiguous **headlines**.
Head Lines is at least a **hextendre**, or a **hexaflexatendre**.

1. Brow furrows
2. Vectors – lines with heads, magnitude, & direction
[slide] 3. Queues to use the restroom
4. Joke punch lines that mess with your head
5. Thin, parallel rows of $\mathbf{C_{17}H_{21}NO_4}$ crystals on a mirror (a well-known Atlanta molecule)
6. Succinct summaries of news stories – **titles**.

The self-referential **headline** (title) of this talk is, of course, **Head Lines**. **Drink!**

Writing Persuasive Headlines

A **persuasive** headline **compresses** a complicated subject or argument into its essential message – its **resolution vector** – using the fewest syllables, in an aesthetically pleasing form.

I meditated on Jewish **haiku**ish.
[slide] Paradox of Zen
If there is no self, then **whose**
Arthritis is this?
(Paradox & self-reference – **drink twice!**)

Our Cultural Department insists on a more elegant rhyming scheme.
[slide] The **G4GL headline challenge**, then – to the extent one actually exists – is to explain an important cultural trend in health & hygiene, using an internally rhyming, symmetric form, with **12** syllables – a **dodecarhyme**!

Philosophy War

The war between **Evolutionists** and **Intelligent Designers** rages on. Someone who believes in Darwin's Theory – that humans evolved from primates – might logically be
[slide] called an **Aper**, like a Truther or a Prepper. An **Intelligent Designer** might be called an **I.D.er**, like a **clever thought** from New Jersey. I'm **coining those words** here today.

Cultural Trend

[slide] There's a clear trend **away** from smoking tobacco, to using **e-cigarettes** as a drug delivery system instead. Cigarette smoke contains over **11,000** contaminants. The chemical mix in an e-cigarette supposedly contains only **nine** contaminants.

Vaping – inhaling water **vapor** laced with trace amounts of nicotine – is becoming a popular fad, as a **safer** way to quit your smoking addiction than by applying epidermal nicotine patches, or chewing **gum** loaded with vile chemicals. Pleasing **flavors** (banana, tangerine) are mixed with the vapor to heighten the experience.

If an Evolutionist were to report this cultural trend from his log notes, our **headline** would be:

[slide] **Aper's papers: Vapers favor vaping safer, flavored vapors**!
That's **16** syllables, but you can just strike out the first four, leaving only **12**.

Gift Exchange
My Gift Exchange item is an efficient **cheese grater**, so you can shred Parmesan on your pasta, and

[slide] **Make America Grate Again!**

Credit / Self-confidence
I want to thank Professor **Ken Brecher** for referring me to a volume of valuable research literature on language & self-confidence, titled

[slide] ***I'm Not Scared*** by **Hugo Furst**.

Here's an important **tail line** for you.

[slide] **FREE TIBET! ***

***** With purchase of one regular Tibet of equal or greater value.

[slide] **Thank you** for your attention & patience! **Speak Clearly!**

[slide] Enjoy the vapors – **drink!**

———

Abstract
Dr. Lew explains important cultural trends with self-referential symmetric rhyme. Enjoy word sequences you may not have thought of, and won't hear anywhere else.

Making Math Visible

George Hart and Elisabeth Heathfield
Stony Brook University

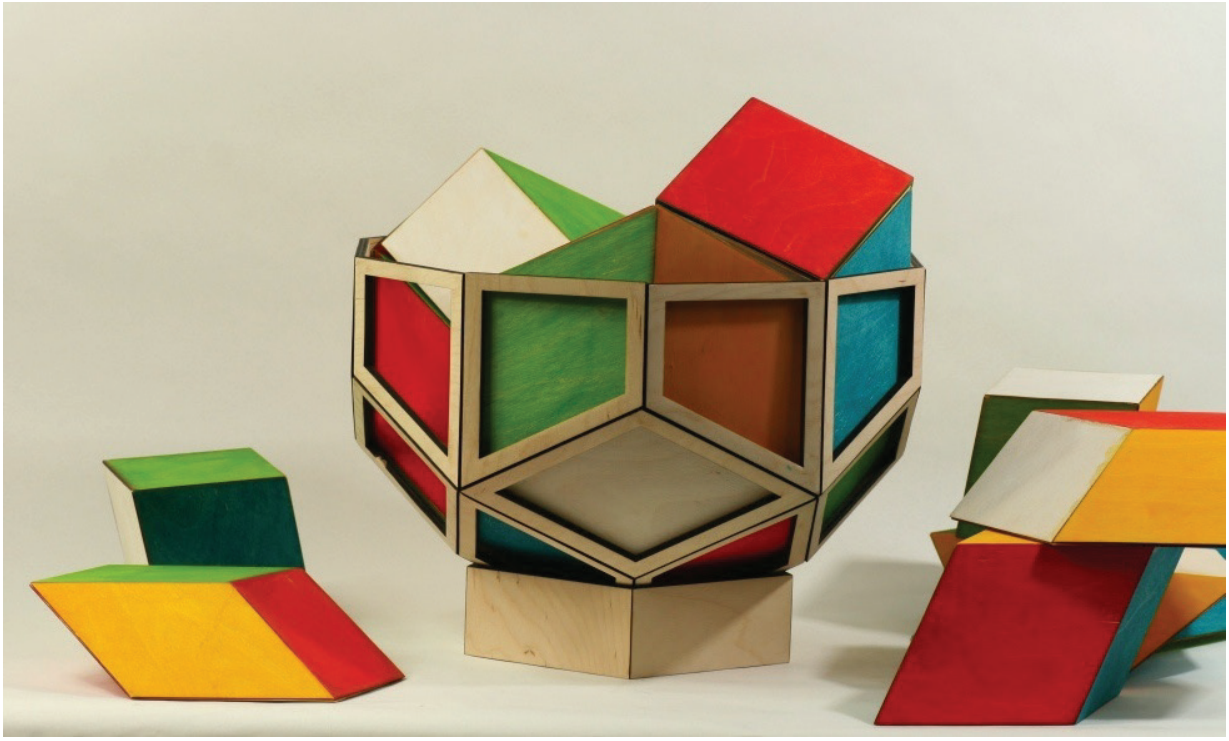
The "Wiggle-Dome" is one of many large, hands-on, mathematical constructions documented at **http://makingmathvisible.com**

The beauty of mathematics is of fundamental importance to professional mathematicians, but is not generally seen by students and the public. We have been working to design and document a series of hands-on mathematical construction activities that result in beautiful objects that can be displayed in schools or other public areas. These include, giant hyperboloids, a giant-scale SOMA set made of cardboard boxes, large polyhedral structures made of CDs, human-scale domes and puzzles, and much more. We are writing detailed lesson plans with lists of materials and step-by-step instructions so people anywhere can replicate these activities and enrich their environment with mathematical beauty.

Just as exposure to great books can entice students to learn to read, beautiful mathematical objects may fuel students' desire to investigate topics in mathematics and give them an opportunity to acquire a positive lifetime perspective towards math. Traditional classroom walls are often overflowing with language-based displays of learning, while the beauty of math is rarely seen.

We believe that creative hands-on activities can informally introduce students to mathematical thinking and get them excited about math. Young people have an inherent curiosity and a willingness to explore that is characteristic of professional mathematicians. Our workshops can foster this natural tendency and show students that "math is cool!" Our goal is to make math visible and accessible by constructing visually engaging, publicly displayed objects that provide a tangible platform for discussion and inquiry.

A giant hyperboloid made from 4-foot dowels and large rubber bands, plus a smaller hyperboloid made from chop sticks and pony-tail rubber bands.



A color-matching dissection of the rhombic triacontahedron. This beautiful, challenging puzzle consists of twenty wood rhombohedra.

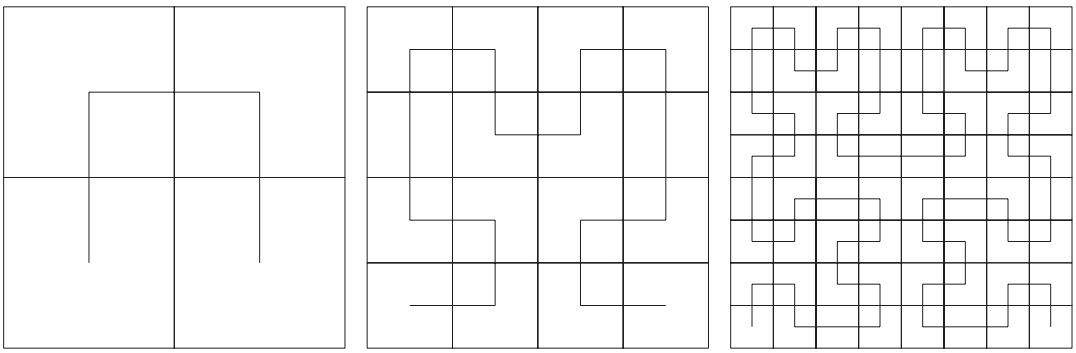
For more information, see **http://makingmathvisible.com**

Space-Filling Curves on Non-Periodic Tilings

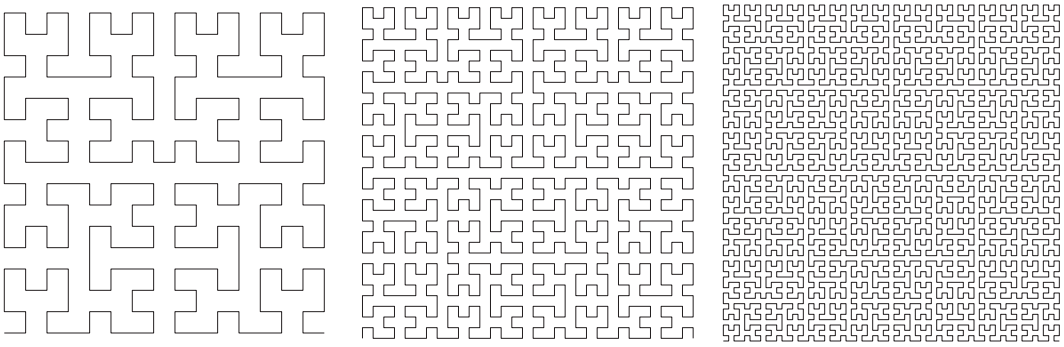
Fred Henle ⟨fredhenle@gmail.com⟩

for G4G12

I first learned about space-filling curves when my grad school advisor showed me the Hilbert curve. Take a square and subdivide it into four squares, then draw a path through the centers of the four squares. That's the first order curve. For the second order curve, subdivide each of the four squares into four smaller squares, each with a first order Hilbert curve (two of them rotated sideways) and connect them to form a path that passes through the centers of the sixteen small squares.

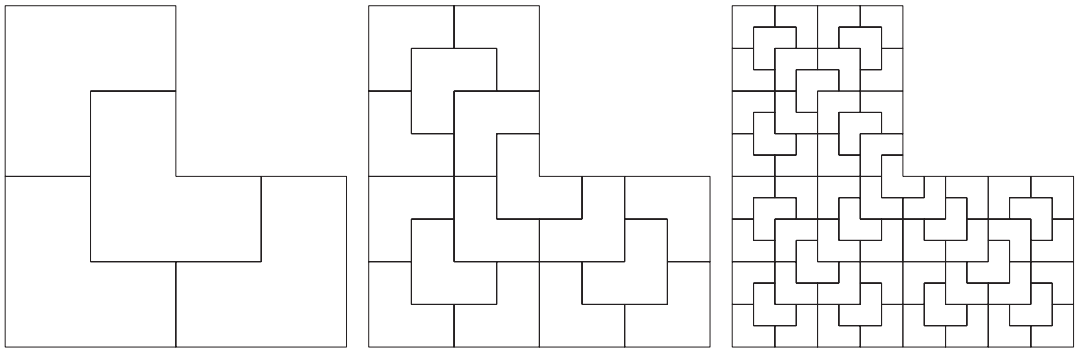


Continuing to iterate but without explicitly drawing the subdivisions:

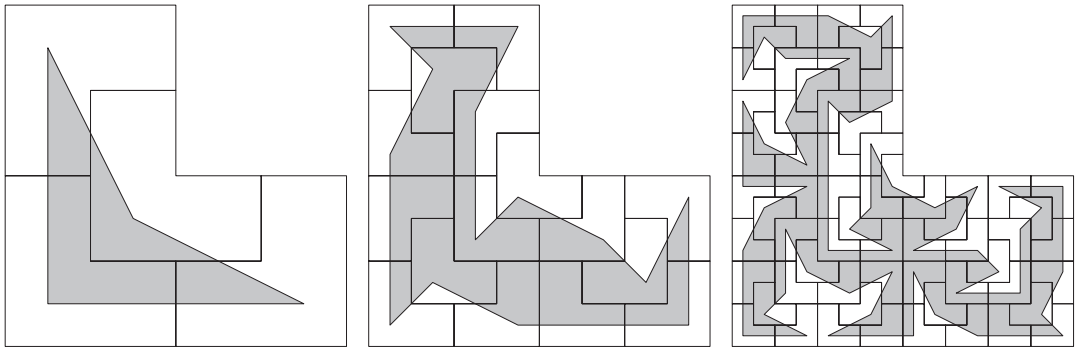


In the limit, the curve passes through every point in the square, hence the term "space-filling." The Peano curve does the same thing by subdividing each square into nine instead of four smaller squares.

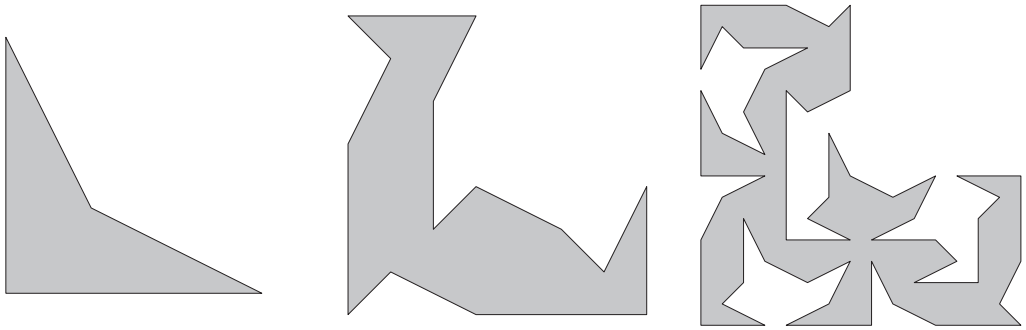
I decided to attempt to create space-filling curves on more interesting shapes, in particular shapes that can be tiled aperiodically. Here is one, called the "chair tiling." The shape is an ell; think of it as three quarters of a square. It can be subdivided into four similar ells in three different orientations:



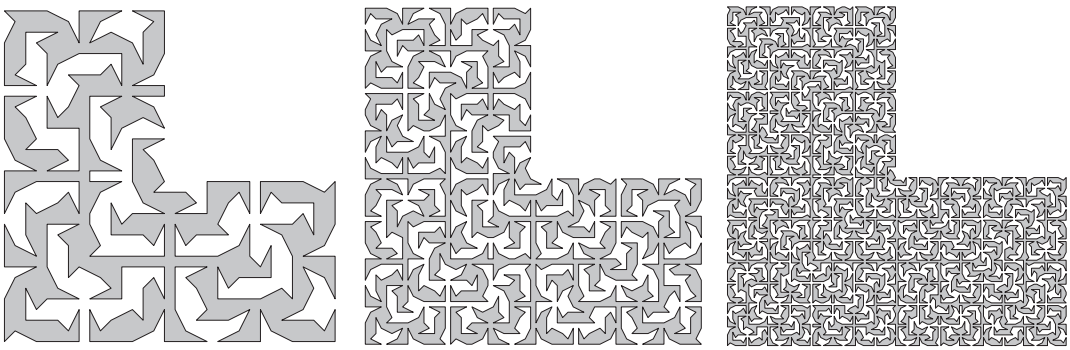
The space-filling curve then traverses each subdivision in turn. I've filled it with gray because the curve itself forms a loop:



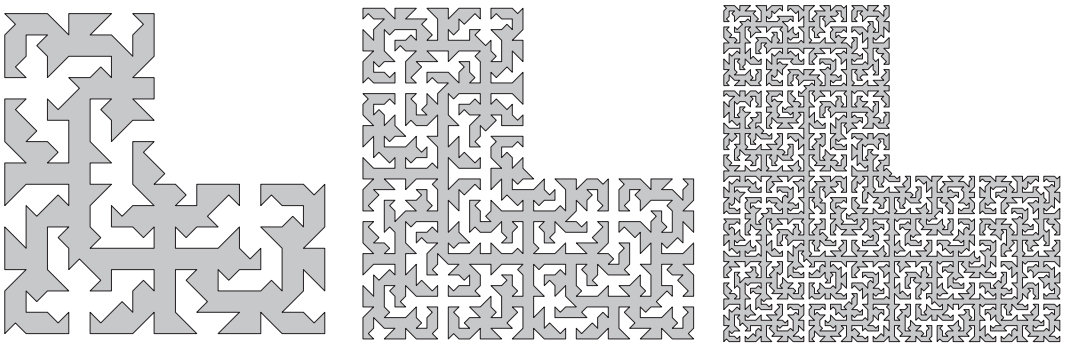
Here is the curve by itself:



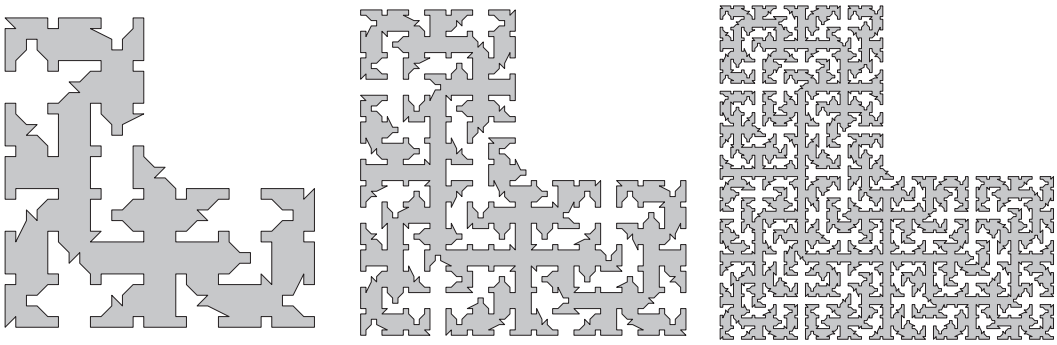
Let's keep going a few more iterations:



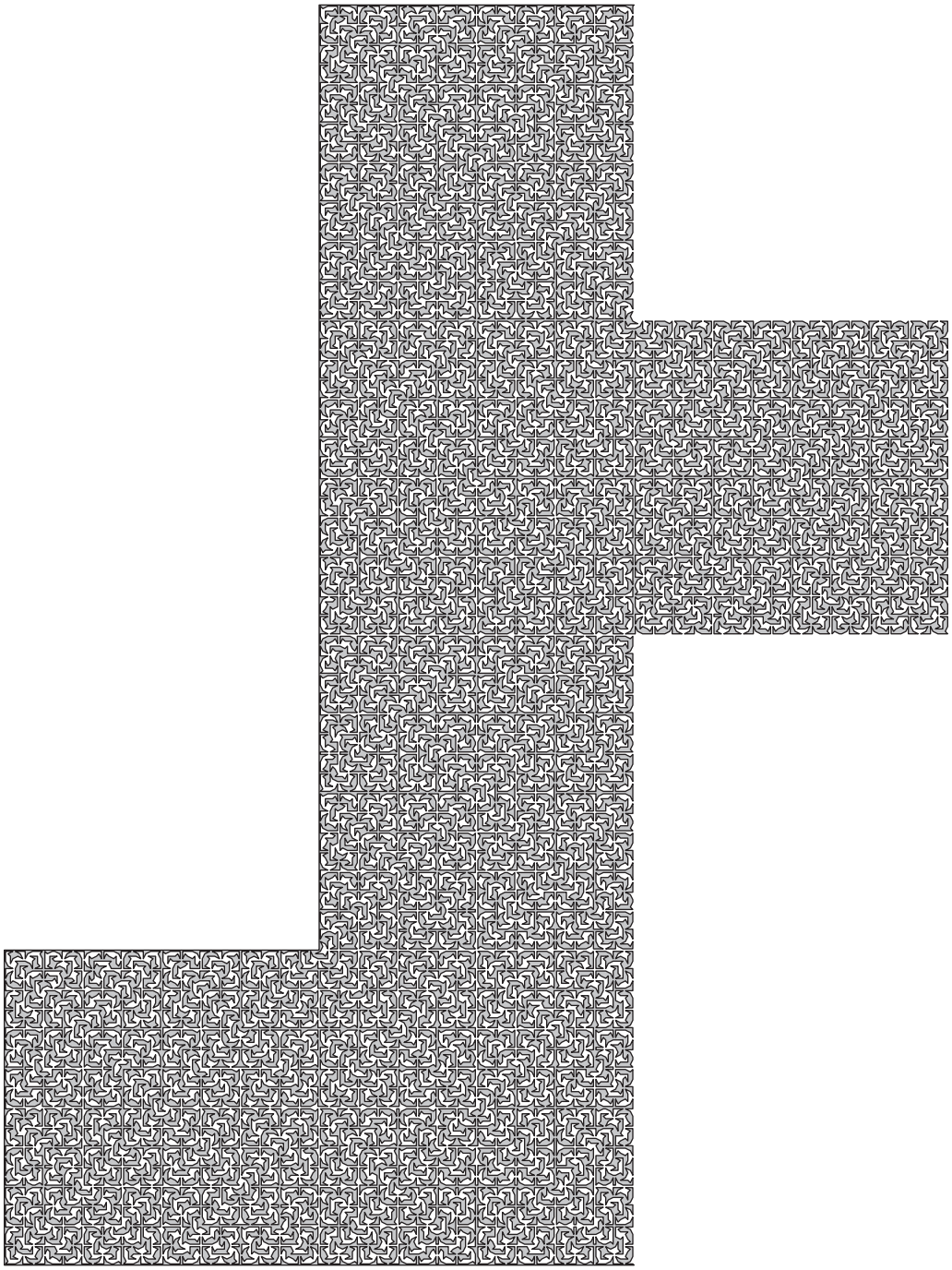
Here's a variation where the path passes through the reflex angle of each ell instead of the center of the middle square:



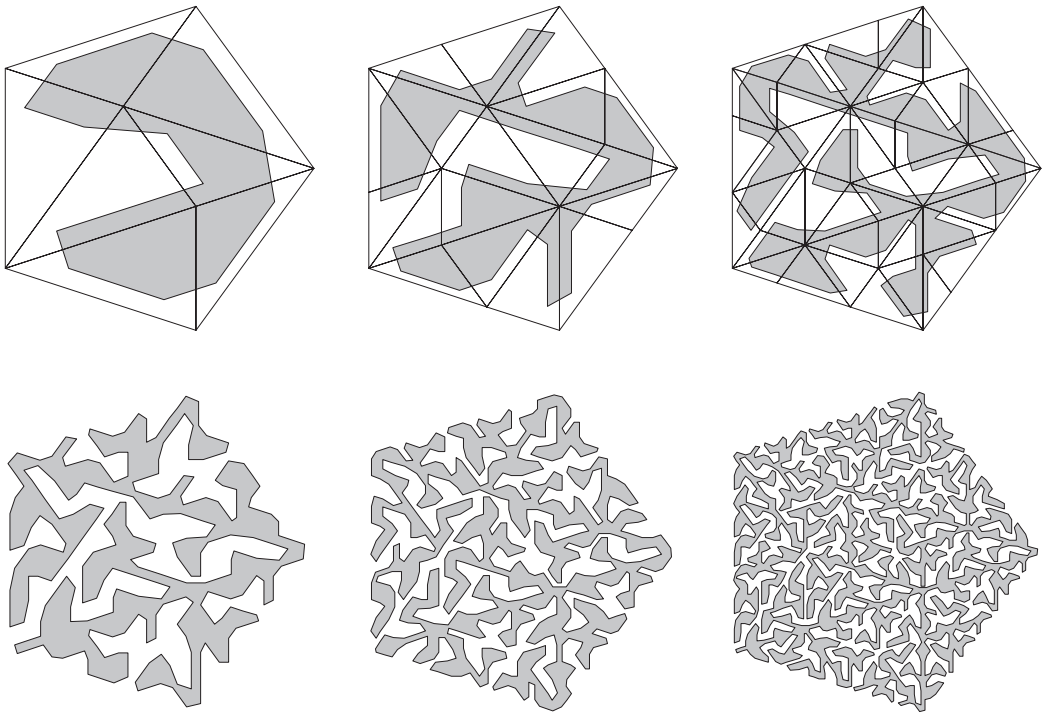
Here's a variation where the path passes through the center of the missing square (outside the ell):



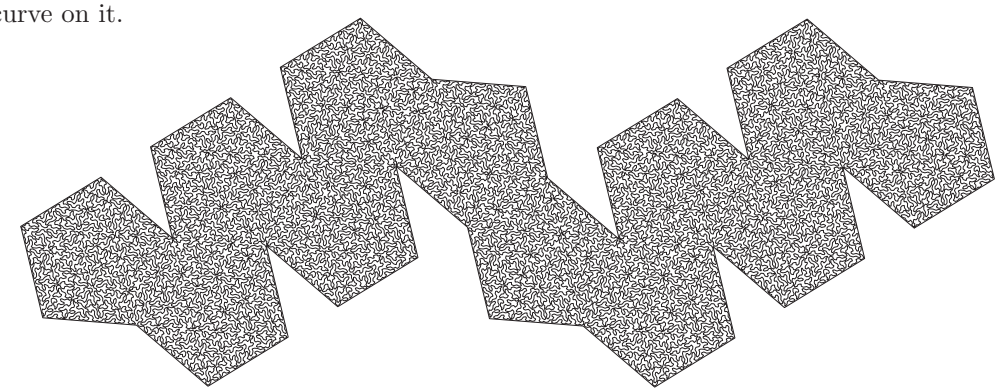
Here are two ells side by side with a single path (filled) to form the net of a cube. Cut the shape out and make five creases, then tape it together to form the cube.



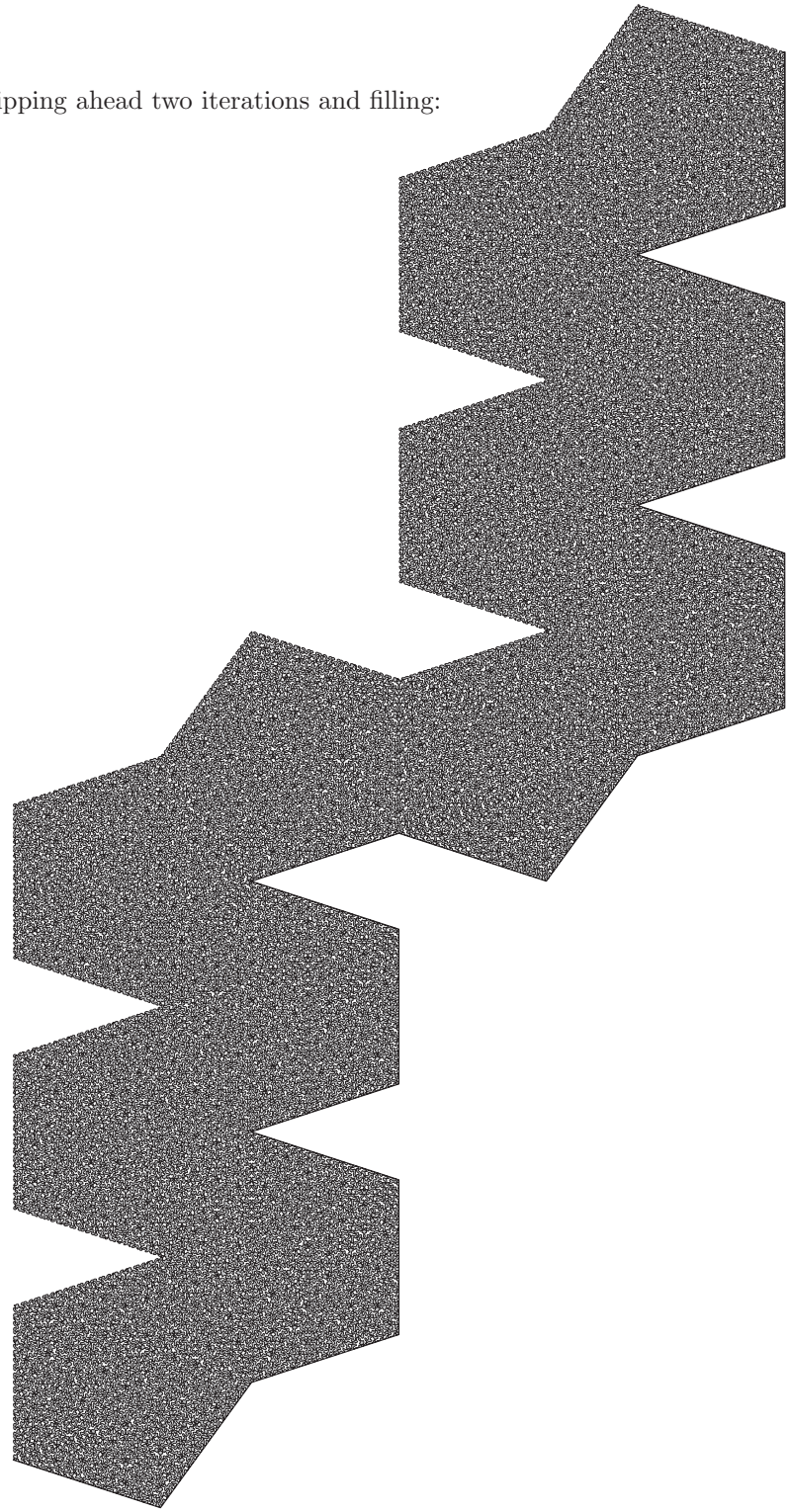
This is cool, but what I really wanted to tackle was the Penrose tiling of kites and darts. Unfortunately, the deflation rules for Penrose tiles aren't bounding volume hierarchical; when you subdivide a kite or a dart you get smaller kites and darts that extend beyond the borders of the original kite or dart. Half-kites and half-darts (also known as Robinson triangles) can be subdivided cleanly, however, and when I realized that a pentagon can be divided into a half-kite and two half-darts, I knew what to do.



For G4G12, here are twelve pentagons arranged in the net of a dodecahedron, with a space-filling curve on it.



Skipping ahead two iterations and filling:



As far as I know this work is novel, but I have yet to consult with experts in the field. In order to produce a space-filling curve on the chair tiling I had to work out rules for five different cases (depending on whether I wished to traverse the ell from opposite corner to opposite corner or some other pair of corners) but I'm happy with the result. I'm a little less pleased with the space-filling curve on the pentagon; I had to make some uncomfortable compromises, such as crossing from one region to another region bordered only at a vertex and not an edge. This seemed unavoidable, however. The fact that the graph has vertices of high degree also means that the curve keeps approaching the same point, which is less aestheticly pleasing to me.

I learned about the chair tiling and the Robinson triangle decomposition of Penrose tiles respectively from Wikipedia articles:

- `https://en.wikipedia.org/wiki/Aperiodic_tiling`
- `https://en.wikipedia.org/wiki/Penrose_tiling`

I also found `https://en.wikipedia.org/wiki/Space-filling_curve` useful background reading. Finally, I used the net of a dodecahedron that I found at `http://www.se16.info/js/circumnavcubetetra.htm`.

I will put some version of this paper at `http://fredhenle.net/g4g12/`. If you would like to correspond with me on this or any other topic, please send me email at *fredhenle@gmail.com*.

Gwenbeads presents

Toplogically Equivalent Surfaces

Topologically Interesting Felt

How to make a Möbius bracelet with wool

by Gwen Fisher

Seifert Surfaces of Trefoils

Materials

Materials: Wool roving, fine natural fiber fabric that is easy to needle (silk georgette or cotton cheese cloth)

Tools: Barbed felting needle, scissors, thick foam (upholstery), soap, water (preferably hot), towel.

Instructions

First, learn how to make a Möbius band in felt.

1. Cut a rectangle of fabric. To make a bracelet, medium size for a woman, cut the rectangle 9" by 1". Expect the wool to shrink 10% to 25%. Also expect the unexpected because the exact shrinkage of wool during the felting process is challenging to control.

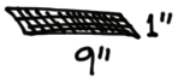


Figure 1

2. Place fabric flat on a piece of foam.

3. Layer wool onto the fabric. To do this, pull the wool with your fingertips, little tufts at a time. Spread apart the fibers in the tufts, and layer them over the fabric. Slightly overlap all of the edges of fabric with wool. Make a few layers of wool that go in different directions. Try to make the wool pile evenly thick, just barely thick enough that you can't see the fabric underneath.

4. Needle felt the wool into the fabric. To do this, use the needle to stab through all of the layers to push the tufts of fiber into the fabric. Do this evenly across the entire piece, rather than focusing all on one spot. You have poked enough when the fibers all hold together well enough to withstand a few shakes.

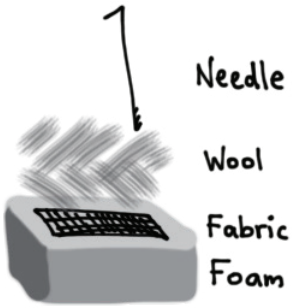


Figure 2/3/4

5. Peel the whole strip off of the foam. Flip it over. To neaten up the edges, roll the fiber up and over the edges. Do not do this at the joins. Anywhere you will join two edges, leave the fiber fluffy there. Needle felt the edges.

6. Repeat steps 3, 4, and 5 to add wool to the other side. You can use wool that is the same color or a different color. If you use different colors on opposite faces, neaten the edges without rolling the fiber over the edge.

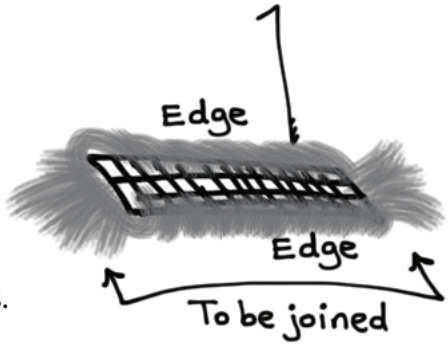


Figure 5

7. To make a Möbius strip, twist the band with a half turn, and join the ends. To join the ends, peel apart the layers at the ends, neatly layer the fibers together, and needle felt through all of the layers to secure. Add more wool if necessary to secure the join and make it look neat.


Figure 6

8. Once all of the wool is held into place, put away your needle. The combination of needle felting and vigorous wet felting permanently binds the fibers together, making the finished object very durable. Actually, you can needle felt without wet felting, but wet felting is much faster. Plus, wet felting is magical to watch and do with your hands. So, get the piece wet. Hot soapy water works fastest, but you can still wet felt with just cold water.

9. Gently rub a single layer of the surface between your fingers, keeping the flat parts flat. After you rub the whole piece, lightly crumple it into a ball, and roll it a little.

10. Open the piece back up, tugging on the surface to pull it back into shape. Repeat step 9, this time with a bit more pressure.

11. Repeat step 10, this time with more pressure still. Keep agitating the wet wool to felt it as much as you'd like. There is a point when the wool is fully felted and shrinks as much as possible, but you can stop before that if you want.

Figure 7

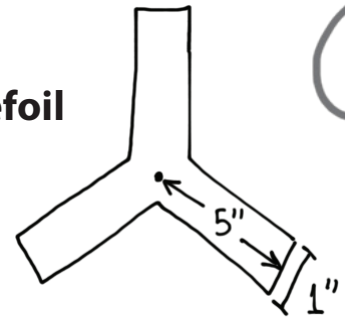
12. Stretch and tug the piece into the shape you want it to be. Let it dry.

Variations & Inspirations

Use a fabric armature in different shapes or join multiple strips together to make other topologically interesting surfaces. Use a Y-shape as shown to make a Seifert surface for a trefoil knot.

Couch yarn with a needle and thread along the edges to emphasize the edges of pieces with more than one edge.

Trefoil



Cut Fabric


Twist and Join

Up Down Coin

by Alexa Meade

A Year of Pentomino Tilings

G4G12 Exchange Gift

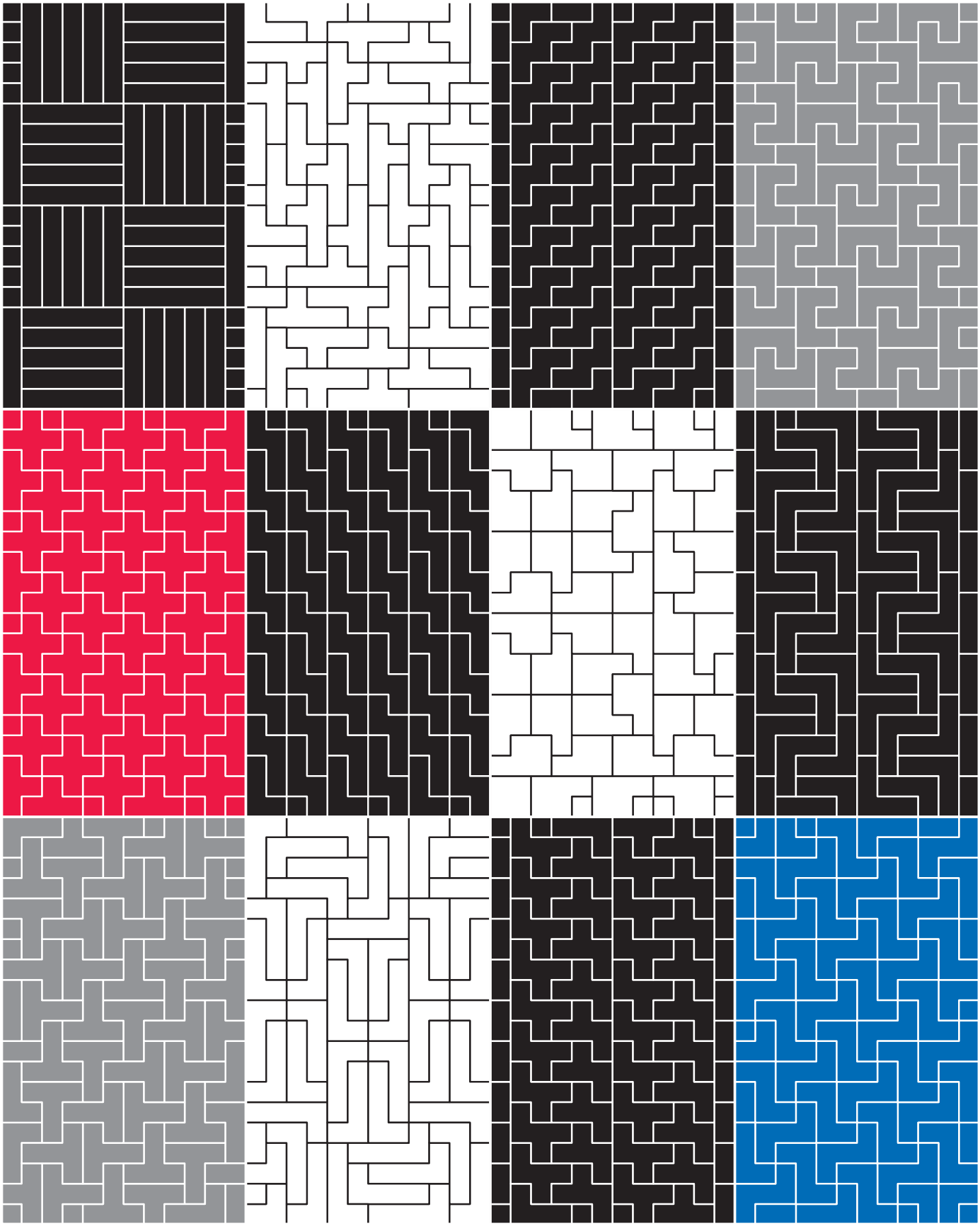
Margaret Kepner renpek1010@gmail.com

A Year of Pentomino Tilings

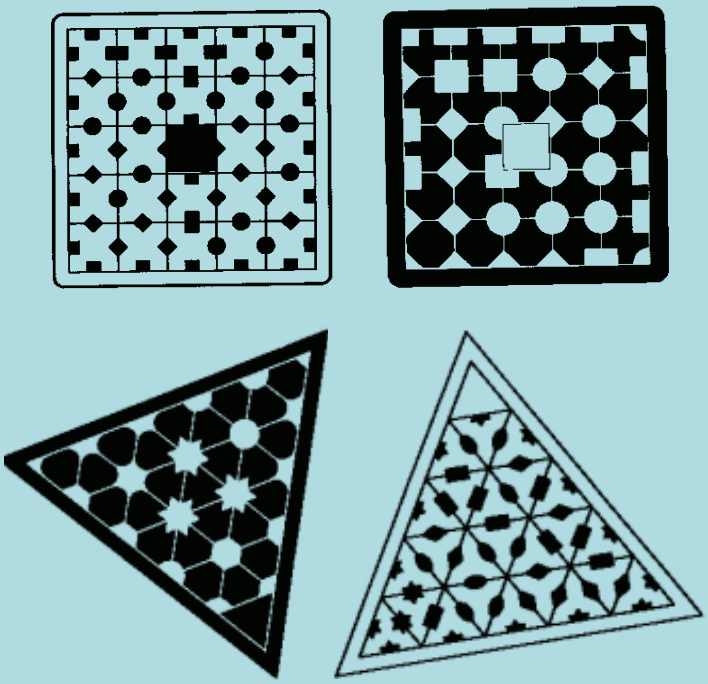
G4G12 Exchange Gift

Margaret Kepner

Each of the 12 pentominoes can tile the plane. That is, copies of a single pentomino can be fitted together, without overlaps or gaps, to tile the 2D plane indefinitely. In this gift, there are 12 business cards, each of which shows a particular tiling with one of the pentominoes on the front side, while the reverse side displays a monthly calendar. For example, a tiling using the Z-pentomino is paired with a calendar for the month of October 2016. The coloring of the tilings relates to the method used: infinite strips, rep-tile recursion, etc. A cover card displays a packing of the 12 pentominoes into a 6 x 10 rectangle.



GAMES



Gamepuzzles! | Kate Jones | Page 70

Black or White

Yunhao Fu

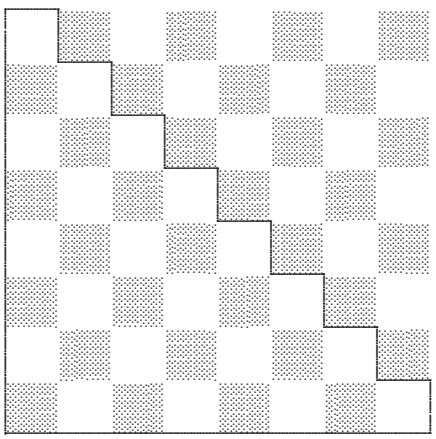
Graduate Student, Guangzhou University and

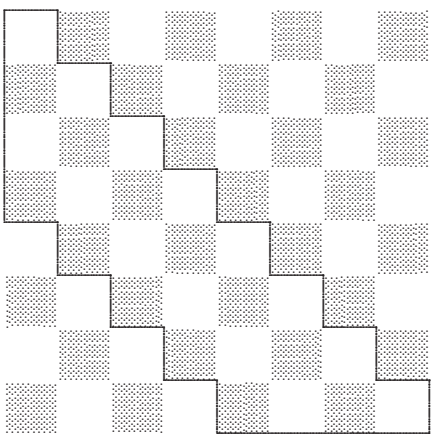
Ryan Morrill

Graduate Student, University of Alberta

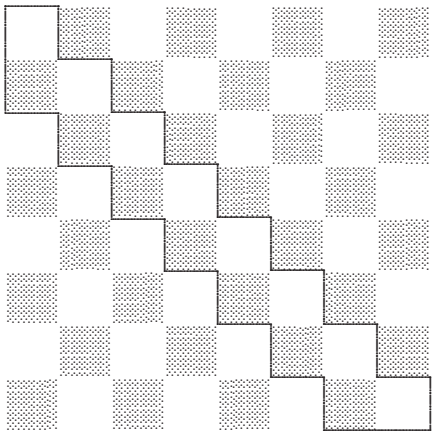
Anna chooses a cell of a standard 8×8 chessboard. She challenges Boris to deduce whether the chosen cell is black or white. He may choose a subboard consisting of one or more cells, bounded by a closed polygonal line with no self-intersections. She will then announce whether her cell is inside or outside this subboard. Naturally, Boris wants to minimize the number of subboards needed for accomplishing his objective.

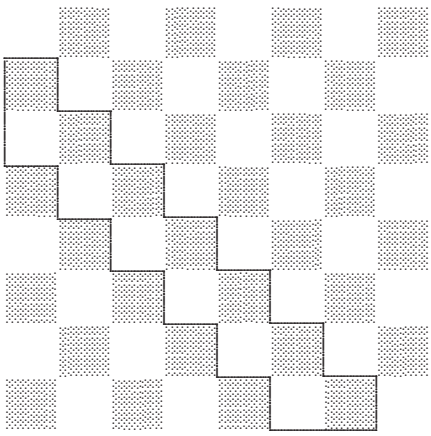
His first approach takes advantage of the fact that all cells along any diagonal are of the same colour. He comes up with the following method which requires four subboards. The first subboard is shown in the diagram below on the left. We may assume that Anna's announcement is "Inside". This narrows down the chosen cell to eight diagonals. Then the second subboard is shown in the diagram below on the right.





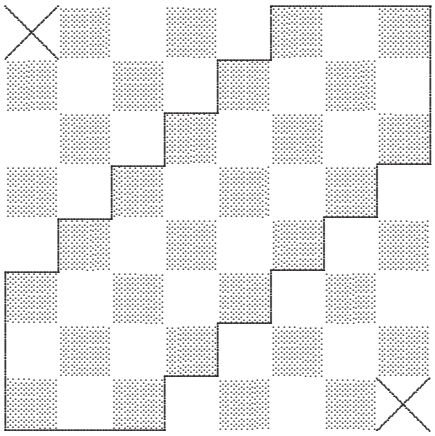
Again, we may assume that Anna's announcement is "Inside". This narrows down the chosen cell to four diagonals. Now the third subboard is shown in the diagram below on the left. We may assume that Anna's announcement is still "Inside". This narrows down the chosen cell to two diagonals. Finally, the fourth subboard is shown in the diagram below on the right.

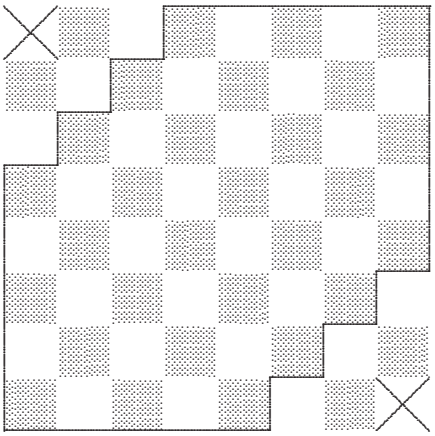




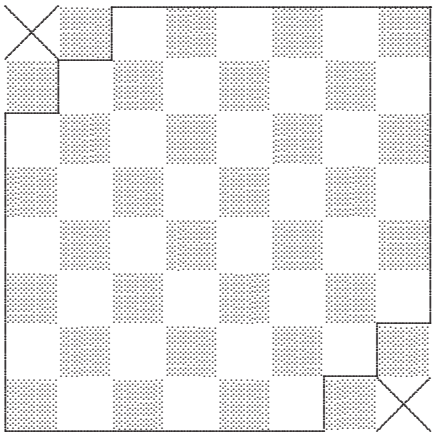
If Anna's announcement is "Inside", then her cell is black. Otherwise, it is white. In any of the preceding subboards, had Anna's announcement been "Outside", a similar and simpler situation arises.

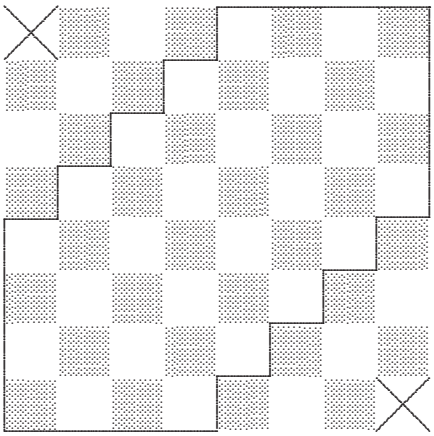
Rather pleased with what he has accomplished so far, Boris tries to simplify his approach and reduce the number of subboards necessary down to three. To his chagrin, he is unable to do so. His method only works if two cells in opposite corners are deleted from the chessboard. Then the first subboard is as shown in the diagram below on the left. If Anna's first announcement is "Outside", then the second subboard is as shown in the diagram below on the right.



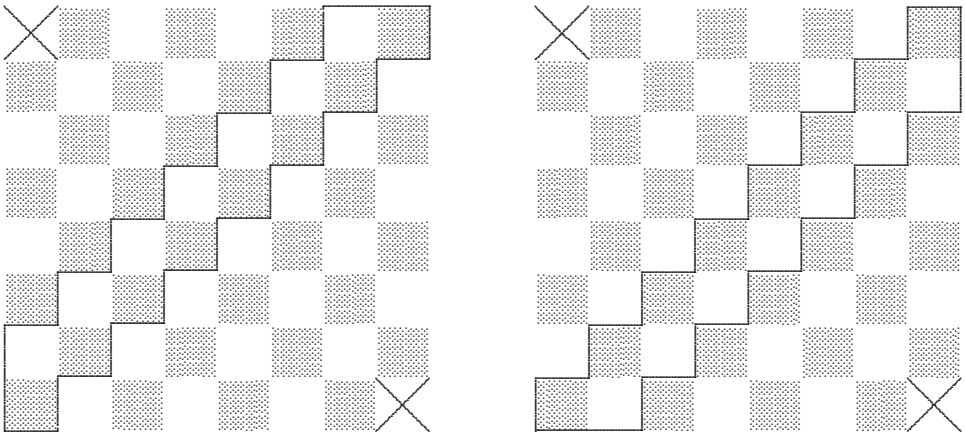


If Anna's second announcement is still "Outside", the third subboard is as shown in the diagram below on the left. If Anna's second announcement is "Inside", the third subboard is as shown in the diagram below on the left. In either case, "Outside" means black and "Inside" means white.

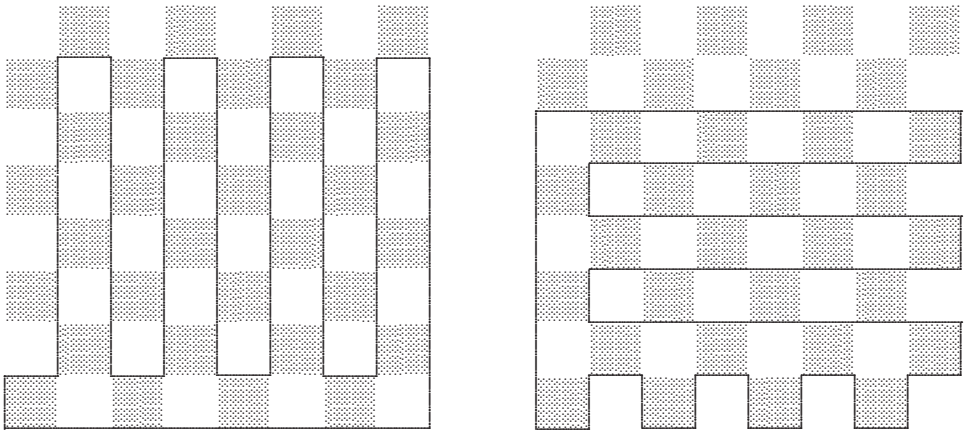




Suppose Anna's first announcement is "Inside". Then the second and the third subboards are as shown in the diagram below. If Anna's second and third announcements are the same, the cell is black. If they are different, the cell is white.

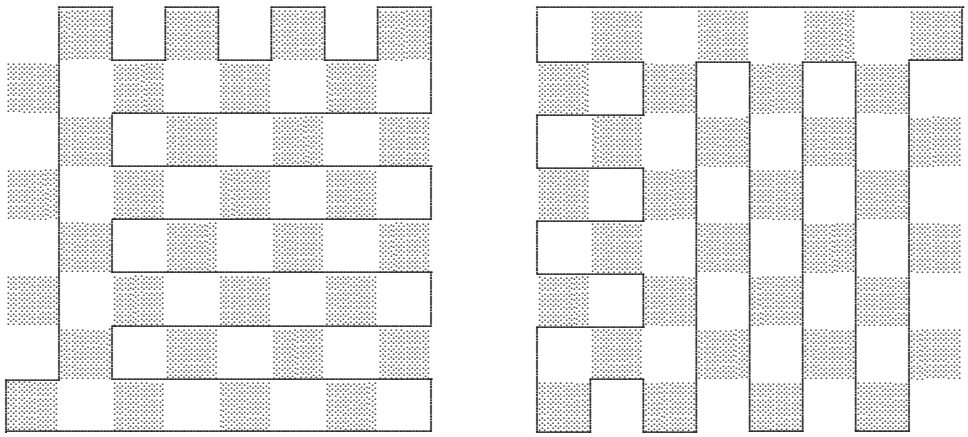


This is not entirely satisfactory. So Boris thinks of taking advantage of the fact that the black cells and white cells are symmetrically situated. In his new approach, the first subboard is shown in the diagram below on the left. By symmetry, we may assume that Anna's announcement is "Inside". Then the second subboard is shown in the diagram below on the right. If Anna's announcement is "Inside", then her cell is black. Otherwise, it is white.

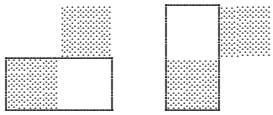


Clearly, one subboard will not be sufficient, since such a subboard must separate all the white cells from all the black cells. So the best possible result has been achieved. However, Boris is still not satisfied. In both approaches so far, he must wait for Anna's announcement before he can choose his next subboard. This is known as an *adaptive solution*.

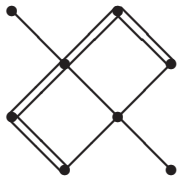
What Boris would like is a non-adaptive solution, in which he would present all subboards to Anna at the same time, and make the deduction upon receiving all the announcements simultaneously. After a while, he comes with one. The two subboards are as shown in the diagram below. If Anna's announcements are the same, the cell is black. If they are different, the cell is white.



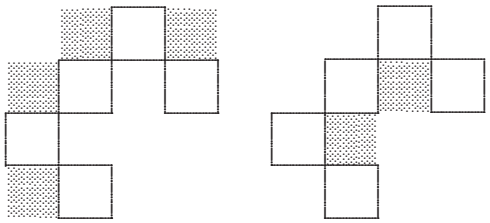
When Boris finally gets around to answering Anna's challenge, Anna is duly impressed. She has not thought that the task can be done using less than four subboards. So she tries to see if she can duplicate the accomplishment of Boris. She starts with the 2×2 chessboard, and there is indeed a non-adaptive solution using only two subboards. They are shown in the diagram below. If the two announcements are the same, the cell must be black. Otherwise, it is white.



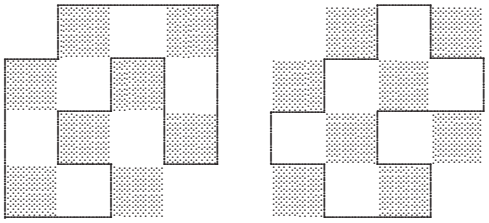
Anna now moves onto the 4×4 chessboard. There are eight white cells. She constructs a graph where each vertex represents a white cell, and two vertices are joined by an edge if a Bishop can move directly between the two white cells they represent. The longest Bishop path without including four vertices forming a square is indicated by a doubled line with three segments. The five white cells represented by the vertices on this path will be inside both subboards. The reason why a square is forbidden is because the four white cells represented by the vertices forming a square will enclose a black square, which must then appear inside both subboards also.



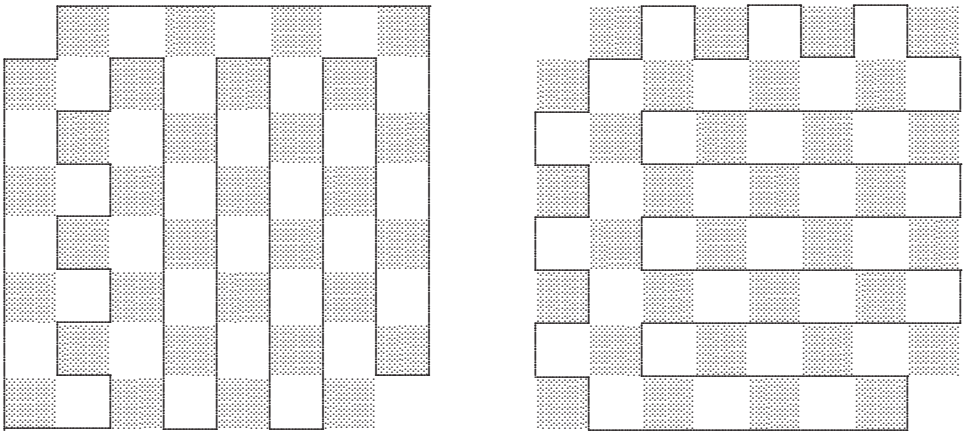
These five white cells may be connected by two disjoint sets of black squares, as shown in the diagram below.



The two subboards are shown in the diagram below. They have been extended so that each includes one of two black squares near the bottom right corner. If the two announcements are the same, then the chosen cell must be white. If the announcements are different, the cell must be black.



This scheme may be generalized to work for any $2n \times 2n$ chessboard. The diagram below shows the solution for the 8×8 chessboard.



The Fall 2013 A-Level paper in the International Mathematics Tournament of the Towns is the source of the challenge from Anna.

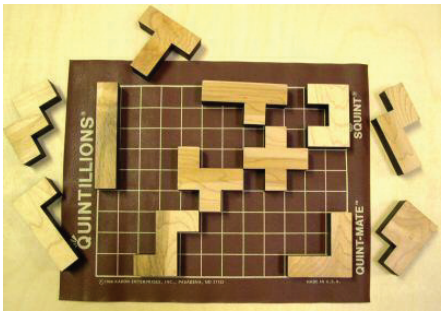
Kate Jones

Celebration of Mind 2015

honoring Martin Gardner (1914-2010)

October 24-25, 2015 at Ye Olde Gamery, Maryland Renaissance Festival (one of hundreds of official Celebrations of Mind world-wide)

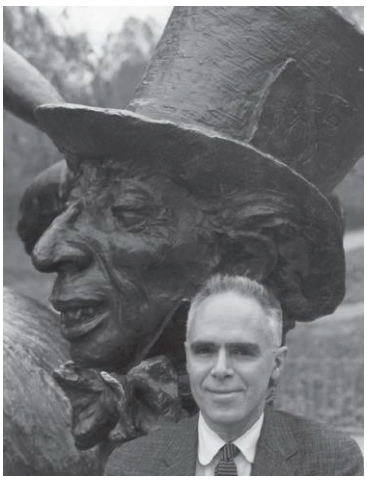
Martin Gardner was one of the most beloved personalities in the areas of recreational mathematics, magic and puzzles. The influence of his work is immeasurable. In a writing career that spanned seven decades, he authored more than 65 books and countless articles, ranging over the fields of science, mathematics, philosophy, literature, and conjuring. His best-selling book was *The Annotated Alice*, an analysis of Lewis Carroll's *Alice in Wonderland*, followed by a sequel, *More Annotated Alice*. He wrote two novels—*The Flight of Peter Fromm* and *Visitors from Oz*. His quarter century of *Scientific American* columns are collected in fifteen volumes. *No-Sided Professors* is a collection of his short fiction.



Martin inspired and enlightened three generations of readers with the delights of mathematical recreations, the amazing phenomena of numbers, magic and puzzles, the play of ideas. It was Martin's article on *pentominoes* in 1957 that popularized this set of shapes and led, through an amazing series of events, to the founding of Kadon Enterprises, Inc. We hold him and his life's work in a special place of highest regard.

We were honored when Martin offered us the opportunity to design and publish his two games. The first of them, originally a feature in *Games* Magazine, was the **Game of Solomon**. We enhanced it with additional games and puzzles, and we styled its rule book as a scroll. See more about it in our website, www.gamepuzzles.com, under Historical Games. See a quaint Shakespearean write-up in our Renaissance catalog.

Martin's work on Lewis Carroll's *Alice in Wonderland* led to his defining suitable rules for a word game alluded to in Carroll's diaries. This became our **Lewis Carroll's Chess Wordgame** where letters move like chess queens to form words. We styled it like a red picnic tablecloth. See full details about it in our website, www.gamepuzzles.com, under Historical Games, and a whimsical poetic write-up in our Renaissance catalog.



Martin and the Mad Hatter

Martin Gardner, Renaissance Man

Martin Gardner was born October 21, 1914, in Tulsa, Oklahoma, the son of a geologist and oil producer. He graduated at the University of Chicago in 1936 with a major in philosophy. Before World War II he was a reporter on the *Tulsa Tribune*, later a writer in the University of Chicago's press relations office.

After four years as a yeoman in the Navy, Martin returned to Chicago where he began his freelance career by selling short stories to *Esquire*. After moving to New York City, he became a contributing editor for eight years to *Humpty Dumpty's Magazine*. This was followed by 25 years as the writer of the "Mathematical Games" column in *Scientific American*.

After living in the western mountains of North Carolina for many years, he returned to Norman, Oklahoma, in 2004, his 90th year. He continued to write until his death on May 22, 2010, at the age of 95. There is an excellent entry about Martin on the Wikiverse website, part of Wikipedia, an ever-growing, open-content, online collection of all of human knowledge.

A bi-annual celebration of Martin Gardner's life and work has been held in Atlanta, Georgia, since 1994, founded and hosted by Tom Rodgers, a businessman, scholar and Renaissance man. Martin himself attended the first gathering. Since then, the Gathering for Gardner continues to be an invitation-only get-together for mathematicians, magicians and puzzlers who enjoy sharing their work and play inspired by Martin's writings. Attendees bring something to share, such as articles, new puzzles, ideas and theories—a joyous grab bag for each participant, and one copy of each would be sent to Martin, and since his death to the Gathering for Gardner Foundation. Contributed articles are anthologized in a souvenir book. Selected articles have been reprinted by A. K. Peters in Tribute books dedicated to Martin Gardner. His memory and inspiration live on.

Kadon's philosophy and artistic vision in designing "gamepuzzles" sum up as a celebration of mind ... the joy of thinking ... playable art ... truth and beauty. And the catalyst for this lifetime of creation was one man: Martin Gardner. We celebrate him and invite all our visitors to join in and get in the spirit of puzzling, gaming, creating and designing with our wonderful collection of original gamepuzzles.

Fair Poker with Marked Cards

William Liles
lilesw@gmail.com
March 5, 2016

This is a puzzle. How can a number of poker players play a fair game in which each of the players assumes that any of the other players will bring a marked deck to the game?

We will start with a few assumptions.
1) The only cheating method employed is the use of marked cards.
2) The cards are marked so that each player can only read the marks of the deck they supplied, and even throughout the play of many hands, the other players cannot determine the markings on the cards from the other players decks.

Notation:
A - Ace
J - Jack
Q - Queen
K - King
S - Spades
H - Hearts
D - Diamonds
C - Clubs

A - K means the cards are arranged from top down A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K.

K - A means the cards are arranged from the top down in reverse order than A - K (the order is K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2, A).

Decks are numbered 1 to N.

Thus AS represents the Ace of Spades. If a number follows the card designation, such as AS2, that means that the card comes from the deck associated with the suffix digit.

There are many variations of poker, some of which have cards that are dealt face-up so that all players can see their values. Do markings matter in such circumstances? Yes, since there is usually a round of betting prior to the face up card(s) being dealt. If the card is dealt off the top of the deck, the dealer, and perhaps others, could bet with the knowledge of the top card's markings. One could deal by removing cards from the deck center, thereby removing that visible top card advantage, but one still knows the top card is not in play, and that changes the odds.

As shown in the above paragraph, there are many considerations. In this paper we will only address games in which cards are dealt face down and the top card is dealt so that the back cannot be seen in advance to being dealt. (This can be accomplished by placing a Joker face-up on top of the deck. For example, to cut the deck, a face-up Joker is inserted into the deck, the

deck is cut at that point, leaving the Joker on the top of the deck. Then by dealing the second cards of the deck, their backs are not shown in advance. This leaves the situation where the visible back of the top card prior to the cut can be read. There are ways to handle this condition, but they will not be discussed in this paper.)

If there is one marked deck and only one player knows the markings, then it is clear that player has the advantage. But what if there is one deck consisting of cards taken from N marked decks when there are N players? For the time being, we will ignore the case of one deck consisting of cards from N decks with M players, where N does not equal M.

The issue of back design needs to be addressed. If all N decks have different back designs and those designs are associated with particular players, then a player will know which of the cards they are holding can be read by which other player. Their best hand of cards will have only the backs associated with that player. A particularly bad hand of cards will have all the backs associated with only one other player, as that other player will know all of the cards in that hand.

How can this be made fairer? What if each player only receives cards from the deck they supplied? Assume that N players supply N decks and each player's deck is shuffled, cut and dealt by other players. Then each player receives all their cards from the deck they supplied. That way no player can read another player's cards. This impacts the game and would require an additional rule, because duplicate hands can now exist. Who wins in a game of five players when all five players hold four aces?

Also certain conditions that cannot exist in a single deck game can now occur. For example, one player holds four aces and another player holds a royal flush. In another example, one player holds a 10 high straight Spades flush and another player holds a 9 high straight Spades flush.

Given the constraint of duplicate hands, using N decks for N players provides a fair outcome and is easy to implement. Note: with this procedure and the previously stated assumptions, there is no advantage to having the cards marked, except if a player can see the top card of their deck and the game includes drawing cards from the top.

Are there other ways to implement a fair game using only one set of 52 cards? If yes, how does one form that set?

Let each player bring to the game their own deck of marked cards. Cards are removed from each deck to form one deck of 52 cards with unique faces. Let the N decks be arranged top down as:
A - K S
A - K H
A - K C
A - K D.

The objective is to prevent any player from gaining more knowledge of another player's hand than the other players have of that player's hand. This forces certain constraints on how the new deck is built.

For example, if N is four and the new deck is made by taking cards in turn from each A-K stack, then the following set is formed:
AS1, AH2, AC3, AD4, 2S1, 2H2, 2C3, 2D4, 3S1, 3H2, 3C3, 3D4,
4S1, 4H2, 4C3, 4D4, 5S1, 5H2, 5C3, 5D4, 6S1, 6H2, 6C3, 6D4, and so on.

A good result from this arrangement is that four of a kind, a full house, three of a kind and a pair cannot be determined. Even a straight cannot be determined, except if it is a straight flush.

One problem with this arrangement is a player could get a flush all from the same deck and the owner of that deck would know that, while the other players would not. This would not be a fair situation.

This problem can be addressed by saying that if one gets a flush, it counts for nothing, and one can remove all of their bets. That works for the holder of the flush, but other players might have placed bets based upon the flush holder's bets. Does one declare that all hands are invalid? The holders of really good hands would not like that, especially if they had been losing prior to the current hand.

In the most equitable scenario each player could only determine the value of at most one card in any other player's hand. There is no way to guarantee this outcome.

Assume five players with five decks assembled into one deck, then it is possible that all five cards in one hand come from different decks. It is also possible for all five cards to come from the same deck.

One way to form the new deck is to take a card from each deck in order of uniqueness to produce the new deck in order A-KS, A-KH, A-KC, A-KD. For the example, the new deck top down will start out as:
AS1, 2S2, 3S3, 4S4, 5S5, 6S1, 7S2, 8S3, 9S4, 10S5, JS1, QS2, KS3,
AH4, 2H5, 3H1, 4H2, 5H3, 6H4, 7H5, 8H1, 9H2, 10H3, JH4, QH5, KH1

An advantage of this arrangement is that there will never be a flush hand with all the cards from the same deck, thus solving the problem with the prior arrangement. Also, any four of a kind will always have cards from four decks. Three of a kind, a pair, two pairs and a full house also must have cards from different decks.

Is the problem solved? Not quite. Five does not divide evenly into 52. There is a remainder of 2, thus two decks will each have one more card in the assembled deck than the other decks will. This will give a slight edge to the owners of the two decks that contributed extra cards to the assembled deck.

So what would be a good number for N? 52 can be prime factored into 2 * 2 * 13. That leaves 2 and 4 as possible values of N. 13 is also a possibility, except that would leave each hand with only 4 cards, thus 13, 26 and 52 are not good values for N.

Would two players with two decks assembled as one deck be fair? With two decks, each suit would consist of either six or seven cards from the same deck. In that case, a flush can be dealt with cards all from the same deck, thus tipping the owner of that deck to the hand of their opponent.

With four decks, each suit would consist of either three or four cards from the same deck, thus a flush cannot exist having all the cards from the same deck. This is much better, but still has an issue. If a player notices that another player has four cards of a flush, then what statistical advantage do they have in determining if there is an actual flush? If not a flush, the only other possibilities are high card (which they will know the highest card of the four cards of the same suit) or one pair. This gives an advantage to the owner of the deck that the cards in the flush came from. Over the course of a night's play that situation might never occur, but it is possible.

There is more to explore regarding this puzzle, but this paper was intended to introduce the problem and offer some considerations to a wider audience. For example, not included in this preliminary paper is any discussions of assembled decks having a number of cards different than 52 (a deck of 48 cards allows for six decks and a deck of 60 cards would allow for five decks).

Kate Jones

Gamepuzzles! $4

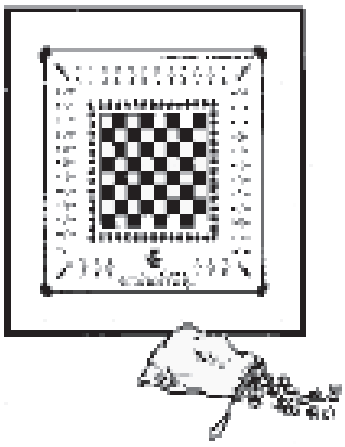
Deductive delights

To play's the thing

The Royal Game of the Goose and The Game of the Labyrinth

This two-fold whimsical adventure treat
Replays the classic tribulations well:
Roll ye the dice, advance along the path—
Grab ye the boons, beware mischance's wrath!
The friendly geese will speed thee straight and true,
But flee the cooked goose, or ye start anew.
The Labyrinth an ancient journey be
Translated from the tablets found on Crete.
The Cosmic symbols trace the path to Heav'n.
Thy four youths must prevail 'gainst ominous fate
To reach the safety of the inmost gate.

Regal: *Grand 24" framed wood board with brass handle, 16 sculptured pawns, wood dice, fabric cover.*
Traveler: *24" canvas banner on rod, with cord to hang, 24 glass marbles, wood dice, fabric sheath.*



Lewis Carroll's Chess Wordgame

A tale is told of wonderlands of mind
Wherein as through a looking glass of thought
The traveler meets with marvels past recount.
A wordgame for a chessboard? Yes, indeed!
So Lewis Carroll's fertile brain opined
And Martin Gardner's skillful sense defined.
Now let the letters stalk about like queens
To range themselves as words upon the board.
Two players vie to weave the "spell" that scored.
"Tablecloth" gameboard, pouch of alphabet disks.

The Renaissance Deck
Peerless artistry of ancient craftsmen in 32 exquisite panels recaptures the tale of William Tell. Played in Europe's finest gaming salons. Fabric pouch, booklet for game rules and historic solitaires.

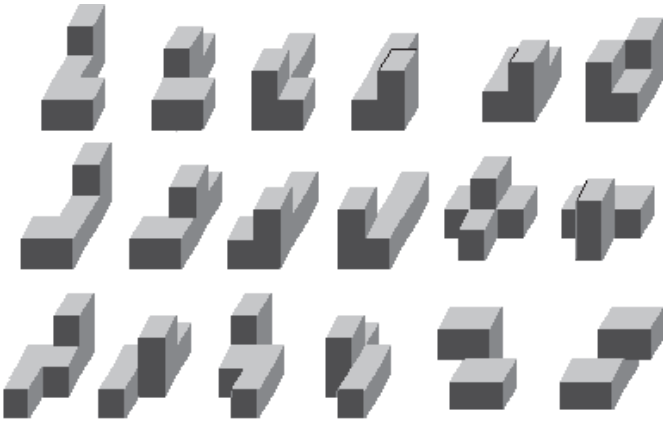
Polyominoes and Polycubes

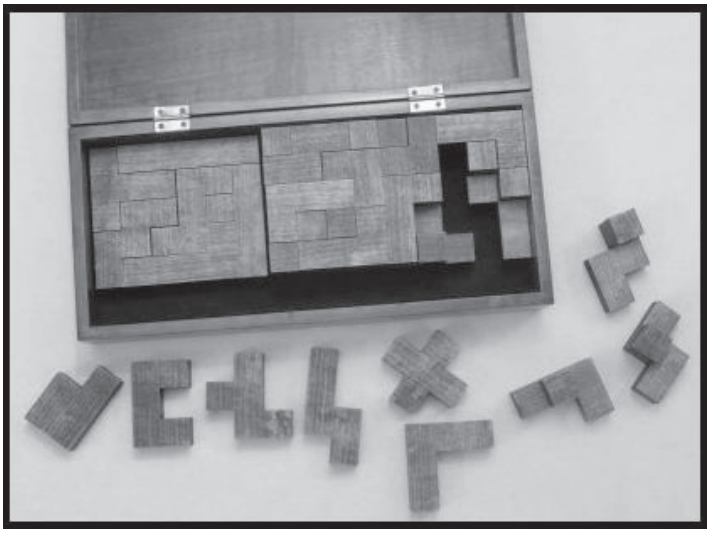
Quintillions®

This comely dozen blocks of polished wood
When joined and fitted in a myriad ways
Invite the playing of five different games
Of skill, and building multitudes of shapes.
The famous "five-ness" of each quint
Made them a *Games* "100 best" in print.
Hand-finished, polished to a silken shine,
Cut from a single board of maple fine.
Its two- and three-dimensionality
Makes for incomparable quality.
With leather-like box, game mat and
80-page booklet; unit cube size, 13/16".

Super Quintillions®

Precision-cut, handfitted superquints
Do mate exquisitely with yon twelve blocks—
A supplement of eighteen shapes comprised.
Forsooth, they are the known world's most sublime
And elegant entertainment of all time.





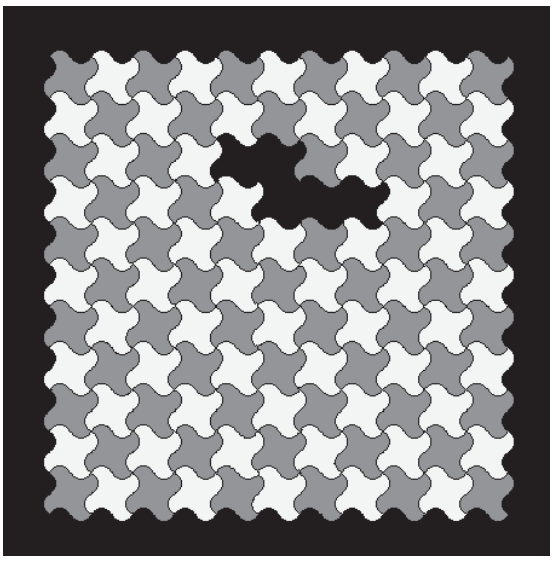
Super deluxe Quintillions®

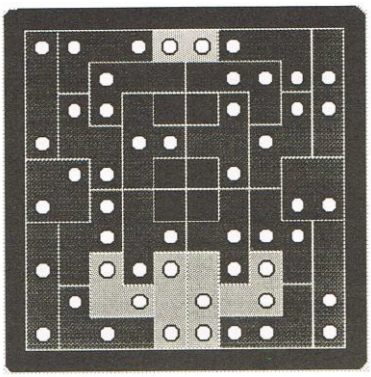
All 30 pieces, quints and superquints,
In one magnificently wrought wood case,
A precious heirloom generations treasure,
To see, to touch, to think, to build, pure pleasure.

Be these "iambic pentominoes"?

Dual Quintachex®

Dismantle this gargantuan field and find
A perfect dozen pairs of five-celled shapes,
Their checkerboarding lavishly affixed
In frosted-mirror sparkle. Wavy bands
Undulate the 11x11 grid.
Therein the dual quints, reposing, pine
For no two mates adjacently recline.
Perchance thy wit and heart be so inclined
To help them snuggle up to their own kind?
Acrylic, in 24" tray, wall-mountable.



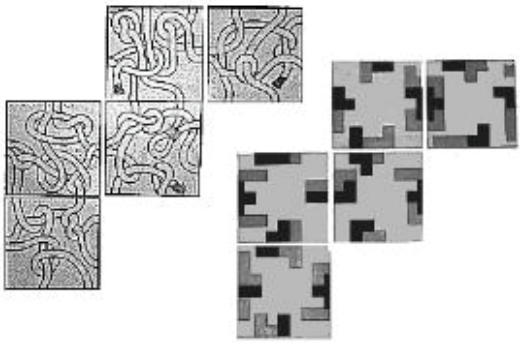


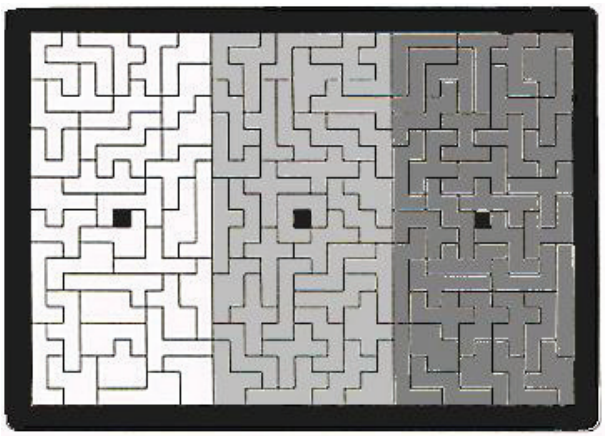
Fill-Agree™

This finely fitted family of tiles,
Wherein two holes in all their patterns ride,
Be formed of 1 through 4 squares joined all ways.
So fill their tray, and all holes neighbored place,
Or line the rows and columns with 5 holes,
Or make them fill/agree with other goals.
In 10" acrylic tray with easel.

Fractured Fives™

Five squares of lasercut and polished wood
Engraved one side with tangled ropes that could
Upon adjacency twelve figures match
Or even run a single line per batch.
Upon the flip side form pentominoes
Where'er an edge against another goes.
Behold a proud self-reference for this dozen,
Their neat solutions will leave brain cells buzzin'.
4" squares with felt storage pockets.



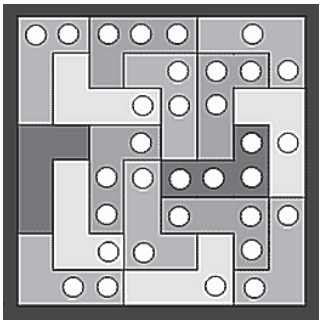


Heptominoes

One hundred eight uniquely shapèd forms
Of seven squares comprised—this fertile field
Of planar coverings attests thy skills
And fortitude to seek gigantic thrills.
In 13x18" tray, all acrylic, sized
to Sextillions and Poly-5.

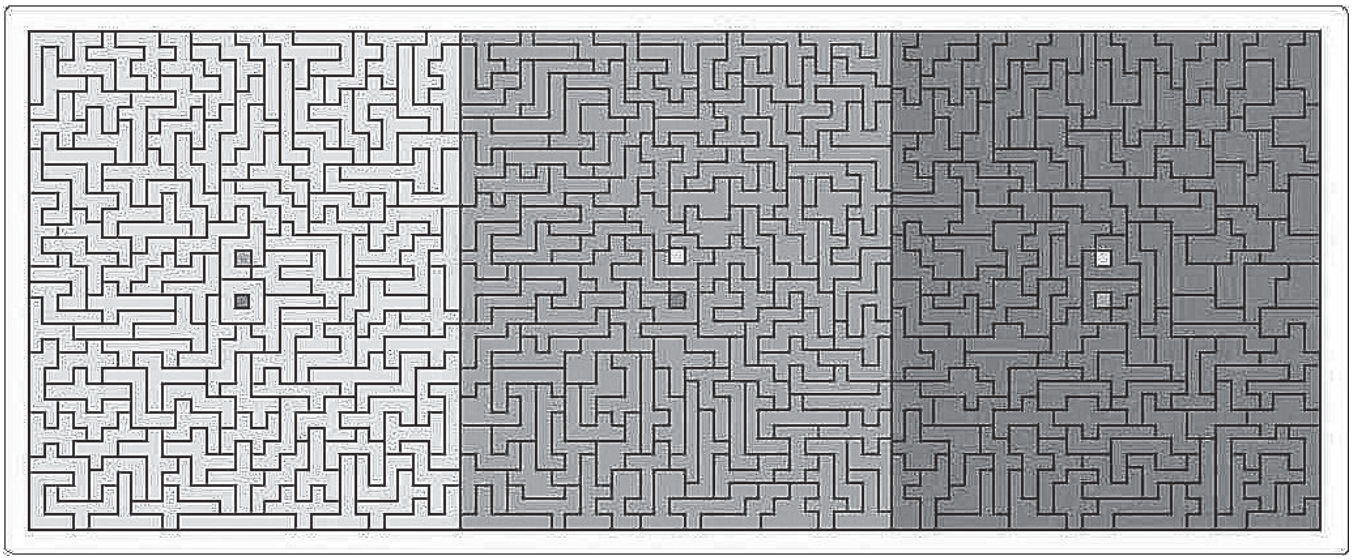
Hexacube

Now to the limits of one's fortitude,
One hundred sixty-six wood blocks allude,
Each block of six cubes differently enscrolled,
A giant 10 by 10 cube all of them will hold,
With four "wild" single cubes therein dispersed.
Be this the grandest set for which you thirst?
In fold-out hinged wood treasure chest...



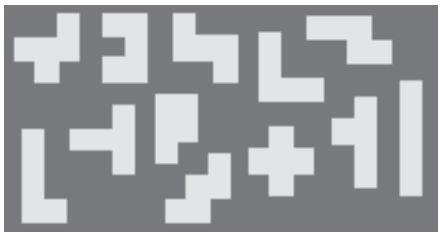
L-Sixteen™

Here be a set that's holey-er than most,
Where tetro-L's have none to four to boast.
Align the holes joined in a single maze
Or struggle till each row four holes displays.
Make zany figures strangely intertwined
And play a game for two to stretch your mind.
4 colors of see-through acrylic in 9" tray.



Octominoes

Now these most daring polyominoes
Three hundred sixty-nine ways 8 squares pose.
Wouldst stretch thy mind and diverse figures build
Or hope to see its giant tray well filled?
Its pieces neatly fit with all its brethren
That we produce, in sizes one through seven.
23x47" tray, 3 colors, all acrylic, with lid for wall-mount or easel display, or built into a table.



Pentomino necklace

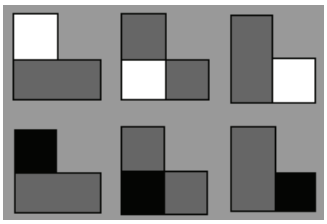
A puzzle as adornment? Wear it proud,
A conversation piece to tease a crowd,
For it will solve a rectangle, indeed,
With chain attached, though table you may need.
Math teachers at conventions are a hit
When they converse with colleagues wearing it.
Gold-tone chain, ½" unit square wood pieces.

Pocket Pentominoes™

12 shapes form countless figures, play a game,
"Chasing Vermeer," the novel, showed the same.
Each piece a different form of five squares grows,
Hence all the world calls them pentominoes.
In drawstring pouch they make fine travel friends,
Wee half-inch scale. The puzzling never ends.
Yellow acrylic. Other colors by custom order only.

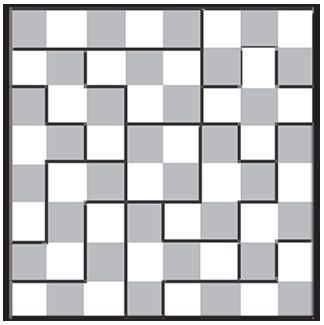
Pocket Vees™

6 Vees (3 white, 3 black) make sweet designs
Or wins the one who first all three aligns.
Red acrylic, ¾" unit squares, in drawstring pouch.



Pocket Quintachex®

The checkered die-cut twelve pentominoes
Reverse for double trouble: your skill shows.
1" scale, white/green rigid vinyl in drawstring pouch.



Poly-5™

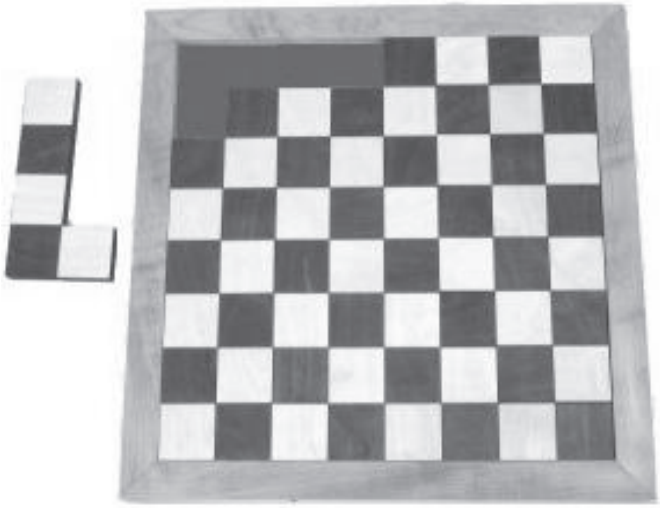
The tidy shapes of 1 to 5 pure squares
Adjoined in all their cosmic variants
Can fill the rickrack tray with tilish tricks
And form endearing patterns large and small.
Six games of thought are played by two or more,
To block or capture or high points to score.
All acrylic, sized to Sextillions, in 6" tray.

Quintachex®

So fair a square of inlaid checkerboard
Ne'er graced the royal gaming halls of yore.
The classic grid in framèd tray disbands
To 13 double-sided checkered parts!
The scores of puzzle figures check your mettle
Whilst three sharp games keep wits in finest fettle.
There be 1294 solutions to the chessboard array alone, whereof only two have the square piece centered. Handfitted of 65 dark and 63 light squares. All acrylic, 8½" tray, game mat.

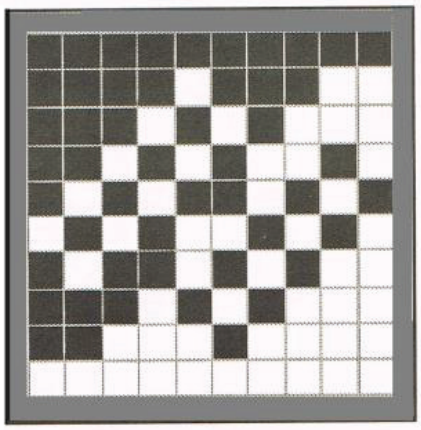
Quintachex® deluxe wood collector's edition

In puzzledom, a connoisseur's true treasure,
Unique in all the world, a sumptuous pleasure.
Handcrafted chessboard of two-sided tiles,
Each of 5 squares comprised, each twice beguiles
As either side its colors lends to match
Within the checkered field a perfect patch.
A master craftsman's care in every line,
It's polished to a silken touch and shine.
20" wood tray, 2" squares fit full-size chess pieces.



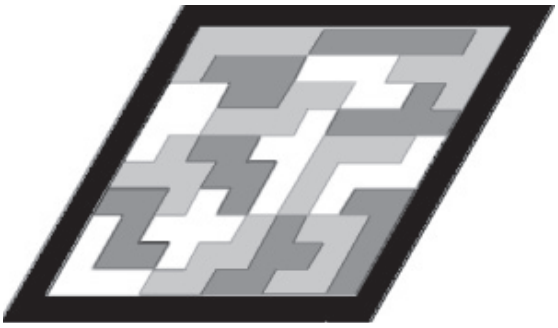
Quintapaths™

A score of 1x5 white sticks be stacked,
Each with a different count of squares of black.
Form paths and patterns, islands max and min
And symmetries to challenge yang and yin.
Their tray beholds a 10x10 array
In thousand-fold profuse variety
While 5x20 makes as yet more plans.
Strategic games 2 players will entrance.
In 10" tray, all acrylic, with easel.

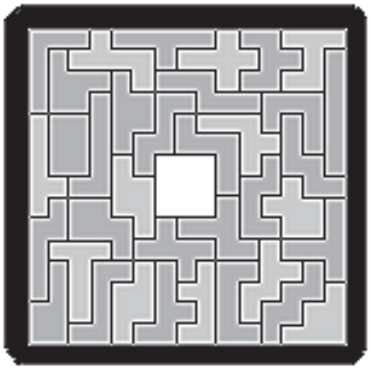


Rhombiominoes™

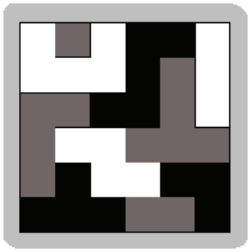
Most clever score of rhombed pentominoes—
Where every tile comports 5 diamonds fair.
There be three colors for to keep apart
Or joined in groups, or scrambled just to start.
We fathom that 8 million ways oblige
Assembling all the pieces in the tray.
Yet even one confounds the solver's quest
And renders every solved array a test.
Diamond-shaped 15" tray, all acrylic, with easel.



Sextillions™

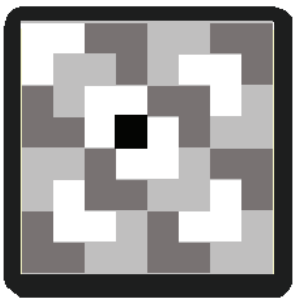


Exquisite set of gleaming shapes of "six"
Squares joined in six-and-thirty ways to prove
Diversity yields yet to union sweet.
Fit free or to its grid, your goals to meet.
Four games for two to half a dozen minds
Whilst puzzlers feast on tasks of sundry kinds.
All acrylic, 8½" tray, game mat.



Ten-Yen™

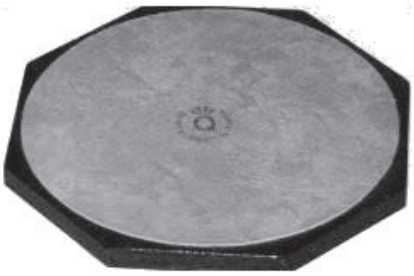
Historic puzzle gleaned from days of yore,
Its ten trim pieces in three hues restore.
Both matched and non-matched schemes invite creations,
Ten thousand patterns wait the puzzler's patience.
In 5½" tray, all acrylic.



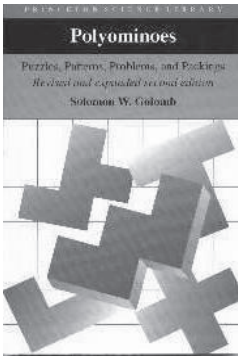
Vee-21™

Thrice seven V shapes clustered squarely place
That their three colors joined or unmatched lace.
The free space springs to any spot at all,
Or, filling grids, the Vees may cross no wall.
In patterns filled or filigreed abide,
Or alphabets and symmetries bestride.
The youngest solvers gleefully succeed
While sharpest puzzlers challenged be indeed.
In 7" tray, see-through tiles, all acrylic with 40-page book.

Turntable for thy 3-D games



Octagonal wood board that smoothly turns
So that thy strategy from all sides learns.
A leather-seeming pad upon the board
Adds luxury to every victory scored.
12" diameter, handcrafted and finished.

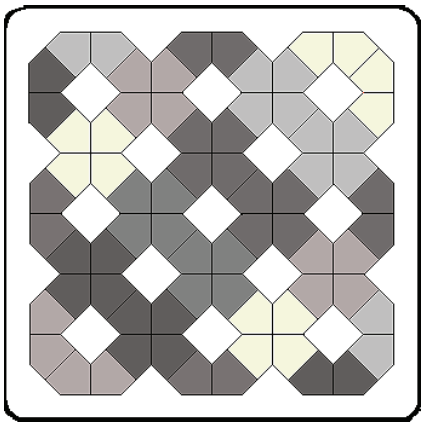


Polyominoes, the book

Well written over sixty years ago,
It is the gateway to what all should know
About the joy of puzzles made of squares,
Whether in groups of fives or fours or pairs.
'Tis the esteemèd Sol Golomb who shows
Why he's the patriarch of polyominoes.
The math, the magic, marvels by the score,
This book reveals what human brains are for.

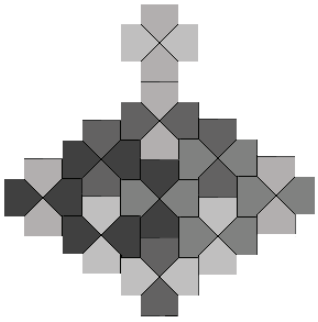
Edgematching Sets

See also **Arc Angles**™ *on the Pentagon Universe stage.*



Boats™

Would these be dominoes reshaped as boats
In rainbow hues that match to sail through moats?
Full hundred figures lattice-laced abound--
Like mandalas their beauty doth astound.
A fleet of 36, each two the same,
Enchants as well with many a game.
The tiles within thy fingers feel so fine
As tactile and the visual arts entwine.
11″ tray, hand-inlaid tiles, with easel.

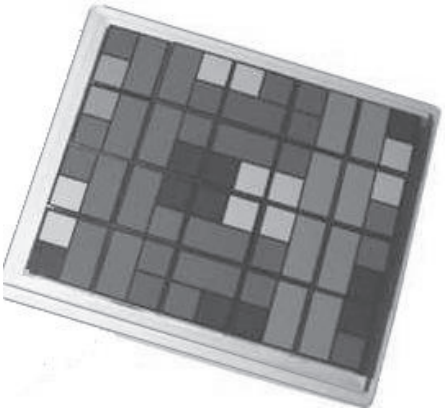


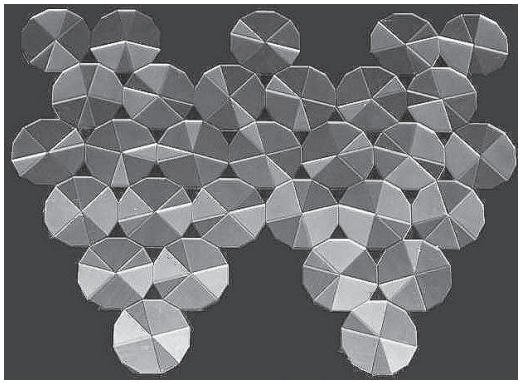
Bowties™

When bowties cross in pairs with colors true,
And all five colors mate with every hue,
Ten different tiles mosaic patterns grow.
Match corners, sides and tips as colors flow.
In drawstring pouch it travels, slim and chic,
To serve what your creative urges seek.

Color Up™ — *Limited edition, while they last…*

These twenty cubes, each with its own three hues,
Bedecked on all six sides with varied views,
Assemble stubbornly with corners matched
For vistas of six colors neatly patched.
A jolly zoo of figures waits thy wit;
A game for two to boot makes this a hit.
In 9″ long wood case.



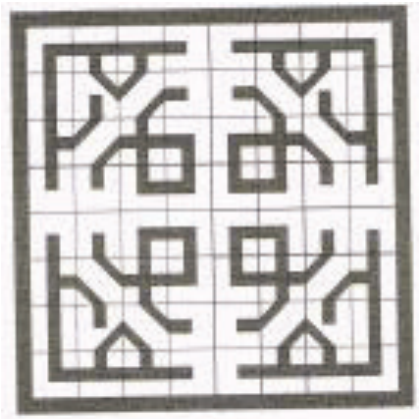


Dazzle™

Twelve-sided circles dance before thine eyes:
To have matched colors touching, be precise.
Symmetric patterns twirl to fit the rules
While dazzles, trails and wedges are thy tools.
Around the pretty gaps three colors flow...
All one, all three, the points they gain will grow.
Thirty 2¾″ tiles, all acrylic, for up to 7 players.

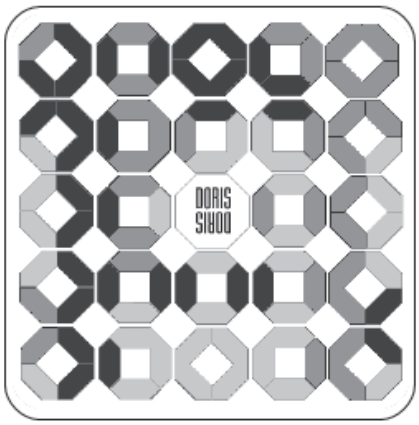
Dezign-8™

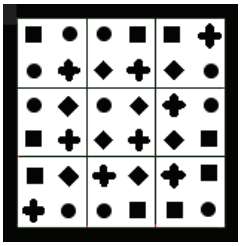
The paths to splendid symmetries abound
Within this 8x8 array of squares.
The inlaid tiles show exits one to four
That work a well-nigh endless wealth of ways
To join in circuits of completed groups
And spike their circuitry with closèd loops.
There be 8 types of tiles, a swarm of each,
That in their joinings games as well can teach.
All-acrylic, 13½" tray, with easel and felt bag.
Ye may request custom colors.



Doris™

This grand and glorious rhapsody of eights
In every mix of colors three awaits
Thy skill to solve and thy imaginings
In forming new shapes, animals and things.
Eleven countries' students vie in tourneys,
A festival of mind worth all their journeys.
The champions receive awards plus fame,
And no two contest entries look the same.
All-acrylic, 14" tray, some mirror colors; leather-like game mats, pawns that serve as astronauts, felt cover and display easel included.



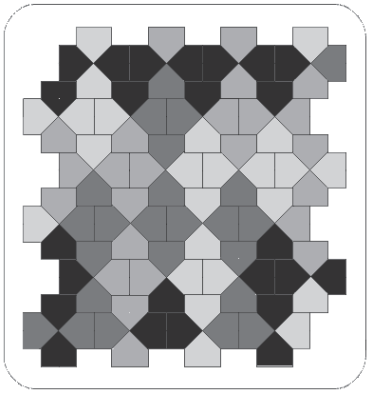


Four on a Match™

Nine squares with four shapes perforated through—
Circle, diamond, square and flower, too—
Crave matching mates on all four corners true.
'Tis greatly vexing. Harken—we've warned you.
This little charmer, cheery red and white,
Serves 1 to 3 sharp players, day or night.
In 5" tray, all acrylic.

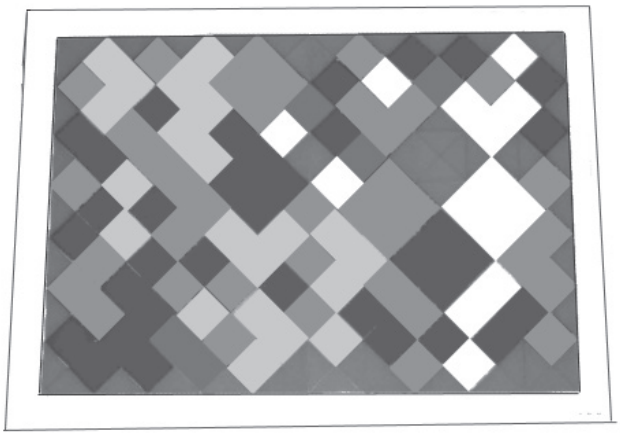
Grand Bowties™

Such grandeur of design and color glow
When bowties twin four colors, row by row...
The petals match on corners, sides and tips
To make ethereal forms like flowers and ships.
Like unto stainèd glass or Moorish tiles,
Each marvelous pattern bringeth joyful smiles.
All-acrylic, 11" tray, with felt cover and easel.



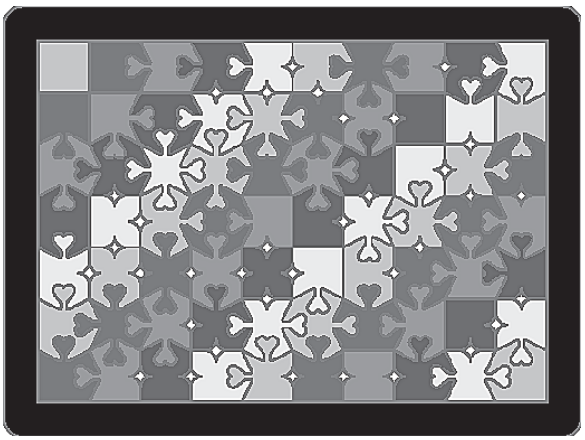
Grand Multimatch® I

What daring and dramatic venture this,
Its 7 by 10 display of four-hued bliss,
That every square be snuggled to its mate,
None left behind or out, none made to wait,
With colors shared and frame a single shade.
Select your own four colors to be made.
13x17" tray, all acrylic, with easel and felt cover; for up to 4 players.



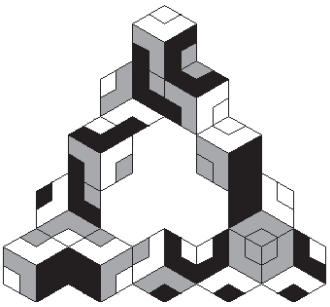
Grand Snowflake™

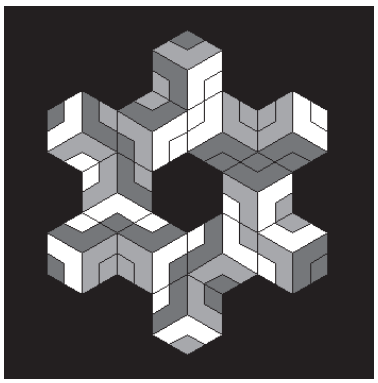
When squares are altered so their edges be
Not straight but now convex, concave, or notched,
Full seventy all-different shapes arise,
Embellished here with twinkles, spaces, hearts.
These snowflake tiles five colors carry through
The 7x10 array. Ah, playful clue:
Straight sides kiss only straight, the twinkles pair,
And hearts fill spaces for a vision fair.
Be warned: this proves most difficult a feat,
But each success becomes a visual treat.
15x18" tray, all acrylic, with easel.



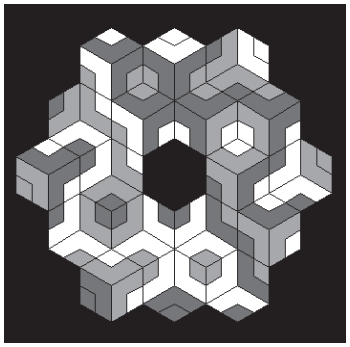
Hexmozaix™

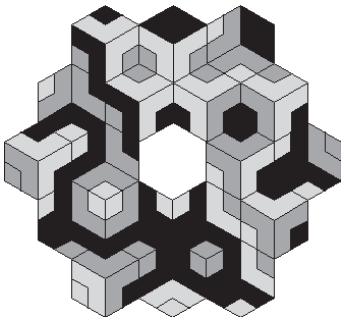
One dozen hexagons, lush and serene,
Bewitch both gamesters' ken and puzzlers keen,
Inlaid with diamond tips and chevron bands
In three-part harmony. Match ye the strands
On borders joined in patterns neat and clean.
2" tiles, all-acrylic, up to 4 players.





Also in gallery wallhanging frame with removable face plate.

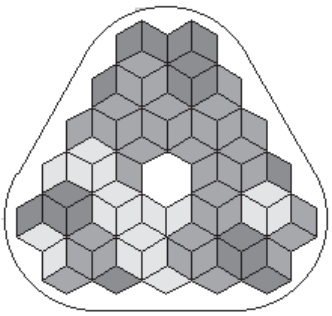




Also in gallery wallhanging frame with removable face plate.

Hexmozaix™ II

Hexmozaix' great companion set,
These eighteen hexagons uniquely hued
Expand the lavish puzzle repertoire
And open vistas of strategic play
When thirty tiles adjoin in grand array.

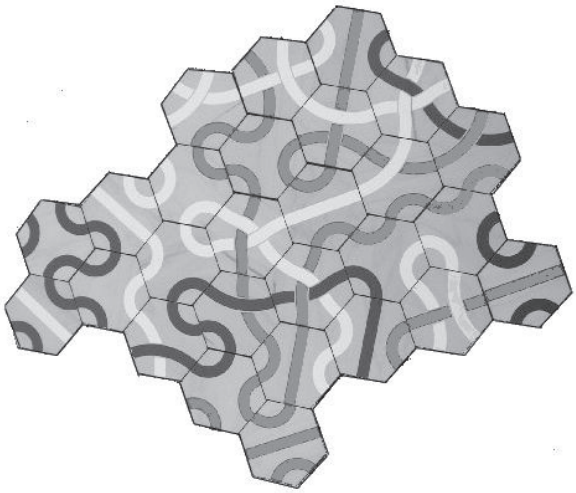


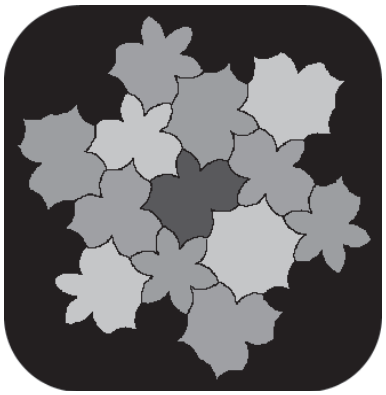
Hexmozaix™ Jr.

Two dozen hexagons inlaid with three
Bright diamonds have four colors mixed all ways.
Combine the hexes colormatched, galore,
In lively patterns ye'll love to explore.
If games thy gambit be, don't feel left out:
Grab two or three friends, then have quite a bout.
7" triangular tray, all acrylic.

Kaliko™

This splendid set of eighty-five wood tiles,
Six-sided, forms fine arcs in colors three.
Path-joining games enthrall from two to four
And diverse puzzle patterns challenge one.
Indeed, this be a masterpiece of thought,
Wondrously conceived and nobly wrought.
Blue felt storage pouch, dainty bamboo screens to hide thy tiles whilst ye ponder thy next great moves.
Five times on "Games 100"; on "Golden Oldies" list, 1995.



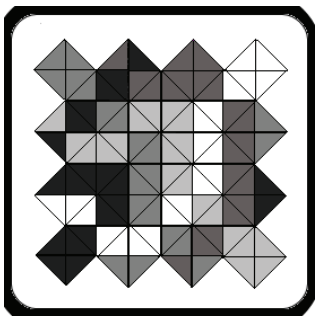


Leaves™

A lucky thirteen shapely tiles appear
So much like leaves,'tis autumn all the year!
Their curvèd edges, notched or peaked or looped,
Adjoin in glorious shapes and colors grouped.
Symmetricals three quads of color sport
While just one oddball singly comes up short.
Behold their tray of triple symmetry;
Ten thousand more form when ye set them free.
11" tray, see-through colors, all acrylic, with easel.

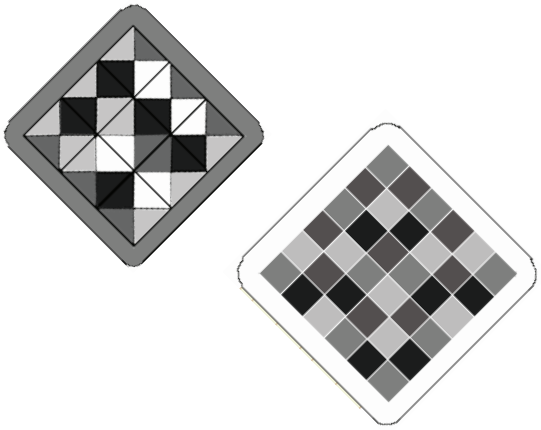
Marshall Squares™

This Aussie set of five-and-twenty squares
Assigns five colors singly and in pairs.
A matching task on figures by the score
Has lots of solving merriment in store.
This shape has over twenty-thousand ways
The colors fully match in bright arrays.
All acrylic, hand-inlaid tiles in 8½" tray.



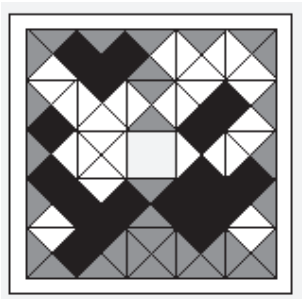
MemorIQ™ *("Memory Cue")*

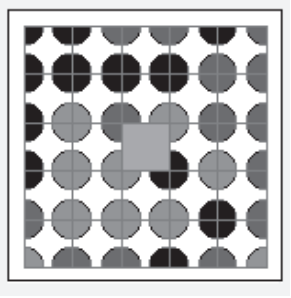
Two dozen hexagons four colors wear
Apportioned in three pentagons so fair,
With three sides solid, three sides split in two
Now match them smartly 'gainst the neighbor's hue.
The lettered tiles be wild, flipped they be bare
A myriad shapes to conquer, if you dare.
Three games of strategy for up to four
Have novel goals, where only nice guys score.
11" circular tray, all-acrylic, hand-inlaid, with easel.

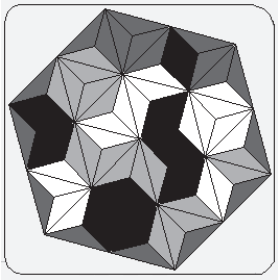


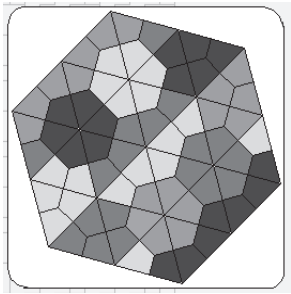
MiniMatch™ I and II

To match or not to match four colors true,
In shapes made of nine squares, is what you do.
Set I has tiles with triangles bedecked—
Adjoin their sides and symmetries detect.
Set II each tile four tiny squares displays
That fit together in amazing ways.
These charming sets befit thy wits, forsooth,
While simpler tasks engage both tikes and youth.
5½" tray, hand-inlaid tiles, all acrylic.









Multimatch® I and II

Two dozen diverse squares display three hues
In all their sundry meetings juxtaposed.
The puzzler's task: assemble all to fuse
Their matching colors in a picture fair.
Set I can frame a single color true
With squares triangularly sectioned through.
Set II sports squares laid squarely end to end
That color patches artfully compend.
Both offer games of strategy and skill
For players, two or more, to share the thrill.
All acrylic, hand-inlaid, in 8½" tray.

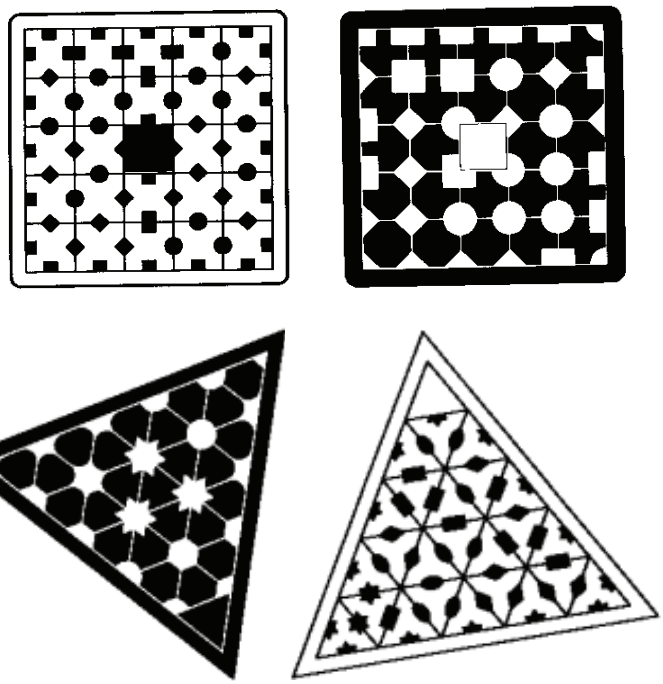
Multimatch® III and IV

Two dozen triangles display four hues
In all their combinations trifold grouped.
The puzzler's venture: match their colors true
That forms in perfect harmony appear.
Set III can frame in solid color pure
With tiles of single-color edge assigned.
Set IV's the very devilment to match
Their 2-part sides, with tips by color marked.
Both offer games of strategy and ploy
For players, two or more, to share the joy.
All acrylic, hand-inlaid, in 8½" tray.

The Fine Touch™ Collection

Our gift of senses—sight and sound and touch—
Be wondrously and variously sparked.
Here now we offer sets whose shapes appeal
Foremost to touch. The tiles be shaped, forsooth,
Of triangles or squares tranformed with art.
Their edges join by contour, matching parts
That leave sweet openings of self-same halves
To make a lattice laced with circles, stars,
Or squares, rectangles, ovals, diamonds fair
Within shaped trays whose borders all may share.
Pure black and white, a filigree for mind
Will please the sighted, challenge well the blind.
In 8.5" trays, all acrylic.

Multi-Touch™ I – edge-formed squares
Multi-Touch™ II – corner-formed squares
Multi-Touch™ III – edge-formed triangles
Multi-Touch™ IV – corner-formed triangles



Rhom-Antics™

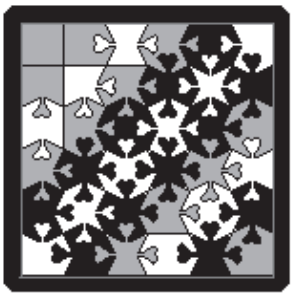
A dazzling decagon of grand design
Embraces ninety rhombi, broad and fine.
Their 72 and 36 degrees
Do show their pentagonish pedigrees.
Such strange, befuddling angularity
Perforce yields scenes of five-fold symmetry,
Or two-fold, or no symmetry at all.
Now mount the rhombs, each edge in turn,
With every combination of three hues
And colormatch them to their neighbors' shade
That patches, ribbons, bands of color flow.
Then play a spiral game of luck and speed,
Or trace ten ladders whither they may lead.
Custom color orders invited. 23" tray, wall-mountable or with easel; all-acrylic construction, hand-inlaid tiles.

Rhominoes™

Five colors pair as twenty-five bright twins
Bedecked by kites and darts like yangs and yins.
To match them in the tray is no mean feat
Though four wild cards extend a helpful treat.
This fiercely tricky set we forged with glee
To celebrate our silver jubilee.
All acrylic, hand-inlaid, in 8½" tray.

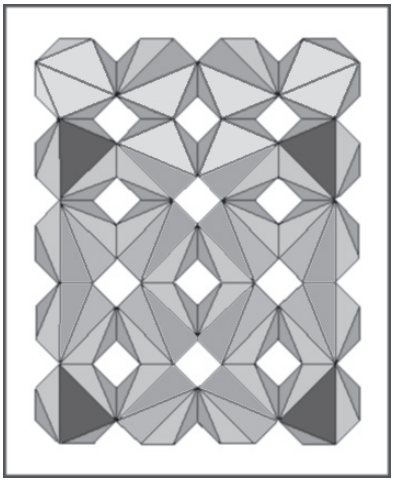
Snowflake Super Square™

A wealth of puzzles, sev'n intriguing games
And loveliness so fine, the heart will leap
With joy at its mere vision. Twenty-four
Transformèd shapes of straight-convex-concave
Embrace their mates in a 6x6 array
Of three jewel'd colors on a white-floored tray.
All-acrylic, in 8½" tray, with game mat.



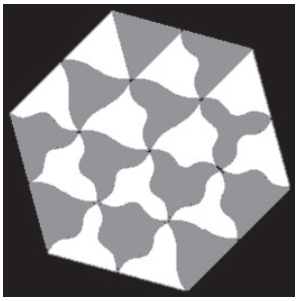
Snowflake Square™

Petitely patterned from its larger frère
With three true colors interwoven fair,
This charmer has but sixteen different tiles.
Its grace and goodly tricks bring many smiles.
All-acrylic, in 5½" tray, with game grid.



Triangule-8™

A moving work of art, of change and flow,
Yon octagons arranged in tray define
The 19 ways to rend them into six
Triangular sections. Through creative hands
Kaleidoscopic patternings can form
As colors fuse and flash in passionate swarm.
All-acrylic, hand-inlaid, 11½ x14" tray, with easel.



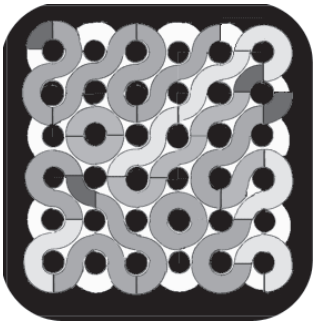
Trifolia™

The graceful curves of valleys, hills, and waves
Transform triangle sides two dozen ways.
Two hues in checkered contrast fill the tray
And serve for games of strategy to play.
All-acrylic, in 8½" tray.

Tri-Jazz™

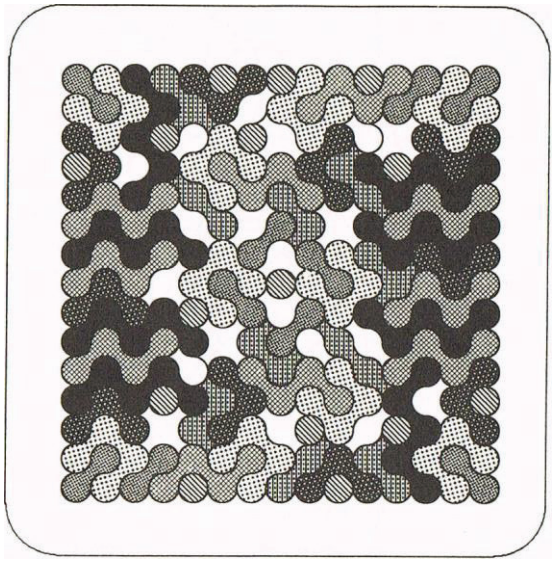
This symphony of shapes and colors three
In skillful hands a moving wonder be.
The four and fifty slanting tiles, in pairs,
Triangles of 3 types affixèd show.
Proportions of precision play duets
As pairs of tiles, in copacetic sets,
Become large rhombs with rectangles imbued
Or rhombs whose every edge is singly hued.
18" tray, with easel, all-acrylic, hand-inlaid.

Polyform Puzzles



ChooChooLoops™

Quartered rings fine arcs and loops provide;
From one to four conjoinèd side to side.
Enclose the islands, some or all in style—
Their wiggly gracefulness will make you smile.
All-acrylic, in 8½" tray.



Grand Roundominoes®

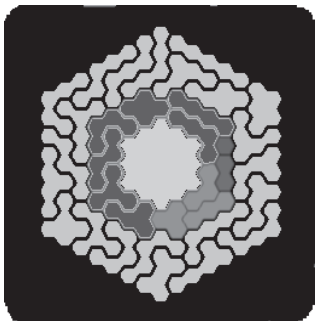
This grandest of our repertoire of "rounds"
Be fondly dubbed the "super dooper" set.
Full nine-and-eighty pieces ply their trade
Of swirling patterns and wild color play.
From one to five rounds fuse in graceful shapes
And open bridges bristle twixt the rounds
Within a giant 15-square array.
The 18 different shapes eight "doopers" hide—
The whimsical cartoon-like duo clumps
That spur the puzzler to inventive jumps.
16" tray, with see-through lid and easel.

Hexnut™ Jr.

'Tis one sweet scheme of hexagons contrived,
All one to four in size, a few of five.
These sixteen can betile the tray, forsooth,
Most varied, colorful and strangely grouped.
Or form a-plenty figures freed from frame,
Or play, for two, a border-skirmish game.
All-acrylic, in 7" tray.

Hex Nut™

A hive of hexes hails the puzzler's quest
Wherein each tile from 1 to 5 comprise
In all their polyform varieties.
The snuggled hex ring swarms forth to suffuse
An endless hoard of splendid puzzle shapes.
Invite thy mates as gaming jackanapes.
All-acrylic, in 8½" tray.

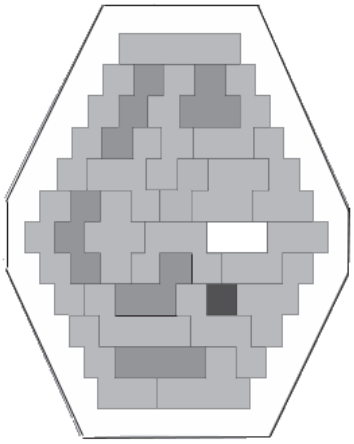


Hexnut™ II

For only hardiest of puzzle fiends
This hexagon-derived ensemble holds
The three-and-eighty shapes of six adjoined.
These hexahexes fill a hexnut ring
With one encapsulated centermost.
Its size be matched to standard Hexnut, too,
Its predecessor made of one through five.
To solve the ring's a *tour de force* – no jive.
15" tray, with easel.

Hopscotch™

Two dozen flagstone patches brick-like fit,
Their rows a half-space shifted snugly sit,
Sized one to four, build figures large and small,
A thrill when thine own hands can tame them all.
When two can play, thy garden path surrounds
The largest territory within bounds.
9x14" tray, all-acrylic, with easel.

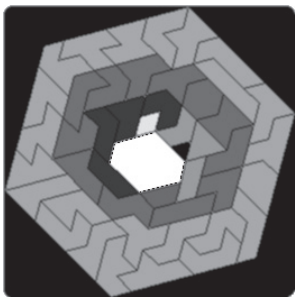


Jamond Hex™

Thy wits will think they've taken leave of thee
When grappling with this most vexatious game,
A mere one dozen shapes—hexiamonds—
The tray can fill in five and fifty ways,
And only one where colors joinèd be
And only one where colors split, all three.
More merry mischief make when midst thy friends:
Share ye this "Hex"—enchantment never ends.
In 5¼ " round tray, all acrylic.

Jamond Ring™

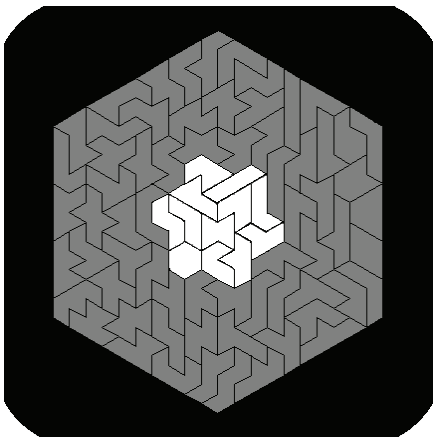
A precious jewel in its hex-shaped ring
Holds tiles from 1 to 7 parts compiled
Of equal-size triangularity.
A fabulous affair of forms and themes
Delights the puzzler, sparks the gamester's schemes.
All-acrylic, 7 colors in 8½" tray.



The road to Heaven is paved with good inventions. — *Kate Jones*

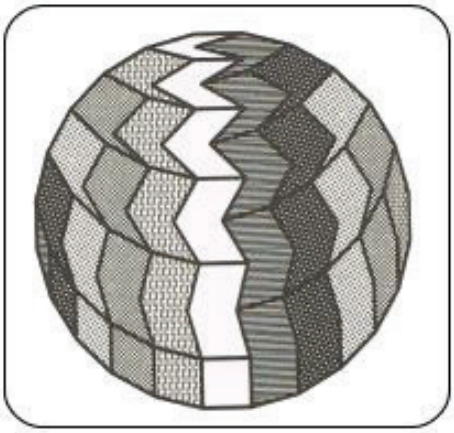
Mini-Iamond Ring™

A cunning feat, nine pieces to deploy
Within a circularity defined.
These wee ones, from full Iamond Ring purloined,
Comport four colors, joined or placed apart.
Or wander ye the diamond place to place:
Nary a spot it cannot show its face.
There be much sport assembling shapes de-trayed
And doubling iamonds orders 1 through 8.
Slip this small beastie to a friend off-guard —
Its harmless look belies it can be hard.
All-acrylic, in 5½" tray.



Octiamond Ring™

This lavish convolution—sixty-six
All-different shapes of triangles conjoined—
Extends the lamond Ring to level eight
Around one hole symmetrically defined.
All in their shapely hex frame finely fit,
'Tis challenge 'nough to please the nimblest wit.
In 14" tray, all-acrylic, sized to lamond Ring, with easel. Bonus: Center hole contains hexiamonds.



Rainbow Rombix®

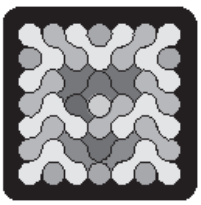
Prismatic color splendor merely hints
At its chromatic purity of form:
In lush complexity of shape and hue
The rhombuses fill four-and-twenty sides.
The thirty-six uniquely shapèd tiles
Entail six measured rhombs, in ones and twos.
Each color group of different shapes comprised
Nevertheless an equal size o'erlays.
Discern more ways to tile the circle's field,
And smaller jewels a partial set will yield.
Review the classic Rombix-16 sides—
And all its deep and subtle properties
Ere venturing to find the treasures told
Within this Rainbow's iridescent fold.
15" black tray, with easel.

Rombix®

Enchanted circle with full sixteen sides
Enfolds the sixteen finely angled tiles.
Their rhombic pairings pave eight ladders' span
In perfect mathematic harmony.
Four colors shift to mime the seasons' pass,
But scramble ye the hues a hundred ways
Or shape the neatest ovals small and large,
Or marvel at their nested-oval charms.
A game for two escapes the tray's confines
While puzzlers ponder parallel-put lines.
All-acrylic, in 8½" tray.

Rombix® Jr.

Wax ye rhapsodic o'er this offspring's charms
As symmetries of shape and hue appear.
Or can ye bring such images to view
As fish and cats and birds and boxes, too?
With ladders, ovals, rhombs it mimes its père.
This darling plays to please all ages fair.
All-acrylic, 4 colors in 5½" tray.

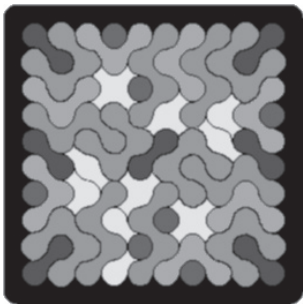


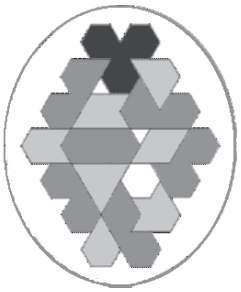
Roundominoes®

A favorite plaything for both young and old --
The curvy shapes of 1-2-3 "rounds" link,
With 1-2-3 bent "bridges" woven through.
These squiggly pieces intricately strew
Their brilliant hues in clever combinations.
Three games for two, plus puzzles that build patience.
All-acrylic, 7 colors in 5½" tray.

Super Roundominoes®

A cornucopia of puzzle joys
And six strategic games for two-to-six!
The ten curvaceous shapes of play beguile
With combinations 1-2-3-4 "rounds"
And "bridges" snuggled close, 10-square arrayed.
Six lively hues, dramatically displayed.
All-acrylic, in 8½" tray.





StarHex-II™

A star-strewn field gives rise to glistening tiles
Each different built of one to four profiles.
They cling or separate in their oval frame
Or, on their grid, engage a latticed game.
Four colors code the hexes that reside
As symmetries the star-crossed patterns ride.
All-acrylic, 6x7½" tray, game grid.

Stelo™

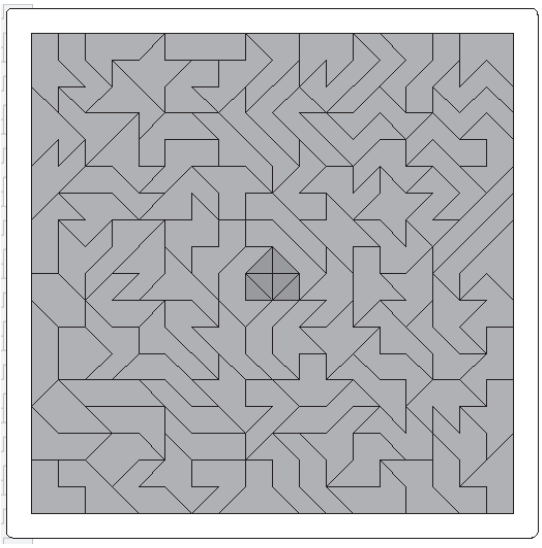
Behold a star, so radiant it can boast
Twelve tiles, each of three triangles composed
In every joining, regular and wide.
What sparkling patterns midst this beauty hide.
A solar cloud of figures glowing bright
Find orbiting on Jacques Ferroul's fine site.
All-acrylic, in 5" tray.

For what is mathematics but the solving of puzzles? – *Solomon W. Golomb*

Tan Tricks™

A classic polyform, of half-squares built,
From one to six in sweet confusion tilt.
Its toughness cannot be denied—it's not
A dove-tailed repartee of Nuke and Spot.
The tiles be cursèd with such angled parts
That they resist assembly from the start.
Six sizes in three squares deploy to fit;
But mix and match them if ye trust thy wit.

Tan Tricks 1 (19 pieces), 5½" tray.
Tan Tricks 2 (30 piees), 8½" tray.
Tan Tricks 3 (108 pieces), 16" tray.



Tetrapentos™

Just seven little tiles, groups 4 and 5
That from our Mini-Iamond Ring derive,
With five wee triangles that interlace,
Will change a hundredfold its twinkling face.
Much lively symmetry invites thy skill:
Solve figures shown or plan thine as ye will.
All acrylic, in 4" round tray.
*See also **Pocket Tetrapentos**, page 37.*

Guest performers

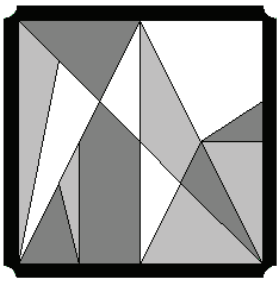
Polyarcs™

First, squares be split by quarter arcs to yield
A convex and a concave half. Then wield
Those halves in twos and threes in every way.
'Tis Henri Picciotto's complex play!
All-acrylic, 29 polyarcs in 5x7" tray.

Poly-Spidrons™

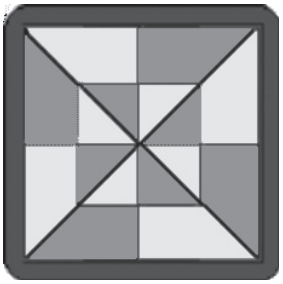
Most convoluted of our puzzle line,
These fractal-trailing triangles entwine
In spiral wizardry, by Jacques Ferroul,
With Daniel Erdely's spidrons as a tool.
For only fearless, well-skilled solvers, lo,
Lest every vortex be thy undertow.
All-acrylic, 12" kite-shaped tray; also embeds Stelo's 12 straight pieces.

Tilings and Designs



Archimedes' Square™ (the Stomachion)

What marvel of antiquity be this,
This fabled square of fourteen parts comprised?
The legends credit Archimedes' wile
With clever cuts that render every tile
An integer, in twelve by twelve reclined.
Solve 18 figures from that early age
And many new designs for youth and sage.
Behold the oldest puzzle ever told,
Our heritage of mind millennia old.
One hundred scholars did decode, with zest,
The Archimedes much-prized *Palimpsest*,
A scroll long lost, inscribed by ancient hands,
A rarest find from Greek and Latin lands.
All-acrylic, in 7" tray.



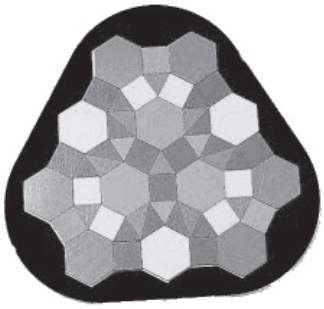
Chasing Squares™

Who'd think a mere eight triangles and "sheds"
Turn filled and empty spaces on their heads?
If ye can form a square whose sides expose
All edges to your touch, that's how it goes.
They link and overlap, in sizes strange.
Can you two dozen all at once arrange?
Or fill the tray with patterns beauteous?
Breathtaking symmetries and figures. Yes!
All-acrylic, in 7" tray.

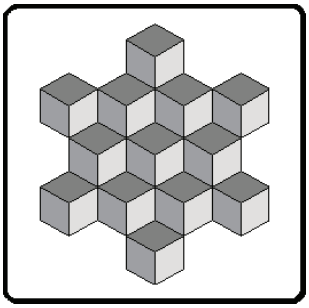
Combinatorix™

The mysteries of Nature are revealed
Through polygons in classic ratios limned:
The Triangle of equal length of sides,
The Square of perfect form and angles right,
The Hexagon all regular and fine,
The Right Triangle called isosceles.
From seeming unity these shapes combine
And grow to infinite complexity.
Thus do 512 wood tiles contrive
To give a solid shape to thought and art
As tiling patterns swirl forth from the chaos
Into enchanting tessellated views.
Full seventeen good games upon three grids
Tempt two to twenty-seven agile wits.
This veritable game room in a chest
Embraces all of Nature's truest and best.
13x26"conpartmented wood case, 3 books,
3 game mats, template. Rope carry handle.

Combinatorix™ Jr.



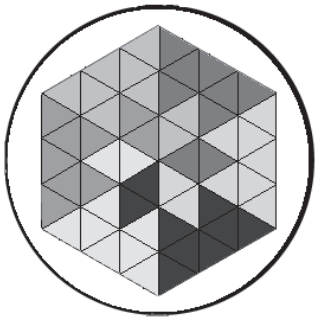
Behold a symphony of polygons,
These be of wood, that four warm colors dons.
Explore the filling of the plane aright
As symmetries of shade and shape delight
Had ye more pieces, ye could pave the world
And far beyond, where galaxies are curled.
53 wood tiles, rounded 11″ acrylic tray, easel.



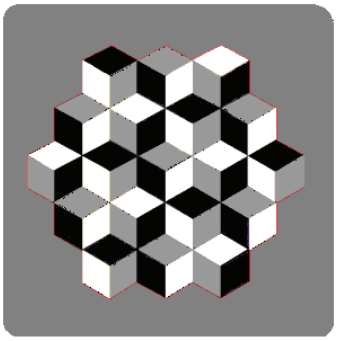
Cubits™

The sixteen diamond-sporting tiles comprise
The fragments—one and two and three—that build
The visual illusions of full cubes
And sheer impossibilities of form obtain.
Their lovely symmetries enchant the sight
Whilst games the clever thinker's wit delight.
All-acrylic, hand-inlaid in 8½″ white tray.

Diamond Rainbow™



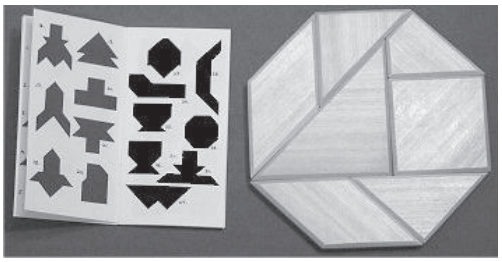
Two dozen diamonds dazzle with six colors bright
That catch the rainbow's regal range of light.
Match tiles not merely by their mutual hue
But twin them with their spectrum neighbors, too.
Create mosaic marvels and behold
Each one a find like to a pot of gold.
In 8½″ round *white tray, hand-inlaid all-acrylic.*



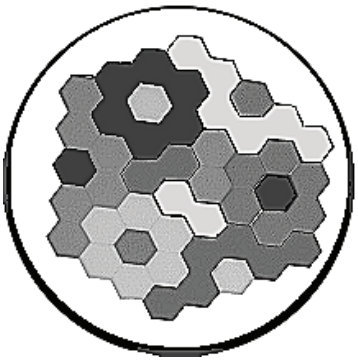
Diamond Star™

Perceive thee twelve stars nestled in the field
Of diamonds in three colors cubically revealed?
The optical illusions please the eye
As galaxies of figures orbit by.
With two or three good mates, play Rhombomania:
Move pawns to meet and bear off— that will gain ya.
All-acrylic, 23 hand-inlaid tiles, 11″ red tray, with easel.

Grand Tans®



Now 'tis a veritable treasure trove
Of puzzlements purloined from ancient lore.
Just seven segments, skillfully designed
To form a hundred shapes that taunt the mind
With most elusive answers. Yet, once caught,
Their beauty satisfies both sight and thought.
Handcrafted hardwood, in felt pouch. Sizes may vary.

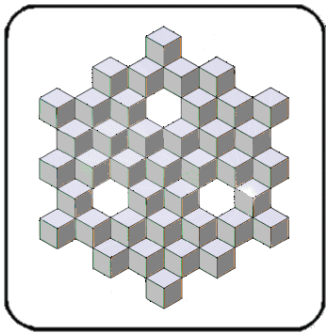


Hexdominoes®

More lively than the classic domino,
These hexagons in pairs six colors show.
What marvelous figures and designs appear!
Matched or unmatched, each goal reached spreads much cheer.
Find ye artistic visions without end,
And play strategic games with a clever friend.
All-acrylic, 27 hand-inlaid tiles in 11" round tray, with easel.

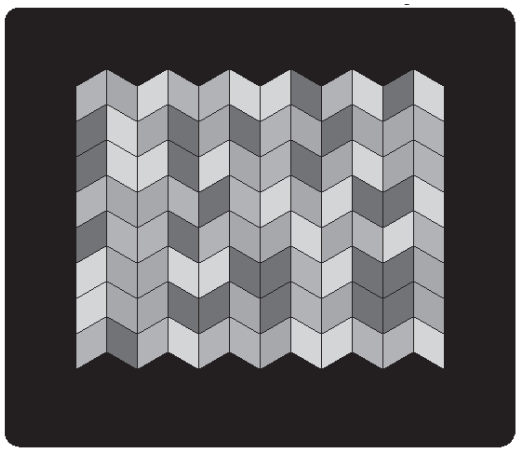
Intarsia™

Such beauty with a mere two parts appears,
Behold, Infinity's vast threshold nears.
The symmetries of every type delight
Most elegantly framed in black and white.
Each pattern on both top and bottom shows.
As several games ye play, the pleasure grows.
All-acrylic, 32 hand-inlaid tiles in 11" red tray, with easel.



Q-Bix™

This optical illusion field of blocks,
In sleek metallic colors really rocks.
The tiles that wriggle like a cubic storm
One, two and triple hexes inlaid form.
A repertoire of figures builds your skill
As each solution brings a visual thrill.
All-acrylic, 15 hand-inlaid tiles, in 11" tray, with easel.

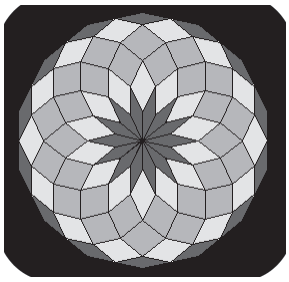


Quadrants™

A plane of iridescence zigzagged gleams
With six-and-ninety diamonds. Therein hide
Seventy-seven groups of two-by-two --
The quadrants -- each a different kind.
Thence four-and-twenty arrow tiles emerge
Each differently 4-color hued. Explore
Their everchanging form, their charm of "four".
All-acrylic, hand-inlaid in 15x18" black frame.
Easel or wall model.

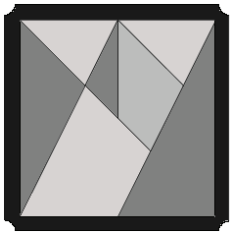
Shadow Play™

Nine shapely tiles make figurines and scenes
As each imagination conjures forth.
Create the form, then title it as suits
Its silhouette: A shadow play of thought.
Acrylic tiles cut with a beam of light
Are pouched for portage. Ask for black or white.



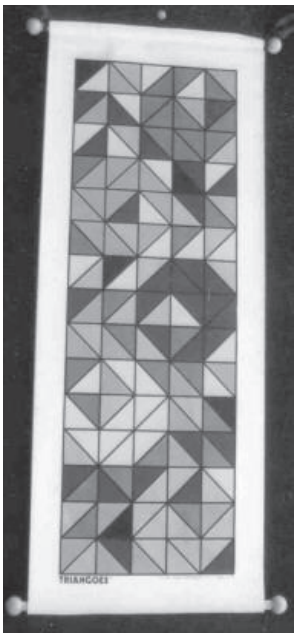
RhombStar-7™

Beguile thine eye with symmetries in Sevens,
A strangely shaped proportion—odds, not evens.
Three glowing jewel hues the tray doth hold.
Or plot ye tilings with illusions bold
And periodic tilings by the score,
Then ponder truths that intellects adore.
All-acrylic in 8½" black tray.
See also ***Pocket Rhombs*** *on page 37.*



Tangramion™

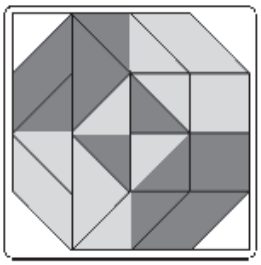
This child of tangrams and Stomachion
A simple seven angled tiles brings on.
'Tis not an easy task to fill the frame
Tenfold, and hundred other figures tame.
Convex, concave, elusive symmetries,
Such beauty all your thinking prowess frees.
All-acrylic, in 7" tray.



Triangoes™

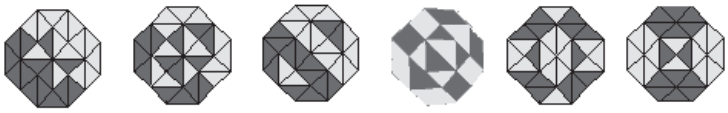
Five colors vie in pairs triangular
To form half-squares, and parallelograms,
And squares, and triangles of double size.
These five clean shapes do fourscore pieces make
That join in marvelous patterns to astound,
And serve as wherewithal for fifteen games.
Herculean task -- to pave with all, in whole,
The regal painting/gameboard/puzzle scroll.
Like leather be the scroll, on rods of wood.
For one to six, diversions great and good.

Two-sided gameboard (one the painting, the other a grid). 80 handfitted pieces in gleaming acrylic. Vinyl board/scroll is 13x36" with cord to hang.

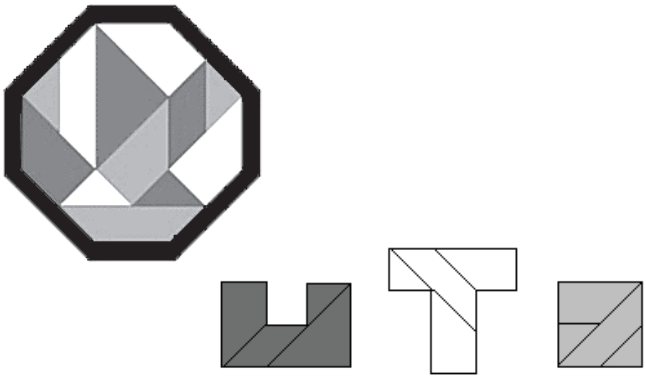


Triangoes™ Jr.

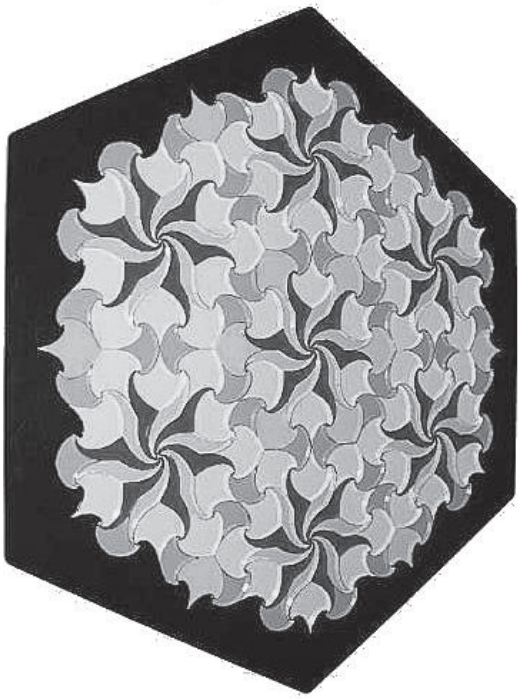
A 17-member subset in two hues
Achieves nigh half the puzzles of the above.
Octagonal frame, hand-fitted tiles to suit,
And simple grid for four sweet games, to boot.
Your artist's soul bestirs itself alive
As brilliant shapely symmetries arrive.
All-acrylic, in 8½" tray.



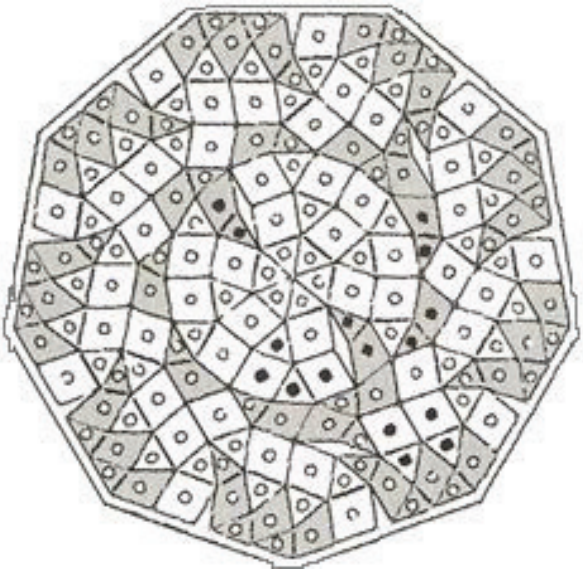
Tiny Tans™ Trio-in-a-Tray™
A quarter century ago were hatched
Three little four-piece puzzles, nicely matched.
Align them, one or all, in symmetries.
Collect all combinations—it's a breeze.
To celebrate their vintage, this array
Enfolds the trio cozy in one tray.
The T, the U, the Square own colors bear,
To mix or separate, a feat most fair.
All-acrylic, in 7" octagonal tray.



Guest performers



Tulips™ *by Daniel Erdely*
Majestic mingling of the spidrons' curl
In vortex-spinning whirlpools they unfurl
As in their field the tulips nestle and sway,
Midst rippling rows of colors cheer the day.
A gorgeous piece of art to grace a wall,
Its bolted covered showcase holds them all.
All-acrylic, 210 tiles in 18½" framed tray
ready to hang on wall or display on easel.

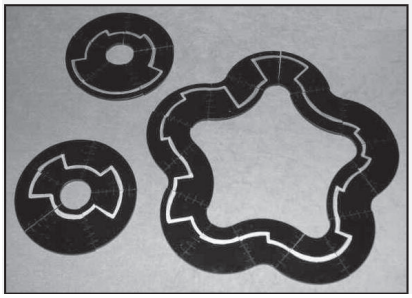


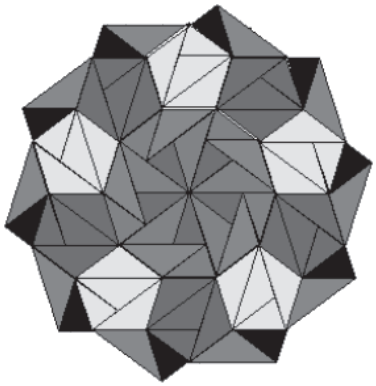
Chaos Tiles™ — *limited edition,very few left*
Four shapes—convex, concave, in mirror pairs—
Full 90 bone-like bakelite tiles conspire
To render chaos into order fair.
Their domino-like pips in colors five
Enrich complexity. For unions strive
That clump the hues and fill all spaces tight.
This nine-fold symmetry set, smooth and neat,
Is puzzle maven Ed Pegg Jr.'s feat.
In gold-embossed leatherette nesting case.

The Pentagon Universe

Arc Angles™

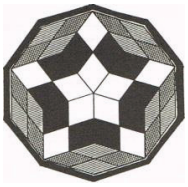
Full 25 arced wedges five rings make
Where all their paths their edge-matched juncture take.
Each tile the golden ratio honors well
As five-fold symmetry its angles tell.
Each edge five notches levels, left and right.
To form one closed loop gives a graceful sight.
All-acrylic, hand-painted 1½" tiles in drawstring pouch.

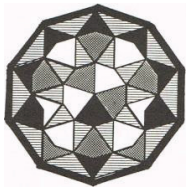




Collidescape™

Two shapes of
The fundamental building blocks portray
Wherewith all regions of a pentagon's
Plane tiling modeled and definèd be—
Stars, pentagons and decagons, and rhombs,
And trapezoids and partial stars to boot
As centuries ago old Kepler showed.
Crisp colors interspersed to mark each field
Kaleidoscopic vistas lush will yield.
25-pod starter set (a decagon display)
53-pod yeoman set (a pentagon array)





Deka-Star™
Deka-Mosaik™
Kite-Mosaik™

This triad tiles their decagon-shaped trays
With four rich hues and pentagon-based shapes.
In Deka-Star find thick and narrow rhombs.
Deka-Mosaik mingles pentagons
With golden triangles—the wide and tall.
And Kite-Mosaik flies the triangle pair
Midst flocks of Penrose kites and darts with flair.
Each set begets a plethora of plans.
Their varied colorations thrill the heart
And turn them into interactive art.
All-acrylic, in 7" trays.

La Ora Stelo™

The pentagon a marvelous model be
Anent geometry's perversity.
Two golden triangles will groupings make
In pairs and trios, thirty-two at stake.
The wondrous figures in which they deploy
Are mathematical and visual joy.
The father of this polyform most cool
Is creative genius Frenchman Jacques Ferroul.

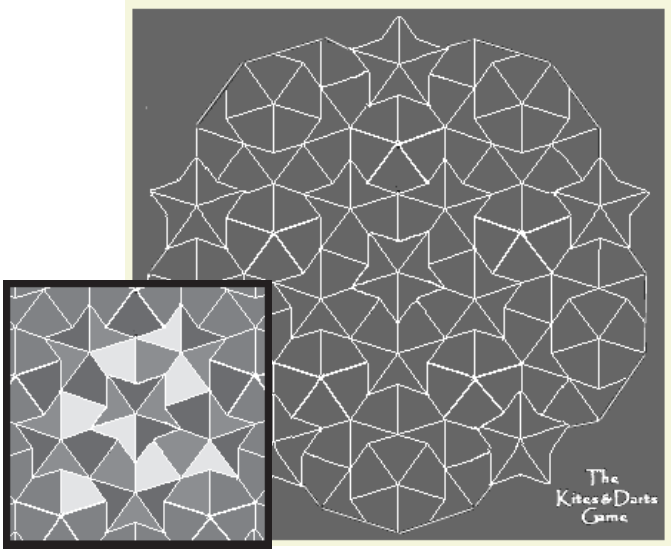
All-acrylic, 12"pentagonal tray is inscribed with 9 concentric pentagons. Easel included.

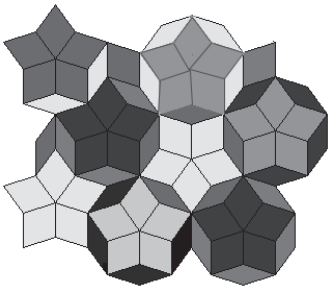
Kites & Darts

The kites and darts two quadrilaterals be
That "tile the plane" with occasional symmetry
But spread forever -- non-periodically.
Sir Roger Penrose tamed their vexing ways
With mathematics' disciplined assays:
The golden ratio rules their share of space.
This patented duo comes in "pods" of eight:
5 kites, 3 darts. A score of pods is best
To start; a hundred nears, not fills, the quest.

- *All-acrylic, 20-pod starter set, 1¼" long sides.*
- *40-pod master set with 16"vinyl game mat.*

*See also **Pocket Star** on page 37.*





Penrose Diamonds

Two shapes of diamonds, *thick* and *thin* by name,
Will tile the plane in non-repeating parts
When joined by laws Sir Roger Penrose writ.
Their golden ratio size and frequency
Form alter states, periodic and serene,
Wherein the beauteous panoramas lie
With shapely color groups to please the eye.

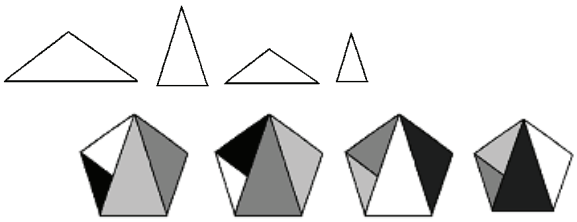
- *All-acrylic, 20-pod starter set, 1¼" long sides.*
- *50-pod development set.*

Pentarose™

Most gorgeous of our stained-glass-like designs
Has nifty shapes like leaves and mice and planes.
Derived from Roger Penrose's prototiles
For non-repeating tilings—famous quest—
They nestle radiantly within their circle tray,
Four beauteous hues besparking every lay.
Much math-ish truth throughout their joinings lies
And symmetries to eager search entice.
All-acrylic, in 11" tray, with easel.

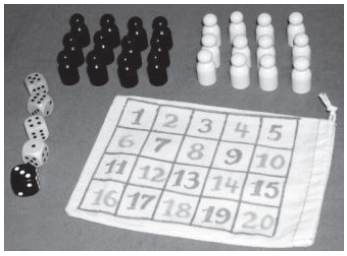
Puzzling Pentagon™

A bit of math arcana -- fear it not:
Four triangles, whose ratios golden be,
Form pentagons. Quartets of these
A larger pentagon with features form
That 'fuddle the most fleet-wit champion.
Or build ye stars and symmetries -- all great.
In pouch they stash, thy challenge to await.
All-acrylic, 4 colors, 4 sizes, largest piece 2" sides.

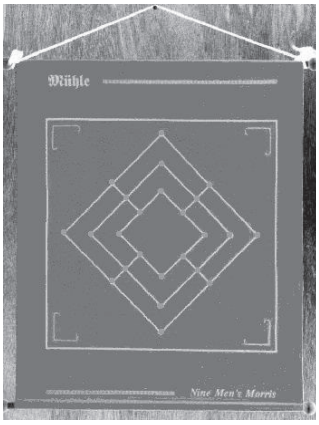


Abstract Strategy Games

A+D+D+D™



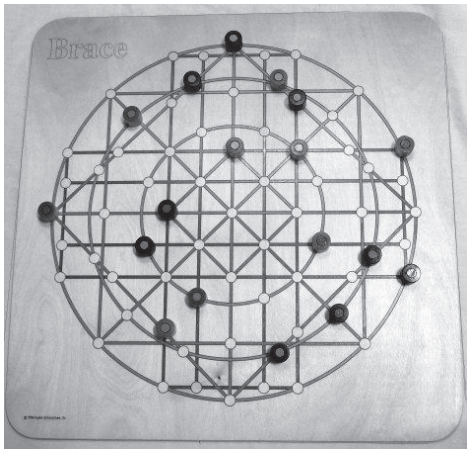
Now here's a tidy little game for two
Where luck plus calculations win for you.
Roll ye five dice, the black one rules in full,
Then from the red a pleasing sum ye pull
That puts a pawn on any niche you wish
The while for rows of 3, 4, 5 you fish.
Think well where footholds on the grid you plant
As intersects some extra scores will grant.
24 wood pawns, handpainted fabric game pouch.



Brace™ and deluxe Brace™

The classic Mühle, Mill or Morris—known
Since pharaohs flourished—fills one side
With 3 concentric squares where threesomes bide.
With nine a-piece to start, hold on with skill.
Reduce by one your foe per row you fill.
A deeper strategy to wield—it's great—
Is: trap opponent in a moveless state.

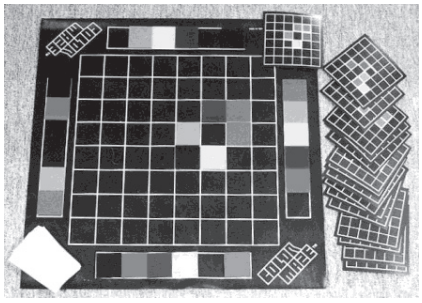
Turn now the other side, to Brace,
Whereon the intricately woven lines
Present a net of multiplex intrigue.
Three sets of pathways, color-marked, form trails
Whereon the players' pieces well-matched slide
To build embraces 'round co-player's disks.
Each "brace" earns points upon the scoring cube
Which flips and rolls to match each player's move.
Ye win if first ye reach the triple groove.
Four other clever games and puzzles, too,
Made Brace an instant hit with us. Now you!

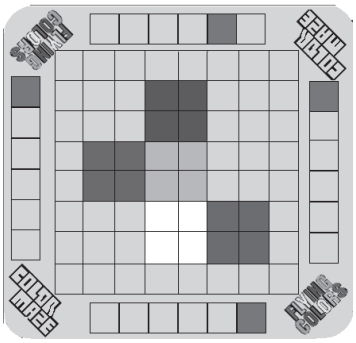


- *24" canvas banner on rods, with cord to hang, acrylic checkers.*
- *Handpainted, 24" engraved wood board, wood cylinders.*

Both styles in custom-sewn fabric bags.

Colormaze™/Flying Colors™

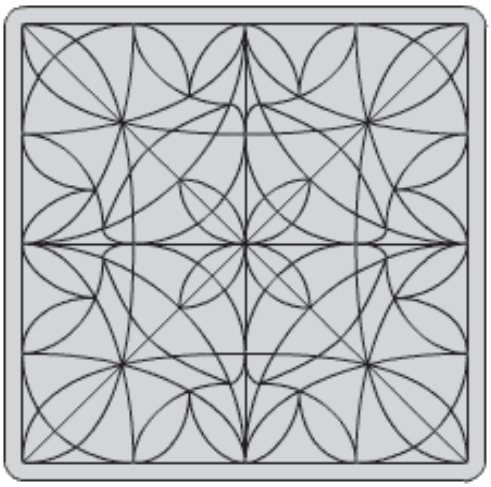




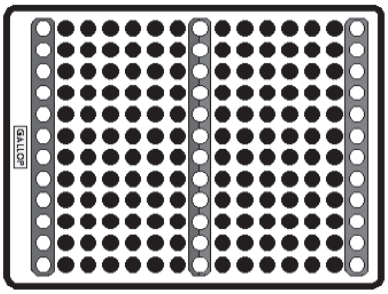
Now yet another maze, of color, this,
Confronts the clever strategist who'll form,
Assembled in the field of patterns strange,
The very six the secret card describes.
Or, place the tiles so nary row contains
Two of a color, then score points perchance.
And more than 100 challenges cavort
In 6-color splendor for the puzzler's sport.
A thousand new games, Flying Colors, marks
Our silver year with Stephen Sniderman's sparks.

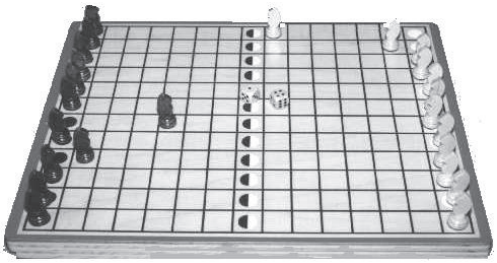
- *18" black leather-like vinyl game mat, 41 tiles, cards, felt pouch in box*
- *Deluxe 18" handpainted wood board, velvet carry bag*

End Point™



Art Blumberg's brainchild, like his Power of Two,
Presents a graceful grid and novel themes.
A complex net of arcs entwined and swirled
Lay out the movement paths for games of skill
As pawns in circling strategies assail
The end points of co-player's starting side.
The pawns with every move flip form and powers
From slides to jumps, from jumps to slides. Beware:
For slides free space, for jumps a mate you need.
Two goals: the end points trap or exit there.
This artful board, in wood engravèd wrought
Bestirs two players' wits and puzzlers' thought.
Regal 24" wood board, 30 pawns, felt slipcase.

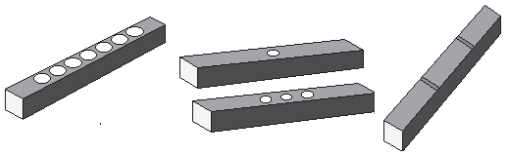




Gallop™

Twelve chargers fleet-foot cross the dotted board
To bring just five within the far-off row.
Two dice direct two pieces on each turn,
The mid-board row mandates an exact stop.
Flee captures but entrap opponent's men.
More games, and much to please the puzzler's yen.
A novel switch—stick dice guide you in "Sidle",
A side-step game that won't let minds go idle.

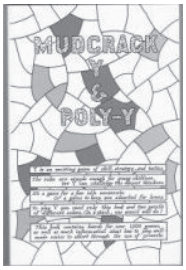
- *Acrylic board, 13x17", wood pawns, felt carry-bag.*
- *Handcrafted 14x17" wood board, wood horses, velvet bag.*



Fox Blox™

Four cubes, as many players as you get,
One alphabet, each roll just four you net;
One line compose, first letters use those four.
To tease alliterations, use ‘em more.
Thy playmates roll anew and rhyme with yours.
Much silly versed imagination lures.
Whether like Seuss or Lewis Carroll sounded,
All players win midst gales of laughs unbounded.
1” jumbo cubes, hand-applied letters, drawstring bag.

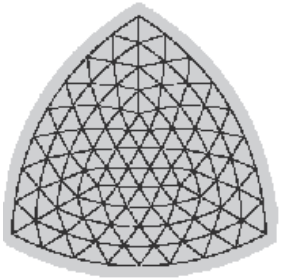
Mudcrack Y and Poly Y
The book of gameboards
ye color in,
Prlogues to Y and *Star
to help you win.

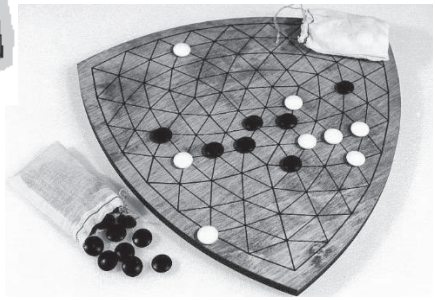


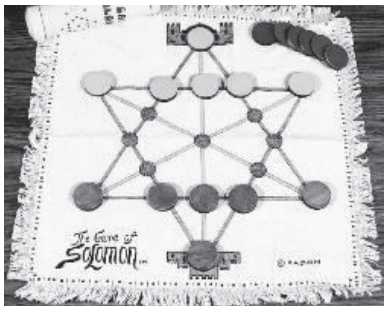
The Game of Y™

This glorious game won first-place honor's rank.
Place stones to link all 3 sides solidly
The while obstruct opponent's linking reach.
A deep and subtle game like unto Go.
Six other games play on the self-same grid,
And solitaires may yield to puzzlers' bid.

- *Deluxe set: Curved 15" footed wood board, velvet bag.*
- *Travel set: Leather-like mat in drawstring felt pouch.*





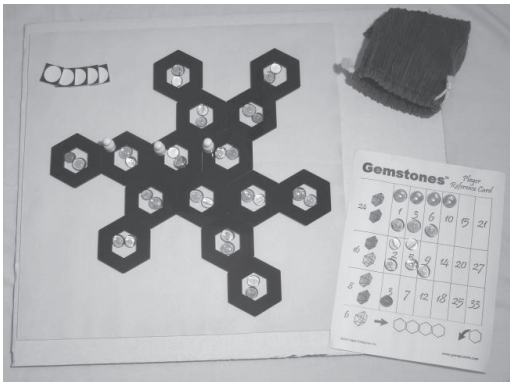


Game of Solomon™

As rumour tells, Sol made this game to keep
His harem playing 'stead of quarreling!
The handsome emblem of his reign of peace,
On fringèd fabric painted, serves as grid
For several games of thoughtful skill for two
And plentitudes for solo ponderings.
If truth be told, the sage's creative partner
Is famous scrivener-scholar Martin Gardner.
Scroll of rules, 18 wood checkers, handpainted 15"game cloth.

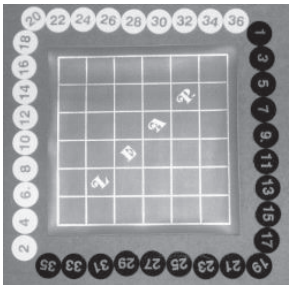
Gemstones™

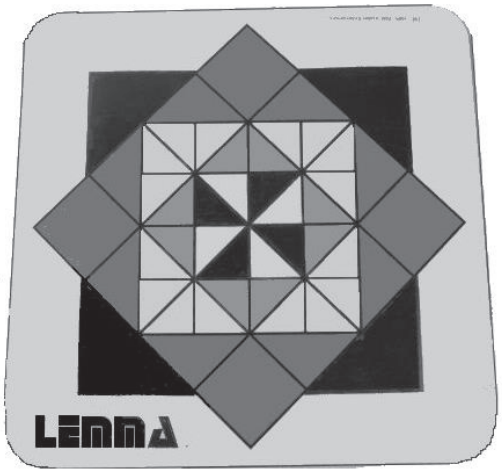
Such shiny treasures to collect’s the game,
Their highest values’ tally is the aim.
In 15 “mines” the gems are strewn as stake.
Three different colors on your turn you take
From three mines linearly placed. Beware
The guardians and replace thy loot with care
As from the bag by random chance you fare.
Then designate who harvests next. It’s not
In order. Favor favorites as you plot
And move some mines to take another spot.
Acrylic hex rings, 102 glass marbles in bag, 5 score sheets, velour storage pouches, ivory leather-like mat.



Leap®

A 6x6 grid gives five leaping games
For two or four to work their cunning skills,
While one can fence with puzzles by the score—
With knights and checkers, magic squares and more.
8½"vinyl mat, 36 handpainted numbered acrylic checkers.
Also part of Six by Six game set.



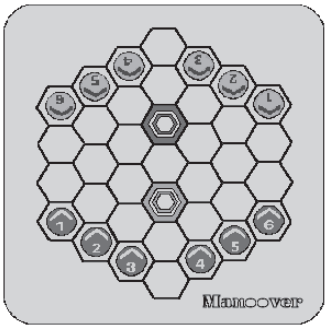


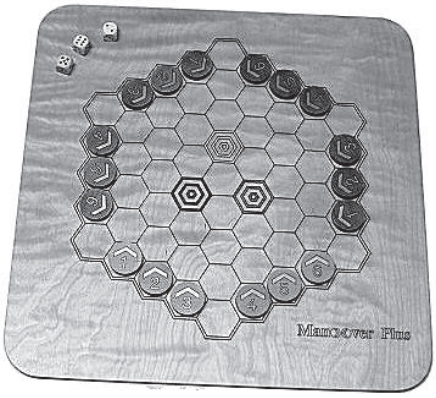
Lemma™

A most revolutionary game idea
Is manifest on this dramatic board.
The players themselves decree the rules of play!
But once decreed, let no one waive those rules.
Much spirited debate may well ensue
As structures of the game evolve and grow.
Thrice hundred problems, too—the puzzler's plight—
Reward with topological delight.
22" engraved, handpainted wood gameboard,
48 disks, felt slipcase.

Manoover™

Luck plays a role, but smarts will have the day...
Six arrowed pieces move or turn each way,
Then slide to reach their portal, push and shove.
Here's strategy plus foresight that you'll love.
Chase thy co-player's disks into thy lair,
Or even push them off the board—it's fair.
Engraved, handpainted 11½" wood board,
felt slipcase, 3 wood dice.





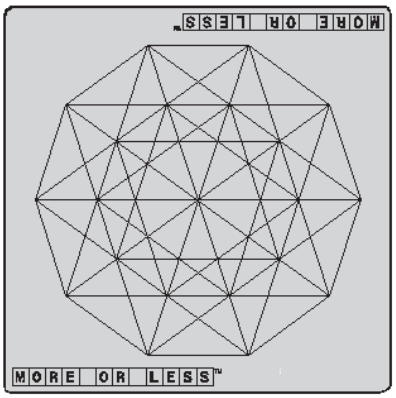
Manoover™ Plus

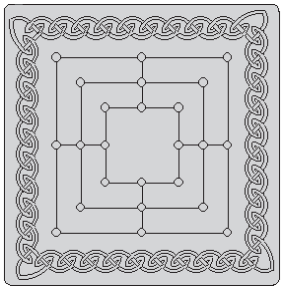
For triple fun and joy, play Manoover Plus.
Flip board for 2 or 3... ingenious!
Roll octahedral dice, collect yon disks:
To move among 3 colors carries risks.
The pay-off's bigger, too, with more to chase,
And trickier tracks to tangle as you race.
17" engraved, handpainted wood board,
felt slipcase, 8 numbered disks each of 3 colors.

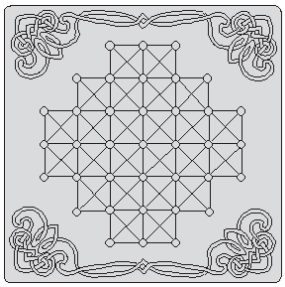
In all the Galaxy, they had found nothing more precious than Mind. ...
-- Arthur C. Clarke

More or Less™

A goal so calculatedly complex
That ye must own far more or ever less
Than thine opponent for a win to count.
Thy jumps convert the pieces, on each bound;
Possession changes hands, both theirs and thine:
Divest, acquire—plan thy tactics' line.
Two games for two, two solitaires for one.
This artful trade-off game is royal fun.
17"engraved wood board, 25 reversible felted wood pawns, felt slipcase.





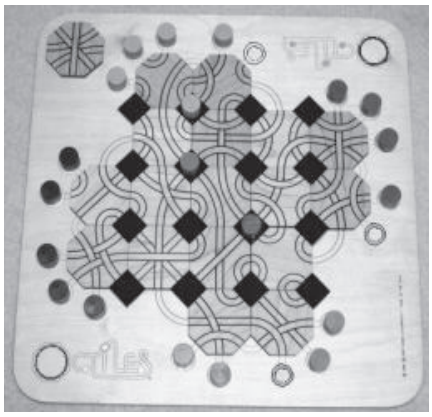


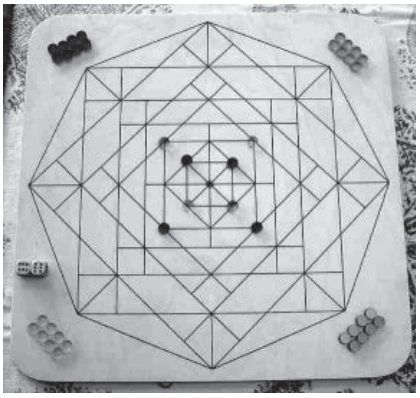
Nine Men's Morris/Fox and Geese

Two classic games engraved upon a board
That flips to show the one and then the yon.
Its ornamented scrollwork is a pleasant sight.
With fine glass stones ye play in black and white.
Now on the Fox & Geese grid, if you will,
Work Solitaire, a jumping task of skill.
11½" engraved wood board, felt slipcase, 18 black and 18 white go stones.

Octiles®

The field of paths is cunningly prepared
To let 5 runners speed across the maze
Or block the foe with loops to win the race.
Engage from one to four participants,
With quality and style to gratify.
'Tis more than games, forsooth a work of art,
And thirty puzzles, intricate and smart.
18" engraved, handpainted wood board, fabric carrybag, 3 game variations, 18 three-inch octagonal tiles.

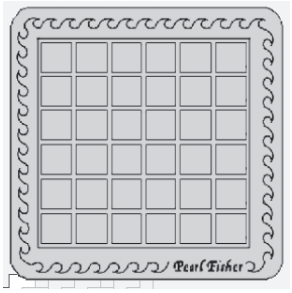


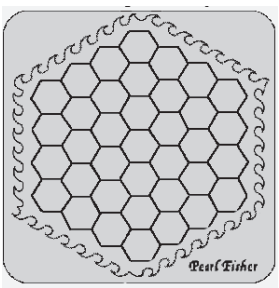


Over-Pass™

A graceful grid engraved upon the board
Sets into motion strategies that ford
Thy ten disks to the edge atop thine host,
Or trade two pieces parallel disposed,
Or exit from yon edge, or slide a space,
Each move told by the dice's generous face.
The jewel'd stones do glow upon the plane
In three smart games for two to four to gain.
17" wood board, 2 wood dice, 40 acrylic disks.

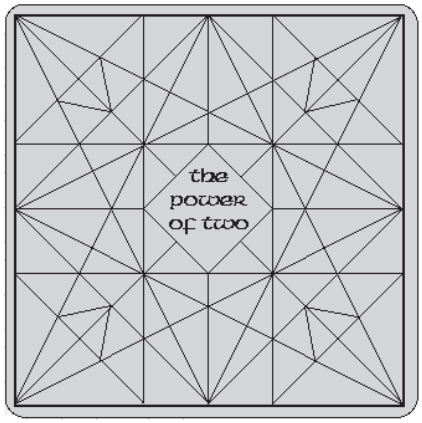
Pearl Fisher™





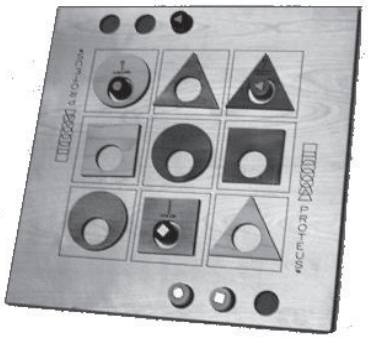
How well can you remember hidden hues?
Eight pearls beneath eight shells—which shell to choose
To move a-right the naked pearl to hide?
Expose a match, gain one point for your side;
Guess wrong, no score, just move the pearl revealed
To an elusive space upon the field.
The winner is the one with largest hoard
When no more new pearls can invade the board.
11½" engraved wood boards, 9 shells, 24 pearls, felt slipcase.

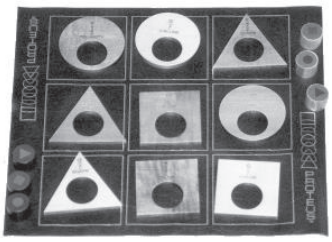
The Power of Two™



This marvelous game did spring full-formed and fine
Forth from Sir Arthur of Ye Gamery's mind.
'Tis played by two with 16 markers each
To be deployed upon the board's array
Of angularly intersecting lines.
Where to be joined a third doth issue forth,
The agèd ones depart the realm apace,
In peace await reentry and new space.
Three actions may play out upon a turn:
To move, perchance to add the newborn spawn
To oust the old that yet may enter new.
The winner's goal: to land all on the field
Or the opponent, moveless left, must yield.
17" engraved wood board, felt slipcase.

Proteus®

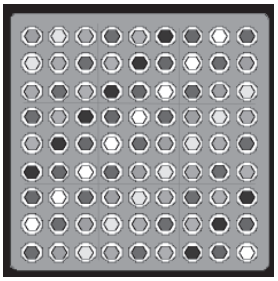




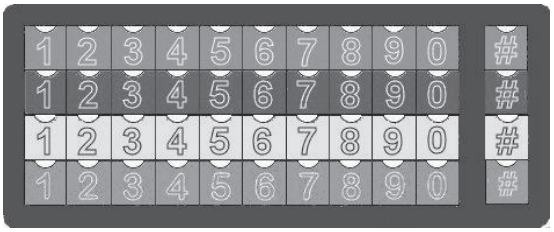
This game of meta-logic played by two
Lets players change its triple set of rules.
So switch yon tiles or move the men that race
To reach to goal writ 'pon the tiles' own face.

- *16"engraved wood board in velvet satchel, maple tiles.*
- *Vinyl game mat in drawstring pouch.*

Pseudo-Coup™



On 9x9 expanse of circled grid
Nine 3x3 plots line up, nearly hid.
Now fill yon eighty-one spots with 9 hues
That nary two the same come into view.
Then add some games of coloring exclusion
For two contenders' tactical profusion.
All-acrylic, 10" board, 2 wood pawns, suction tool.



RunnuRound™

Ye digits 1 through 0, or 0 through 9
In four fine color rows they intertwine.
Four players search for runs of three if found,
Whether left or right or wrapped around.
Place ye thy wager, bonus points to get,
Or slip thy numbers smartly, scores to vet.
Four wild cards make more complex rows to hoe
Where double vision aids what players know.
6x16″ tray, 4 bamboo screens to hide tiles.
All-acrylic or wood handpainted number tiles.

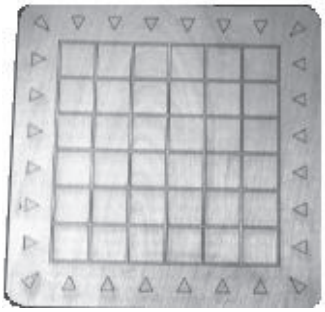
Six Disks™

Whether by art or merry bears adorned,
Six numbered disks will make—ye be warned—
True magic sums on polyominoes...
All same sums or all different, so it goes.
A 3D tic-tac-toe game, built for two,
Moves disks apace till one wins. You, or you?
And sundry other challenges can jump or slide
And move yon disks atop or side to side.
All-acrylic, 4″ handpainted disks, drawstring pouch.
Two styles: abstract symmetries or teddybears.

Six-by-Six™ (and Leap®)

Our greatest repertoire of games for two!
A simple 6x6 grid waits for you,
Engraved, superbly finished board and disks,
And 32 games, fun without the risks.
Plus hundreds of fine solitaires portend
Thy sharp mind into overdrive to send.

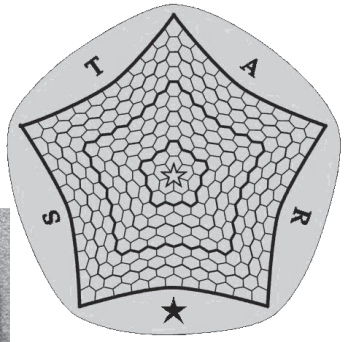
- *Engraved 14″ wood board, 36 numbered wood disks, 24 special wood disks, 3 books, fabric carry bag.*
- *Leap version with 8x8″ vinyl play mat, 2 books, 36 numbered reversible acrylic disks (see page 30).*

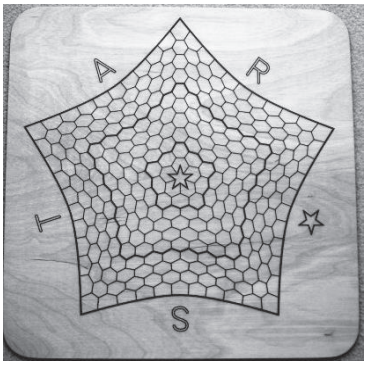


*Star™

Ye have not seen a Star quite like this sight:
A gameboard of such depth and breadth and height
Of subtle strategies that speed the mind
As complex connectivities unwind.
Let stones appear, two players placing true,
That edges join when trails and paths pass through
Blockades and obstacles each sets in place
To slow co-player's plans to win the race.
If ye enjoy and cherish Game of Y,
Then Star, evolved therefrom, will make thee fly.

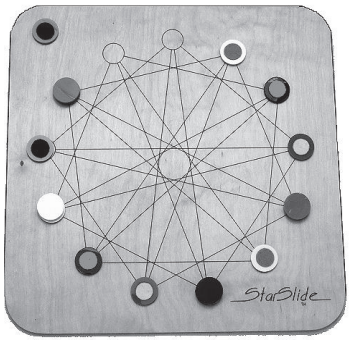
- *Deluxe 17″ engraved wood board*
- *17″ roll-up leather-like vinyl mat*

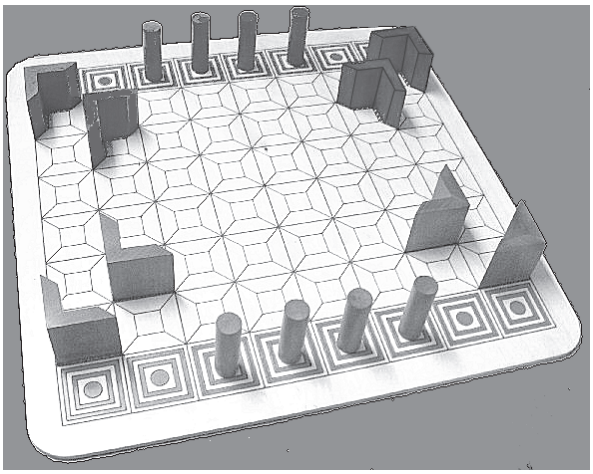




StarSlide™

Slide disks upon the pathways of the star.
They link the thirteen circles near and far.
A disk moved into center blocks long slides
And any vacant circle then bestrides.
A last-moved disk stays put one turn in thrall;
A player blocked from moving loses all.
Engraved 14" wood board, 24 acrylic disks.



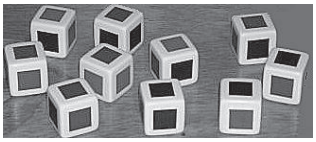


Teleporters™

A hyperspace-traversing game with style,
The ports conjoined a teleport compile
That speeds your traveler's passage to beyond,
Much lively tanglings that divide and bond
And interactions with the other team.
Handcrafted marvel, it's a gamer's dream.
Engraved, handpainted 18" wood board,
4 "ports" and 4 poles for each player, felt slipcase.

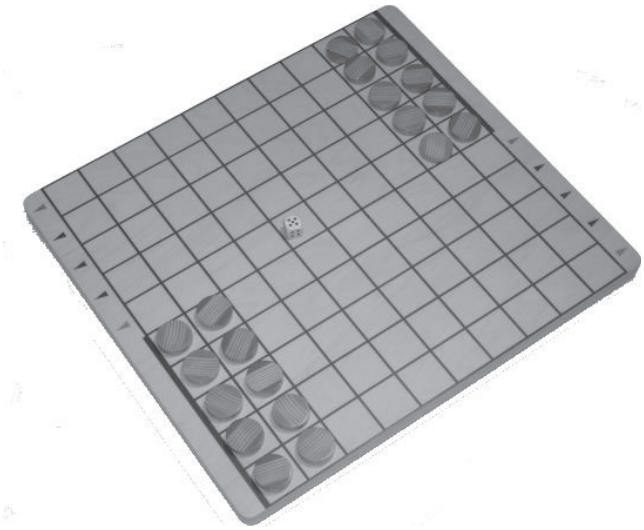
Throw a Fit™

Ten different jumbo dice three colors wear
And tease the players to collect all ten:
In just five rolls a fitting match to throw.
Or stack yon cubes, a larger cube to form,
Or colormatch their faces, to and fro.
In travel pouch they're always fit to go.



Transpose™

A 10x10 grid in the classic style
Enables eight strong gambits played by two:
Depart 8 counters first to win the fray,
Or gain the upper hand atop each stack;
Or gather all thy counters into one —
A tower, or chain, or flock — or cross the board;
Or wall up thy opponent inch by inch,
For Amazons' fleet arrows it's a cinch;
Or circumambulate the board's fair rim.
Beware the switch that throw thee off thy track,
Or plan another switch to bring thee back.
Enjoy, whilst planning out your ploys and risks,
The finest craftsmanship of board and disks.

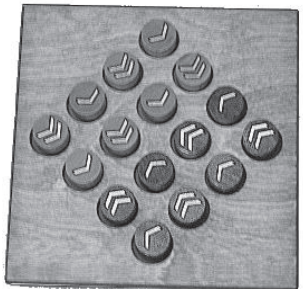


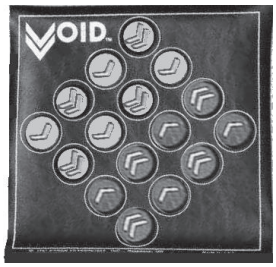
- *Deluxe 16x18" footed wood board, fabric cover, 24 wood disks.*
- *All-acrylic inlaid board and disks.*

Void™

A 4x4 grid doth invite two minds
To five sly games that "point" the way to fun,
Plus solitaires that leap 'bout like a knight
Or tour the board with arrows aimed just right.
The arrow pieces' flip sides turn up blank
And as the nulls among the voids they rank.

- *Deluxe 8" handcrafted wood board*
- *8½" softpack, roll-up vinyl mat*





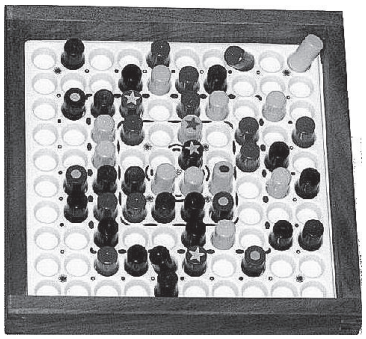
Guest performers

Batalo™

Scott Harmon did conceive this royal rout
A quarter century ago about
An elegant invasion strategy.
This handsome set will surely challenge thee.
12x12" leather-like vinyl mat rolls up to store in tube.

Quantum™

The vast, unfathomable 'plexity
Of each game's starting shake and random start
Tangles the very Cosmos in the fray
As tumbled pieces scuttle into cups,
Thence start upon a journey jeopardous
To gain control against opposing force
Of four most central squares upon the board.
From two to four staunch cohorts brave the row
And leap or step or stride to take the foe.
Deluxe 10½" wood-framed field, uniquely weighted pieces, fabulous historical book by Peter Aleff. On GAMES' 100 best games list, '85, '86



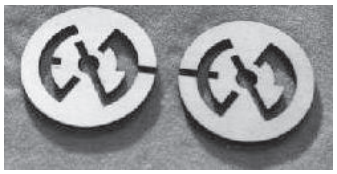
Tara™ *by Murray Heasman*

An Irishman's deep love of Celtic knots
Begat this tiling game of pretty plots.
Whose kingdom shall prevail as forts unite
While seeking other player to divide?
Each turn a tile ye place that paths appear
And bridge their way like snakes from there to here.
10" cardboard field, molded pieces, well-made box.

Puzzles, Just Puzzles

Instant Insanity™

A mere four cubes, their colored faces mix
To form 4 rows of multicolor tricks.
Three generations have been driven wild,
And to this day all puzzlers are beguiled.
We've added further tasks and a game for two.
Keeping your sanity is up to you.
Plastic cubes and tray, in pouch.



Oskar's Disks™

Two 4" wheels mesh notches tenderly
To slip and slide and link 'til intertwined—
A lovely, tactile turmoil for the mind!
Perceive ye that the cut-outs match astride
Save for the entrance slot that's slipped aside.
Its genius design can't be denied.
Hand-polished, lasercut wood in pouch.

Pyramid Puzzles

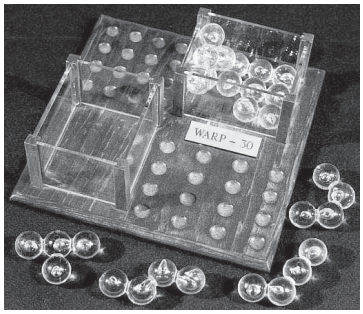
— Limited supplies and vanishing fast…

If ye have fortitude to spare, try these:
Conundrums each containing several shapes
Of close adjoinings of yon spheres.
Solutions sit so fine on sculpted base.
The wood ones are handcrafted with much grace.

- *Surprising (4 wood pieces, wood base, 2 solutions)*
- *Perplexing (6 molded pieces, acrylic base, 2 shapes)*
- *Giant (9 molded pieces, acrylic base, unique)*
- *Kolossal (12 wood pieces, wood base, 47 sol.)*

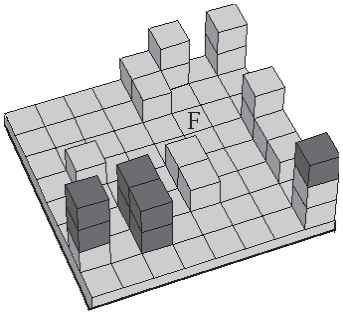
Warp-30™ ***— Limited supplies, end of line…***

A neat quartet of charming challenges
Positioned on a handsome wooden base:
Eight crystal globule clusters nestle close
To build two shapely pyramids with skill
Or two glass boxes level-topped to fill.
The geometric secrets in those mounds
Grow clearer as ye warp the space with rounds.
8x8" handcrafted wood board, felt cover.



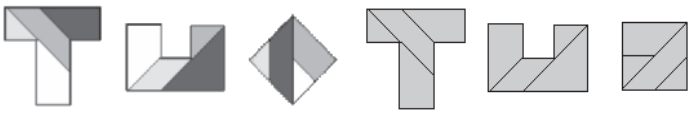
Rolling Block Maze ***— Custom orders only…***

The three red blocks from start to finish move,
With tips and topples shortest route they prove
And midst and over fixed blocks find their groove.
Handcrafted, engraved 9x9" wood board, wood cubes.



Tiny Tans™

Delightful little foursomes tease and please,
And each transfigures into dozens more.
Their symmetries ye seek, in silhouette,
And cheer for other tricky shapes you get.
Choose one or all, the T or U or Square,
And even mix and match them if you dare.



- *Multi-hued acrylics*
- *Handcrafted polished maple*

Pocket Puzzles

These portables in pouches travel well,
Companions at the ready, tile or spell,
Or roll some dice that sums or colors bring,
Amusements for the mind, ah, that's the thing.

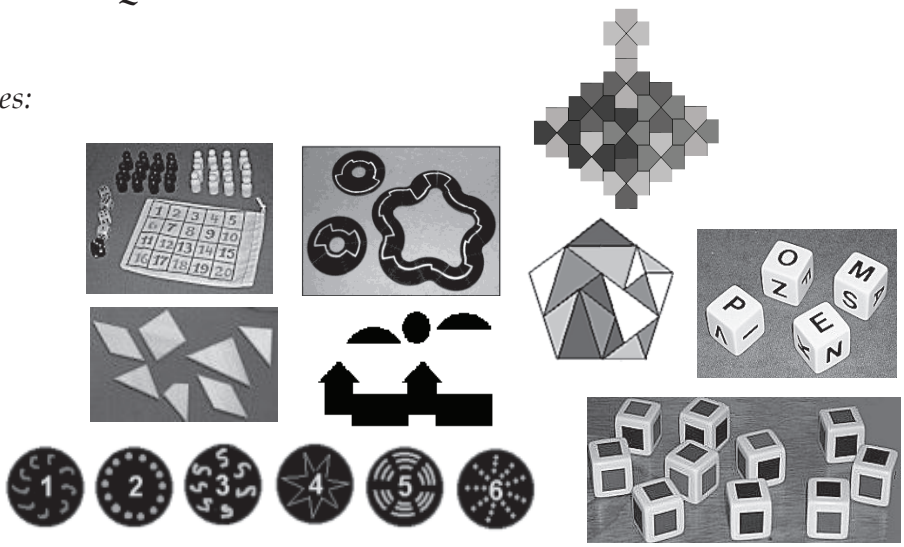
Pocket Rhombs™—Their symmetry is seven-folded art,
So plot ye diamonds free or on the chart.

Pocket Star—Now catch a falling star for rainy days,
The golden ratio tiles have endless plays.

Pocket Tetrapentos™—When polyiamonds four and five unite,
The loveliest lacy patterns come to light.

See also these pocket puzzles on page 5:
Pocket Pentominoes … **Pocket Quintachex**® … **Pocket Vees**™

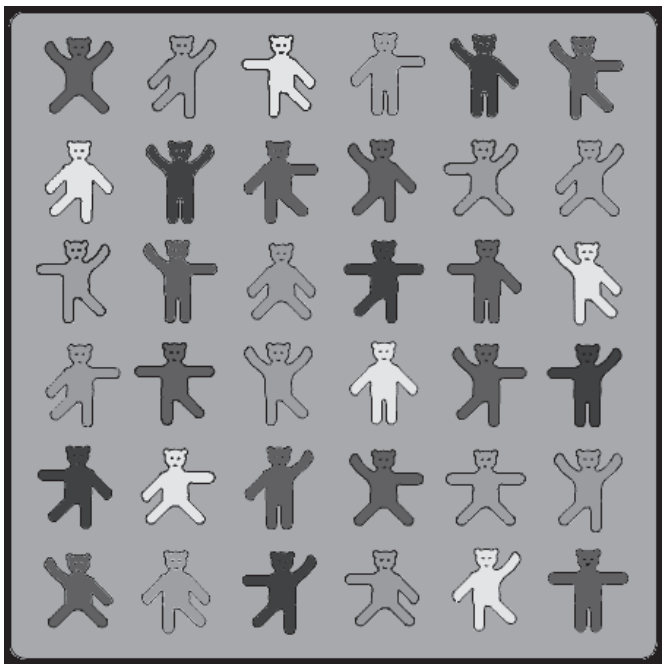
See also these pocket puzzles on their pages:
A+d+d+d™ … 27
Arc Angles™ … 25
Bowties™ … 8
Fox Blox™ … 29
Grand Tans® … 21
Puzzling Pentagon™ … 26
Shadow Play™ … 22
Six Disks™ … 33
Throw a Fit™ … 34

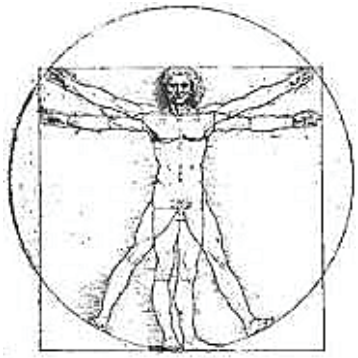


For Youngest Players

Bear Hugs™

For what would suit a child of three or four
So well as hugs and kisses by the score?
A puzzle for so young—how can it be
Ought else but playful bearful imagery?
A family of bears, their poses charm
With 36 varieties of leg and arm.
Six colors let the bruins cavort apace,
Each in its own well-formèd matching place,
Or group them as the youngster's mind invents
And make up fun-filled stories and events.
Then let the math-degreed and scholarly
Reflect upon their cool group theory!
For when twin brothers flip and swap their lair,
Near three and thirty thousand sums they bear,
While tray's reverse 36-factorial ways
The circles beyond countability displays.
All-acrylic, 23" two-sided tray, felt cover, display stand.

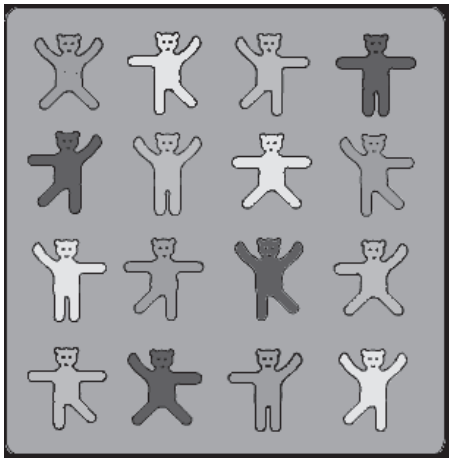




Play is child's work.
Youngest minds are hungriest to learn.

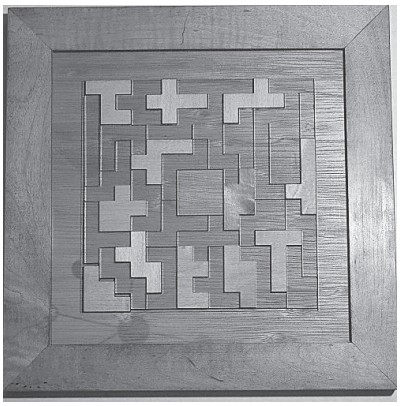
Bear Hugs™ Jr.

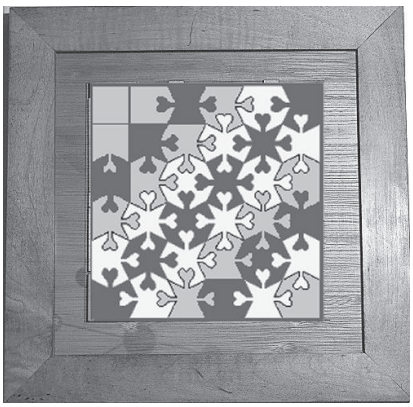
This little sibling of the Bear Hugs band
Has 16 teddies that can sit or stand,
Each with a different pose its space to scan.
A merry mimic of DaVinci's man,
Two arms, two legs each side. They can array
To fill each row and quadrant with all three:
All different colors, arms and legs there be!
Then stories tell and dance moves pose with grace...
Such fun-filled teaching tools all kids embrace.
All-acrylic, 15" reversible tray, display stand.

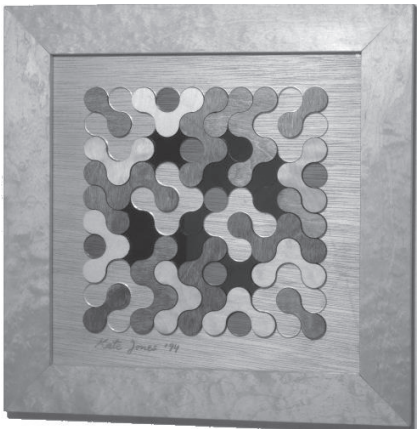


The Rusticana Collection

Elegance beyond compare ...

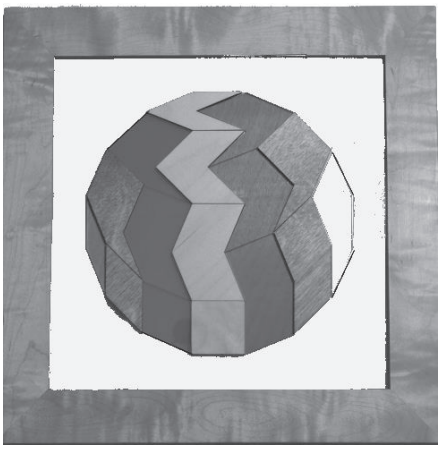


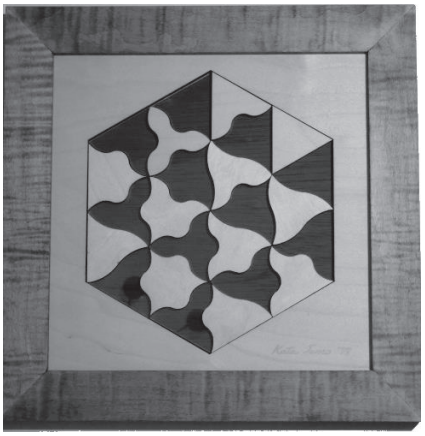




A stately style bestowed 'pon favorite sets
That ye didst see in fake stained glass alone:
Here weaving clear or glittery segments through
A wood mosaic fitted fine and true
Within a lustrous wooden framèd tray.
Some game grids fixed beneath -- just flip and play.
Such elegance of texture, craft and art
Bespeaks a joy for eye and mind and heart.

Clockwise from top left:
Sextillions™, Snowflake Super Square™, Super Roundominoes™, Rombix™, Trifolia™





To create beauty. To capture
a spark of cosmic truth,
to transmute chaos into order.
To explore new roads
as yet not dreamed of,
perchance to create beauty.

Gamepuzzles – a Renaissance in thought and art – as seen at Ye Olde Gamery, Maryland Renaissance Festival, since 1984.

Index

Preamble

Since 1979, the Kadon team
Has brought into the world a wondrous dream –
Original, artistic paradigms
Of good and true and beautiful pastimes.
To celebrate the mind in playful ways,
To find the best path through life's harrowing maze,
To build a monument to harmony,
To solve and understand all that we see,
In microcosms let us search for truth—
The Universe rewards our quest, forsooth.
Now share with us the joy of thinking clear
And let gamepuzzles fill your life with cheer.

— *Kate Jones*

Addendum — Gamepuzzles Renaissance Catalog

Alphabetical price list, 2015

*(Items with * are sold here but are not Kadon products; names with ** are public domain.)*

- a+d+d+d, $15
- Arc Angles, $32
- Archimedes' Square**, $39
- Batalo*, $29
- Bear Hugs, $195 (+$15)
- Bear Hugs Jr., $85
- Boats, $85
- Bowties, $29
- Brace, $65; wood, $115
- Chaos Tiles*, $35 (+$5)
- Chasing Squares, $35
- ChooChooLoops, $59
- Collidescape, $65, $155
- Colormaze/Flying, $75
- Colormaze, wood, $85
- Color Up, $65
- Combinatorix, $325 (+$15)
- Combinatorix Jr., $49
- Cubits, $59
- Dazzle, $135
- Deka-Mosaik, $39
- Deka-Star, $39
- Dezign-8, $175
- Diamond Rainbow, $59
- Diamond Star, $85
- Doris, $135
- Dual Quintachex, $395(+$15)
- End Point, $95 (+$5)
- Fill-Agree, $65
- Four on a Match, $21
- Fox Blox, $19
- Fractured Fives, $45
- Gallop, $95/$135 wood
- Game of Solomon, $49
- Game of Y, $85, $49 (vinyl)
- Gemstones, $65
- Goose**/Lab, wood $195 (+$10), canvas $65
- Grand Bowties, $95
- Grand Multimatch I, $175
- Grand Roundominoes, $195
- Grand Snowflake, $155
- Grand Tans, $39
- Heptominoes**, $175
- Hexacube, $1295
- Hexdominoes, $85
- Hexmozaix Jr., $52
- Hexmozaix, $59
- Hexmozaix II, $95
- Hexmozaix Combo, $155
- Hexmozaix gallery, $135/$195
- Hexnut Jr., $35
- Hexnut, $59
- Hexnut II, $175
- Hopscotch, $65
- Iamond Hex, $25
- Iamond Ring, $59
- Instant Insanity*, $16
- Intarsia, $85
- Kaliko, $135
- Kite-Mosaik, $39
- Kites&Darts master, $149
- KD starter, $65
- KD pods (8pc.), $5
- Kolossal Pyramid, $150
- L-Sixteen, $59
- La Ora Stelo, $65
- Leap soft, $29
- Leaves, $65
- Lemma, $125
- Lewis Carroll Wordgame, $49
- Manoover, $49
- Manoover Plus, $95
- Marshall Squares, $59
- *Medieval Games** (book), $25
- MemorIQ, $85
- Mini-Iamond Ring, $25
- MiniMatch I-II, $35 each
- More or Less, $65
- *Mudcrack Y** (book), $12
- Multimatch (card.), $9
- Multimatch I-IV, $59 each
- MultiTouch I-IV, $49 each
- Nine Men's Morris/Fox-Geese**, $65
- Octette, $59
- Octiamond Ring, $175
- Octiamond Stack, $55
- Octiles, classic, $125
- Octiles, wood, $145
- Octominoes**, $550 (+$25)
- Oskar's Disks, $25 (wood)
- Over-Pass, $65
- Pearl Fisher, $65
- Penrose Dia., starter, $65
- Penrose Dia., develop, $175
- Penrose Dia., pod (8 pc.), $5
- Pentarose, $65
- Pocket Pentominoes, $12
- Pocket Quintachex, $12
- Pocket Rhombs, $14
- Pocket Star, $20
- Pocket Tetrapentos, $12
- Pocket Vees, $12
- Poly-5, $39
- Polyarcs, $36
- *Polyominoes** hardbound, $39
- *Polyominoes** softcover, $25
- Poly-Spidrons, $85
- Power of Two, $65
- Proteus, $95, $75
- Pseudo-Coup, $49
- Puzzling Pentagon, $19
- Pyramid bags, $6, $8
- Pyramid, Giant, $29
- Pyramid, Perplexing, $21
- Q-Bix, $85
- Quadrants, $175
- Quantum, $39
- Quintachex, $85, $295
- Quintapaths, $65
- Quintillions, $85
- Rainbow Rombix, $125
- Renaissance deck**, $15
- Rhom-Antics, $495 (+$15)
- Rhombiominoes, $65
- Rhominoes, $59
- RhombStar-7, $59
- Rolling Block Maze, $75
- Rombix, $59
- Rombix Jr., $29
- Rombix Rusticana, $150
- Roundominoes, $29
- RunnuRound, $66
- Sextillions, $59
- Sextillions Rusticana, $150
- Shadow Play, $12
- Shape by Shape*, $15
- Six-by-Six, $125
- Six Disks, $29; bears, $29
- Snowflake Square, $29

- Snowflake Super Sq., $59
- Snowflake Super, Rust., $150
- *Star, $65 vinyl; $95 wood
- StarHex-II, $39
- StarSlide, $59
- Stelo, $25
- Super Deluxe Quint., $295
- Super Deluxe Q. box, $75
- Super Quintillions, $135
- Super Roundominoes, $59
- Super Round. Rusticana, $150
- Surprising Pyramid, $50
- Tan Tricks, $26, $59, $175
- Tangramion, $39
- Tara*, $40
- Teleporters, $150
- Ten-Yen**, $29
- Tetrapentos, $22
- Throw A Fit, $25
- Tiny Tans, acrylic, $9
- Tiny Tans, wood, $12
- Tiny Tans, Trio-in-Tray, $35
- Tool, $6
- Transpose, $175 (wood)
- Transpose, $125 (acrylic)
- Triangoes, $225
- Triangoes Jr., $59
- Triangule-8, $175
- Trifolia, $59
- Trifolia Rusticana, $150
- Tri-Jazz, $275
- Tulips, $275
- Turntable, $39
- Vee-21, $39
- Void, $55 (wood), $29 (vinyl)
- Warp-30, $85

Procurement Guide

We accept Visa, Mastercard, Discover/Novus, and American Express, by mail and phone and secure online ordering. For check or money order, please make payable to Kadon Enterprises, Inc., in U.S. funds drawn on a U.S. bank. Mail to: Kadon Enterprises, Inc., 1227 Lorene Drive, Suite 16, Pasadena, MD 21122-4645. Phone: (410) 437-2163 (24 hours). We accept purchase orders from educational institutions. We also accept PayPal—payable to kadon@gamepuzzles.com.

Our website is www.gamepuzzles.com. Our email is kadon@gamepuzzles.com.

We ship by priority mail in U.S.A., and by air priority to addresses outside the U.S. We will gladly ship your gift orders. Please add shipping as follows, for each destination: Within U.S., 15% of order amount; to Canada, 20%; other countries, 25%. Maryland residents only, please add 6% sales tax.

In person see us at the Maryland Renaissance Festival each Fall, and at traveling art shows in other times and places. Our show calendar resides in the website.

MAGIC

Paper for G4G12 in Atlanta, 2016

Math Magic Puzzles

Hirokazu Iwasawa
17-45, Mominokidai, Aoba-ku, Yokohama 225-0026 JAPAN
e-mail: iwahiro@bb.mbn.or.jp

Can you do the Puzzler's card tricks below? You do not need any special cards with gimmicks or a card with pencil dots or anything. For most of them, you do not need any sleight of hand, multiple outs, magician's force, magician's choice, etc., either.

Trick #1: 4×5

How it looks overall:

Although the spectator mixes the cards in the packet well, the Puzzler's spells make the cards specially arranged three times.

How it looks in detail:

1. With a packet in his hand, the Puzzler says, "Here are five Clubs, five Hearts, five Spades, and five Diamonds, that is, 4×5=20 cards in this packet, which I have already cast a spell upon."
2. Have the spectator cut the packet (and complete the cut) as many times as she likes, deal any number of cards, counting one by one, from the packet onto the table, and then riffle shuffle the two resulting packets together. The Puzzler notes that the cards are well mixed then.
3. Saying, "However, as I have already cast a spell upon the cards…," the Puzzler shows that every group of four cards from top to bottom consists of a Club, a Heart, a Spade, and a Diamond. He puts the five sets of cards face-up in one pile on the table.
4. The puzzler takes up the pile, turns it face-down and, without any shuffle or cut, casts a spell upon it. Then he shows that every group of five cards from top to bottom consists of cards with five different face values from one to five. While showing it, he notes that some five card sets lack some suits and sometimes even the same suits are next to each other: "So, the first spell has already been broken and only the second spell is working now." He puts the four sets of five cards face-up in one pile on the table again.
5. As if ending, however, the Puzzler continues: "The other day, when I showed these two magical powers to my daughter, she innocently said 'How about a third spell?' Well, as I had used magical powers twice already, little energy was left. But I tried." The Puzzler takes up the pile, turn it face-down and, without any shuffle or cut, casts another spell upon it. Then he shows that every pair of cards from top to bottom has both a black card and a red card.

Comments for magicians:

While the basic principle for this trick is well-known to you, I designed it as I think that causing such three consecutive effects as above is rather unusual if not quite novel.

Trick #2: RH+

How it looks overall:

Although which card is face-up or face-down is subject to the spectator's free choices, only a very special set of cards are finally face-up.

How it looks in detail:

1. The Puzzler hands a packet of, say, twenty cards to the spectator, asking her to shuffle the packet thoroughly.
2. Taking the cards back from her, the Puzzler quickly deals the cards from the packet onto the table while he turns over some cards and makes a pile of all the cards with some face-up and the others face-down seemingly in a completely random order.
3. Then, again, some cards are turned over and all cards are piled again on the table. But, this time, it is the spectator who steers the ship. At every step, she decides "one" or "two." If "one," take a card from the top of the packet, turn it over, and put it on top of the pile. If "two," she decides "turn over" or "leave (as they are)." If "turn over," take two cards from the top of the packet, turn them over, and put them on top of the pile. If "leave," take two cards from the top of the packet and put them, without turning over, on top of the pile. She repeats this routine until all cards from the packet are piled again on the table.
4. The Puzzler deals the piled cards into two piles (left, right, left, right, and so on), takes one of the two, turns it over, and puts it on top of the other pile. Then he spreads all the cards to see that only a special set of cards, say, some royal flush, are face-up and all the rest are face-down.

Comments for magicians:

I know this kind of effect is not new to you. But I feel the way of shuffling by the spectator described above conceals the trick better than the classic way of shuffling.

Trick #3: ERR

Different from other MMP tricks, this trick requires some sleight of hand.

How it looks overall:

Although the spectator hides her card well in the deck, the Puzzler finds it by seeing the faces of cards in the deck.

How it looks in detail:

1. The Puzzler hands a full deck to the spectator, asking her to shuffle it thoroughly.
2. Taking the deck back from her, the Puzzler quickly shuffles it once.
3. The Puzzler instructs the spectator to do the following: While the Puzzler turns away, the spectator takes about one fourth of the cards of the deck up together as a packet from the top, holds it in her right hand, looks at the bottom card, remembers it, uses her left hand to cut the remainder of the deck in the middle into two parts, inserts the packet in her right hand between them, and squares up the resulting deck.
4. The puzzler turns back, investigates the faces of the cards in the deck, and finds the card the spectator remembers.

Note:

To do this trick, you need to peek at the faces of a minimum number (two, in fact) of cards during Step 2, perhaps while you perform some false shuffle.

Comments for magicians:

This may remind you of Dai Vernon's Emotional Reaction, but the trick is quite different.

Trick #4: One Packet, Two Spells

How it looks overall:

Two spells selected by the two spectators do work at the same time upon one packet and the Puzzler's two predictions come true.

How it looks in detail:

1. The Puzzler asks two spectators to help him. He hands a full deck to one of the spectators, asking her to shuffle the deck thoroughly.
2. Taking the deck back from her, the Puzzler says, "I take about six cards from the top," takes cards, and put them onto the table.
3. The puzzler hands the rest of the deck to one of the spectators, saying, "To prevent me from knowing how many cards are in the pile, while I turn away, take about six cards, maybe seven, eight, or five cards, together from the top of the deck and put them on top of the pile." While the spectator does so, the Puzzler writes down two predictions on a piece of paper.
4. Have the two spectators select their respective spells almost freely from, say, city names. Here, as an example, let them be "New York" and "Los Angeles. (While the exact number *n* of cards in the pile is unknown, two spells must be selected so that the number of letters of either spell must be smaller than *n* and the total number of letters of the two spells must be larger than *n*.)
6. Have the first spectator deal from the pile onto the table, one card for each letter of "N-E-W-Y-O-R-K," and then put the remainder on top. Have the second spectator deal from the resulting pile onto the table, one card for each letter of "L-O-S-A-N-G-E-L-E-S," and then put the remainder on top.
7. The Puzzler notes that the deck was shuffled thoroughly and that he does not know even the number of cards in the pile. Have the first spectator do the same spelling routine yet again with "N-E-W-Y-O-R-K," and then turn over the top card of the resulting pile. The card agrees with the Puzzler's first prediction. Have the second spectator do the same with "L-O-S-A-N-G-E-L-E-S," and then turn over the top card, which agrees with the second prediction.

Note:

To do this trick, you do not need sleight of hand, but you need to peek at the faces of a minimum number (two, in fact) of cards before Step 3.

Comments for magicians:

I designed this trick as I think that two different spells upon one packet at the same time is rather unusual.

Two Person Telepathy: A Hidden Gem in Anneman's *Practical Mental Effects*

Doron Levy[1]

About three decades ago I visited New York City for the first time. Living magic, doing magic, and thinking about magic, I could not have missed on the opportunity to walk into Tannen's magic store, New York City's oldest magic shop.

In these pre-internet days, visiting a magic shop was one of the only practical ways to buy magic items. Though dealers were always present in magic conventions, and the US mail still functioned, in these pre-internet days, the first-hand experience of visiting a magic store was the best way for amateurs and professionals alike to talk about magic.

Before stepping into the store, I expected to leave several hours later with a bag full of tricks. In reality, I did emerge after several hours, but without any new tricks. Instead, I walked out holding two books: Corinda's *13 Steps to Mentalism*, and Anneman's *Practical Mental Effects*. Not knowing anything about mentalism, the art of mind reading, but eager to learn – I chose the right path: reading books. Many years have passed, and my library includes hundreds of books on magic, but I always enjoy rereading my first two books. A Dover edition of Anneman's book sells for about $10 on Amazon, while Corinda's book is about twice as expensive. Both of them were published over half a century ago. However, they are full of gems, which was acknowledged, for example, by the recent release of a 6 DVD set by Richard Osterlind, in which he systematically demonstrates material from Corinda's book.

Over the years, I have been using a lot of material from Anneman's book in my shows. In honor of the 12th Gathering for Gardner, I would like to revisit one effect: a two person's telepathy routine. This routine is called "A Practical Card Code". It is attributed to Orville Meyer, and it appears on Page 292 in the last chapter of the book, the chapter on Psychic Codes. Many manuscripts were written on two person telepathy, including Step #8 in Corinda's book. Examples of other resources include the book by Ron and Nancy Spencer, *Telepathy Personified,* and the manuscript by R.T. Stark, *Cipher Speak*.

Meyer's *Practical Card Code* is a very simple, yet quite clever, coding system, which allows two people to communicate information about a chosen card with minimal effort. The original layout of the code is shown in Figure 1. The idea is straightforward and can be played out in several ways. The code is known to the magician and the assistant. The assistant can be blindfolded and is sitted in front of the audience. A volunteer picks a card, which can then be shown to the entire audience. Since the assistant is going to reveal the card, there is no need to hide the card from the magician. The magician then asks the assistant to reveal the card, and the assistant does so successfully.

The identity of the card is "transmitted" by the magician to the assistant through the secret code shown in Figure 1. For example, if the chosen card is the 3 of Hearts, the magician will say: "Tell me what is the chosen card". "Tell me" sets the suit of the chosen card as "hearts," and "what" tells the assistant that the card is between an Ace and 6. In reply, the assistant says: "The card is RED," at which point the magician says "Yes," which will then direct the card to be in the group A,2,3. The

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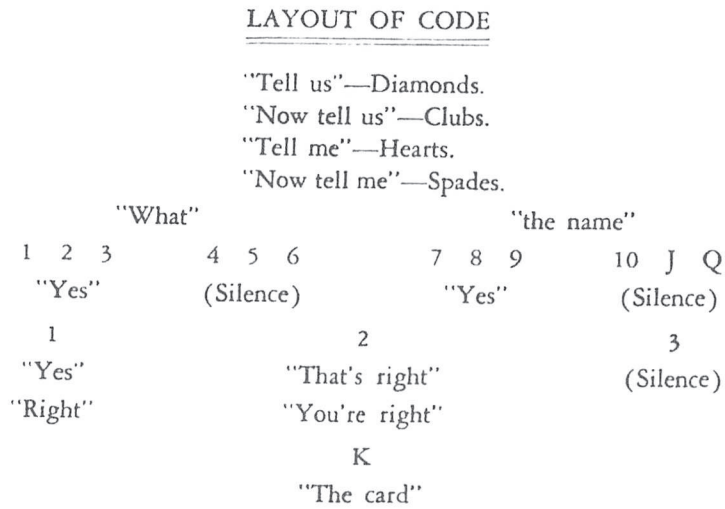


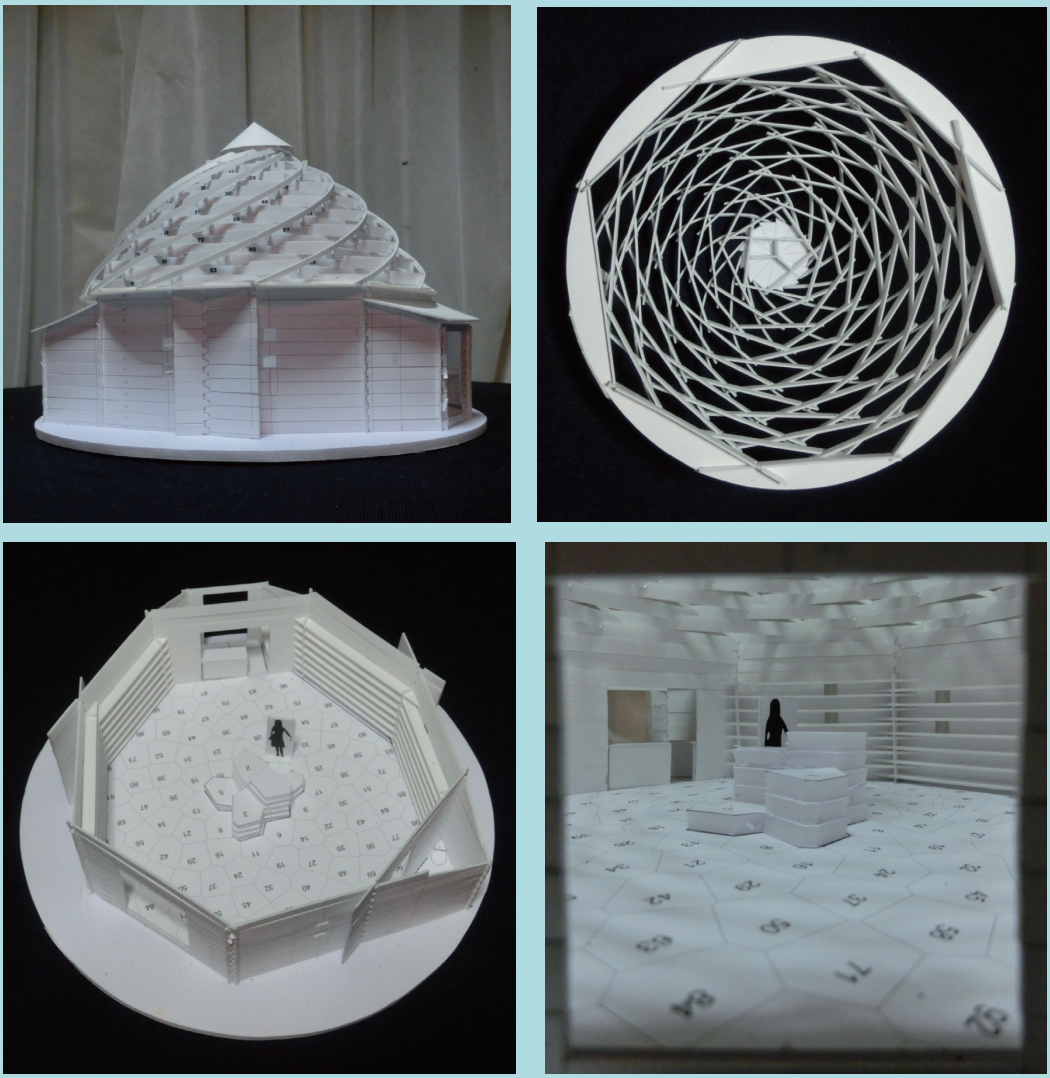
Figure 1: A Practical Card Code by Orville Meyer, as appears in Anneman's Practical Mental Effects.

assistant now knows that the card is A♡, 2♡, or 3♡, but the information is revealed gradually. The assistant will say: "The card is a HEART," to which the magician will reply by nodding his head, but saying nothing. This directs the assistant to the third option, and she can conclude with: "The card is a THREE OF HEARTS".

The code requires several clarifications: First, the deck is assumed to have no Jokers. It is easy to add a special code sentence for the joker, but that is not part of the original code. Also, there is a special code sentence for the King. For example, the King of Spades will be "transmitted" using the sentence: "Now tell me the card". The assistant knows that the card is the K♠ and can then chose either to immediately reveal this information or to gradually reveal it, similarly to the non-King cards.

I probably performed this effect hundreds of times. It is easy to master and leaves a very strong impression. I was never a big fan of the "silence" part of the code. This can always be replaced by a different word, such as "good."

Just like many other magic tricks, I would not recommend repeating this effect more than once, at least not in its original form. However, with some simple modifications, the trick can be repeated several times in front of the same audience without revealing the method. For example, starting from the second card, I would recommend modifying the code in a cyclic way. That is, once the first card was revealed, one can use this information to encode the difference between the known card and the next card. For example, one can assume an ordering of the suits, say ♢, ♣, ♡, ♠. Then if the first card, e.g., was a heart, and the second card is a club, the magician has to encode the number 3, that is skipping 3 suits in the list (in a cyclic fashion). The first word in the communication to the assistant can encode this message: "Fine," "Good," "Let's," "OK." The face value of the card can be encoded in a similar way by encoding for the difference from the first card. I am not providing here exact rules, but the principle is straightforward. It is important to note though, that if the effect is to be repeated several times, some variations should be made in how the card is selected and also precisely what the magician sees. For example, while the first card is chosen by a spectator and is shown to everyone, including to the magician, the second card can be hidden from the magician (i.e., forced in your favorite way). Repeating effects makes sense only if the overall sense of mystery gradually increases. Enjoy!



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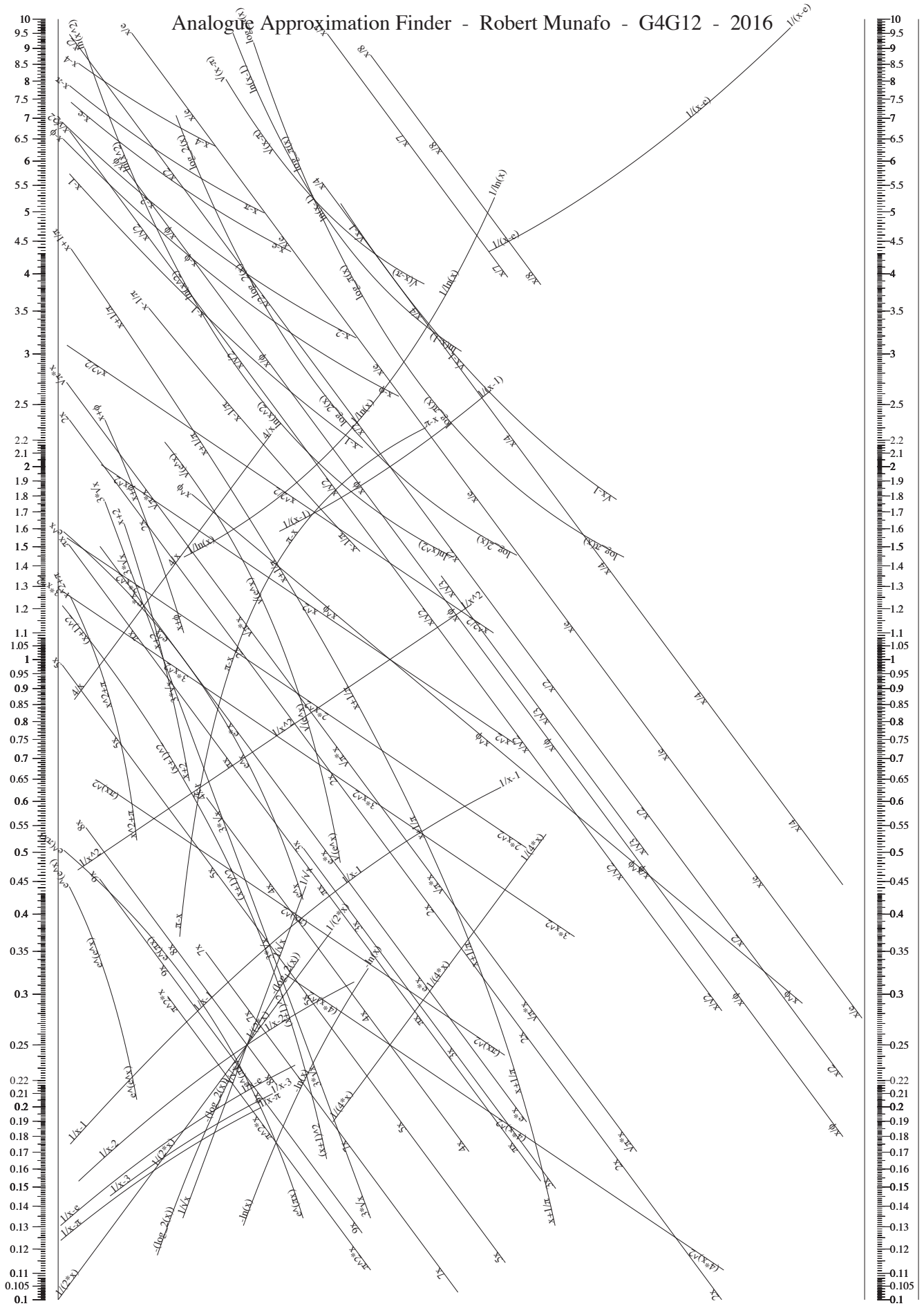
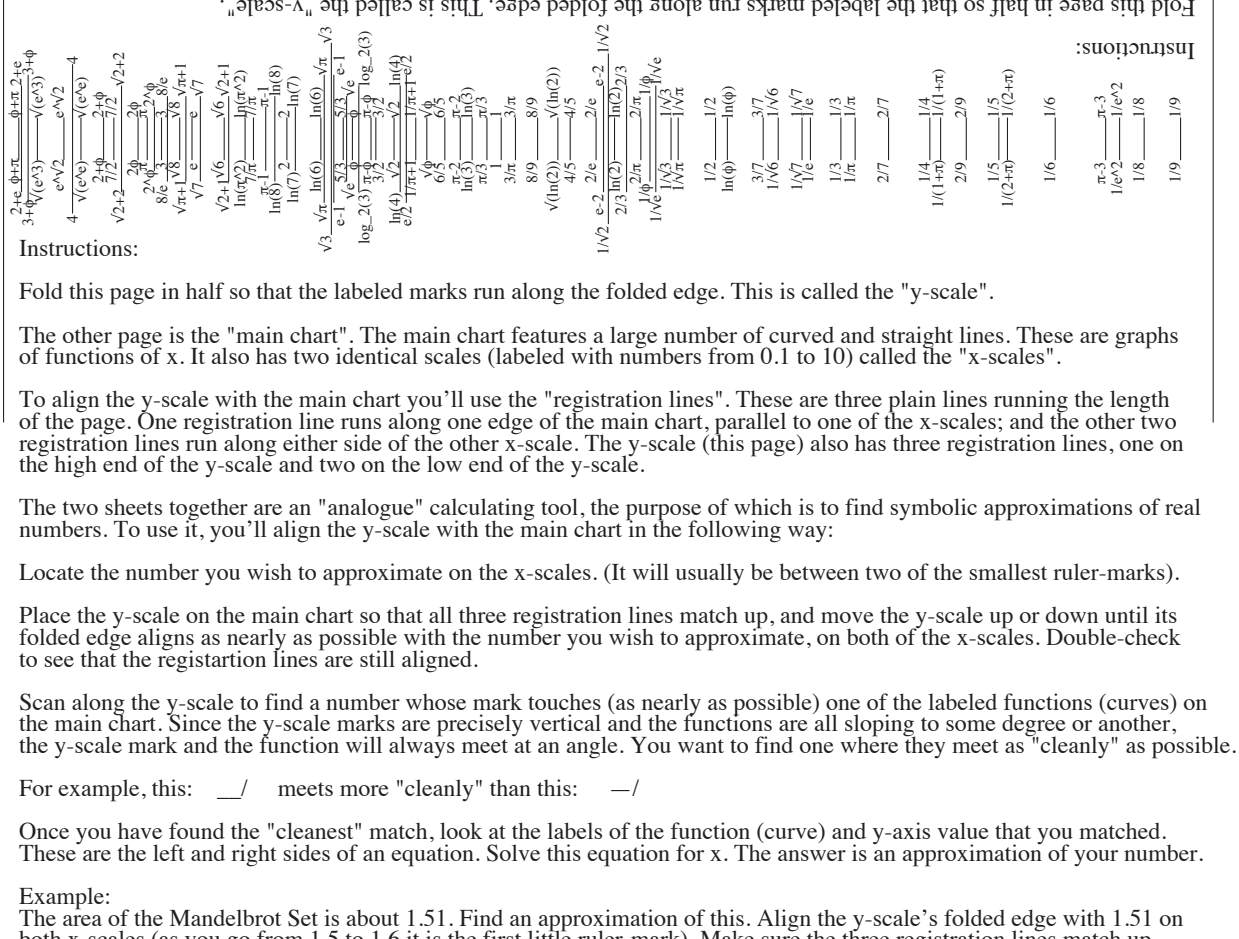

Analogue Approximation Finder - Robert Munafo - G4G12 - 2016

Analogue Approximation Finder - Robert Munafo - G4G12 - 2016



Instructions:

Fold this page in half so that the labeled marks run along the folded edge. This is called the "y-scale".

The other page is the "main chart". The main chart features a large number of curved and straight lines. These are graphs of functions of x. It also has two identical scales (labeled with numbers from 0.1 to 10) called the "x-scales".

To align the y-scale with the main chart you'll use the "registration lines". These are three plain lines running the length of the page. One registration line runs along one edge of the main chart, parallel to one of the x-scales; and the other two registration lines run along either side of the other x-scale. The y-scale (this page) also has three registration lines, one on the high end of the y-scale and two on the low end of the y-scale.

The two sheets together are an "analogue" calculating tool, the purpose of which is to find symbolic approximations of real numbers. To use it, you'll align the y-scale with the main chart in the following way:

Locate the number you wish to approximate on the x-scales. (It will usually be between two of the smallest ruler-marks).

Place the y-scale on the main chart so that all three registration lines match up, and move the y-scale up or down until its folded edge aligns as nearly as possible with the number you wish to approximate, on both of the x-scales. Double-check to see that the registartion lines are still aligned.

Scan along the y-scale to find a number whose mark touches (as nearly as possible) one of the labeled functions (curves) on the main chart. Since the y-scale marks are precisely vertical and the functions are all sloping to some degree or another, the y-scale mark and the function will always meet at an angle. You want to find one where they meet as "cleanly" as possible.

For example, this: __/ meets more "cleanly" than this: —/

Once you have found the "cleanest" match, look at the labels of the function (curve) and y-axis value that you matched. These are the left and right sides of an equation. Solve this equation for x. The answer is an approximation of your number.

Example:
The area of the Mandelbrot Set is about 1.51. Find an approximation of this. Align the y-scale's folded edge with 1.51 on both x-scales (as you go from 1.5 to 1.6 it is the first little ruler-mark). Make sure the three registration lines match up. Scan along the y-scale. You will see that "sqrt(7)" on the y-scale aligns with the function labeled "4/x". This suggests that 4/x equals the square root of 7. Solving for x gives: x = 4/sqrt(7). Using a calculator you can verify that 4/sqrt(7) is 1.5118... which is indeed a good approximation for 1.51.

Breaking Squares and Cubes

Paul Clarke

4 January 2015

The following problem appeared in a collection of puzzles [1] by Martin Gardner in 1977.

Forty matchsticks are arranged to form the skeleton of an order-four checkerboard. Remove the smallest number of matchsticks that will break the perimeter of every square.

The reader will quickly find a solution with nine matchsticks. To see that nine matchsticks is a minimum, note that there are sixteen 1×1 squares and the removal of a matchstick out of the middle 24 can break the perimeter of at most two of these squares. This requires at least 8 matchsticks, and we need to remove another one to break the perimeter of the 4×4 square.

Gardner goes on to pose the problem for any $n \times n$ checkerboard with $2n(n+1)$ matchsticks. Henceforth we let $\mu(n)$ denote the number of matchsticks in a minimum solution. Generalising our argument from the $n = 4$ case, we can easily prove that when $n \geq 2$ is even,

$$\mu(n) \geq \frac{n^2}{2} + 1 \tag{1}$$

When $n \geq 3$ is odd, we can show that

$$\mu(n) \geq \frac{n^2+1}{2} + 1 \tag{2}$$

Proving the inequality for the odd case is slightly harder than for the even case. We leave this as an exercise for the reader.

A general procedure due to Gardner for achieving the minimum of (1) for all even n is illustrated in Fig. 1. The analogous procedure for the odd case is shown in Fig. 2. So in some sense, Gardner's original problem has been completely solved.

What about three dimensions? We can rephrase Gardner's problem as follows.

Matchsticks of length one are arranged in the form of an $n \times n \times n$ cubical lattice. Remove the smallest number of matchsticks that will break the boundary of every cube.

By "boundary" we mean the set of matchsticks that lie on any face of the cube. To help visualise the lattice, it is useful to imagine thickening an $n \times n$ two dimensional checkerboard into an $n \times n \times 1$ cuboid. The $n \times n \times n$ lattice is then formed by gluing n of these cuboids together.

In a similar way to the two dimensional problem, we let $\mu_3(n)$ denote the minimum number of matchsticks we can remove that will break the boundary of every cube in an $n \times n \times n$ cubical lattice.

The solutions to the two dimensional problem rely on the existence of a 2×1 "domino" that can tile the checkerboard in an efficient way. The analogous domino for the three dimensional problem is a $2 \times 2 \times 1$ cuboid. Extending this analogy, we can see that the removal of a matchstick can break the boundary of at most four $1 \times 1 \times 1$ cubes in the $n \times n \times n$ lattice. Furthermore, this can only be achieved if the matchstick does not lie on any of the faces of the outer $n \times n \times n$ cube. Therefore, for n even, we have that

$$\mu_3(n) \geq \frac{n^3}{4} + 1 \tag{3}$$

We now describe a procedure that solves the problem in this many matchsticks for all even n. Suppose that n is even. Split the $n \times n \times n$ cube into $n \times n \times 2$ cuboids (Fig. 3). Then, tile each cuboid in the same way with $2 \times 2 \times 1$ dominoes leaving a single $2 \times 2 \times 2$ cube in the centre (Fig. 4). Remove the centre matchsticks of each of the dominoes. We are now left with an unbroken $2 \times 2 \times n$ tower in the centre of the $n \times n \times n$ cube. To break the boundary of all of the cubes in this tower, we tile the tower with dominoes as shown in Fig. 5. We can then break the boundaries of the remaining four $1 \times 1 \times 1$ cubes and the boundary of the $n \times n \times n$ cube with two more matchsticks.

It is clear from Fig. 4 and Fig. 5 that this procedure will always break the boundary of every cube in the lattice. Furthermore, in total, this procedure requires the removal of

$$\frac{n^3 - 4}{4} + 2 = \frac{n^3}{4} + 1$$

matchsticks. Therefore, (3) becomes an equality, and we see that the three-dimensional problem has been solved for n even.

We now briefly turn our attention to the odd case of this problem. The case where $n \equiv 1 \pmod 4$ appears to be hard, but the author has made some progress on the case where $n \equiv 3 \pmod 4$. We present a solution for the $n = 3$ case in Fig. 6. Here we leave out two $1 \times 1 \times 1$ cubes on opposite

corners and also the centre $1 \times 1 \times 1$ cube. Then we tile the remaining $3 \times 3 \times 3$ cube with $2 \times 2 \times 1$ dominoes. The boundaries of the three left out cubes are then broken one by one.

We believe that this procedure can be generalised for all $n \equiv 3 \pmod 4$. We also believe that this solution is optimal, as there does not appear to be any other efficient way to tile the $n \times n \times n$ cube while leaving out three $1 \times 1 \times 1$ cubes. Thus, we are led to conjecture that when $n \equiv 3 \pmod 4$,

$$\mu_3(n) = \frac{n^3 - 3}{4} + 3 \tag{4}$$

The natural question that now arises is whether or not this problem can be generalised to four, five or higher dimensions. In the four-dimensional case of n even, one can partially visualise applying a procedure similar to the solution of the even case of the three-dimensional problem. Future work will focus on making this notion concrete.

References

[1] Gardner, M. (1977) "Mathematical Carnival". New York: Vintage Books.

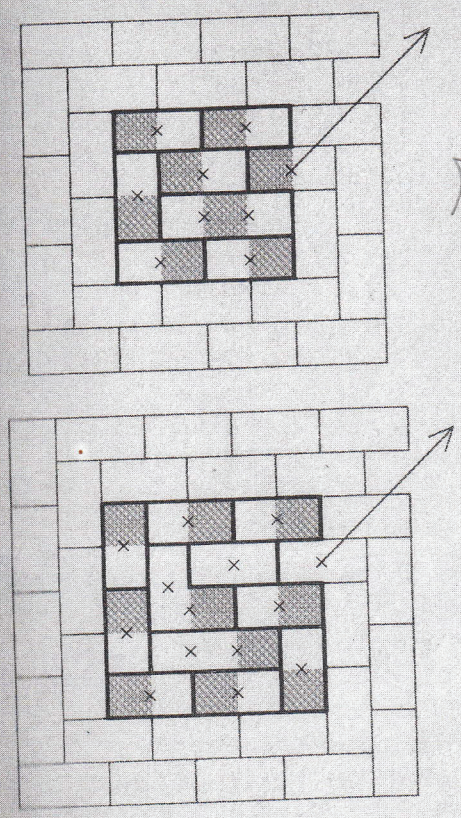


Fig. 1

Fig. 2

FIGURE 62
Solutions for toothpick problems

broken square higher than order-one. It can be shown that this is never possible, so that the minimum is raised to $\frac{1}{2}(n^2 + 1) + 1$. The lower drawing in Figure 62 shows a procedure that achieves this minimum for all odd-order squares.

D. J. Allen, George Brewster, John Dickson, John W. Harris, and Andrew Ungar were the first readers to draw a single diagram that could be extended to display all solutions rather than separate diagrams as I had found, for the odd and even cases.

David Bienenfeld, John W. Harris, Matthew Hodgart, and William Knowlton, attacking the companion problem of creating rectangle-free patterns, discovered that the *L*-tromino plays in this problem the same role the domino plays in the square-

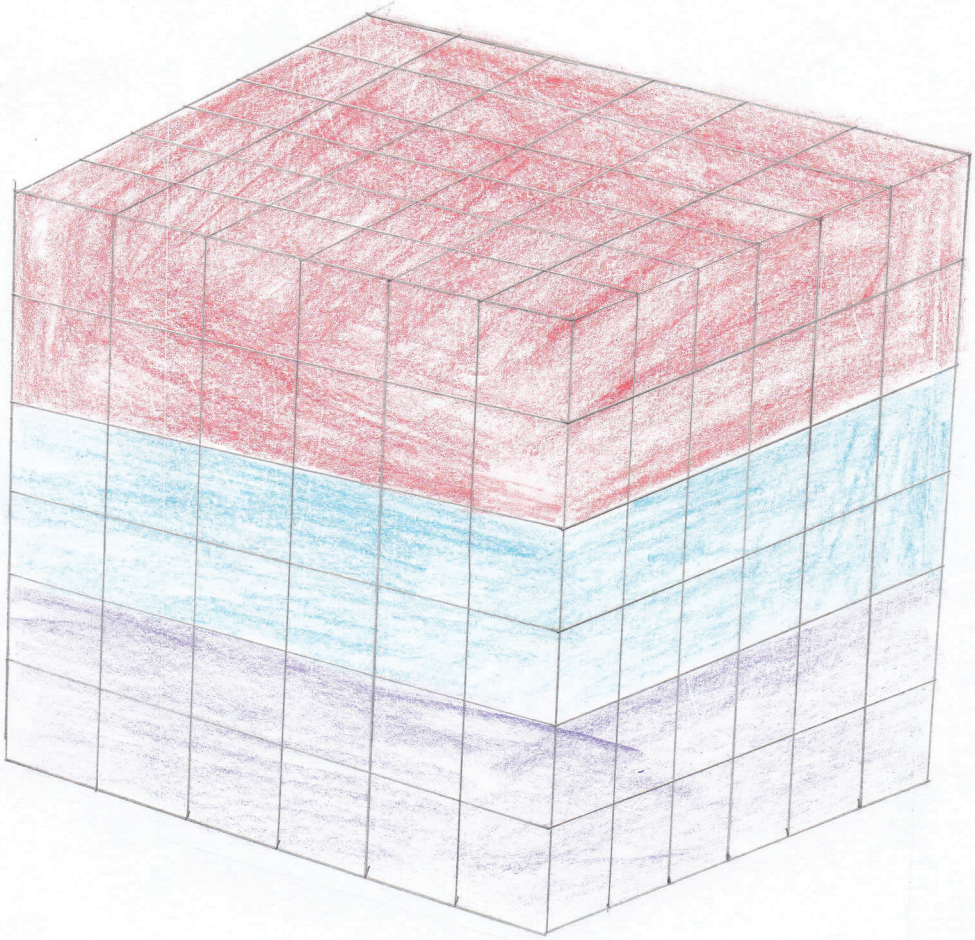


Fig. 3 : Splitting the $n \times n \times n$ cube into $n \times n \times 2$ cuboids for $n = 6$

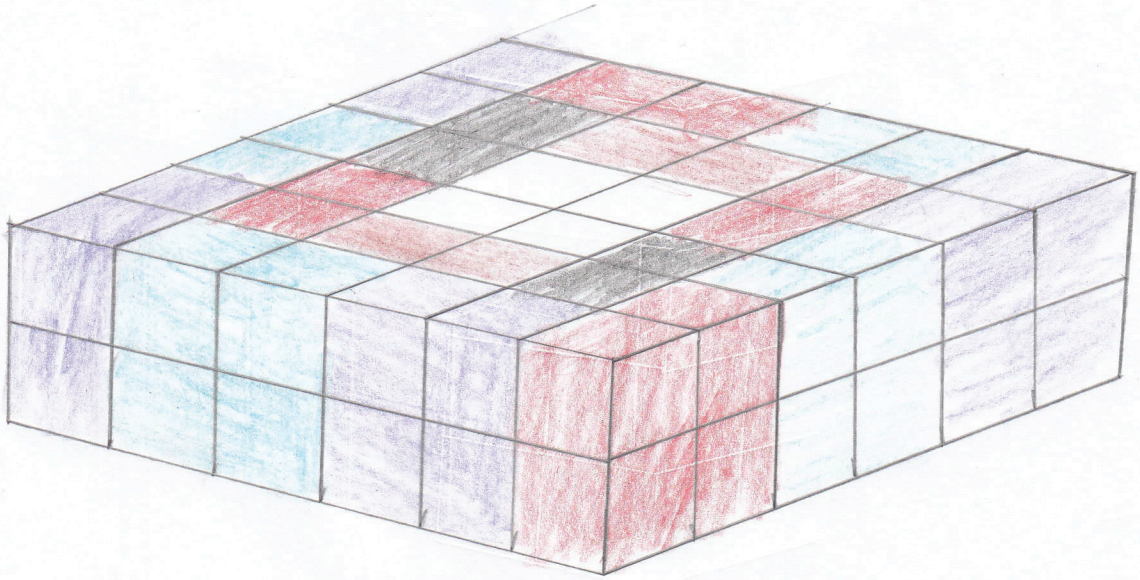


Fig. 4: Tiling an $n \times n \times 2$ cuboid for $n = 6$

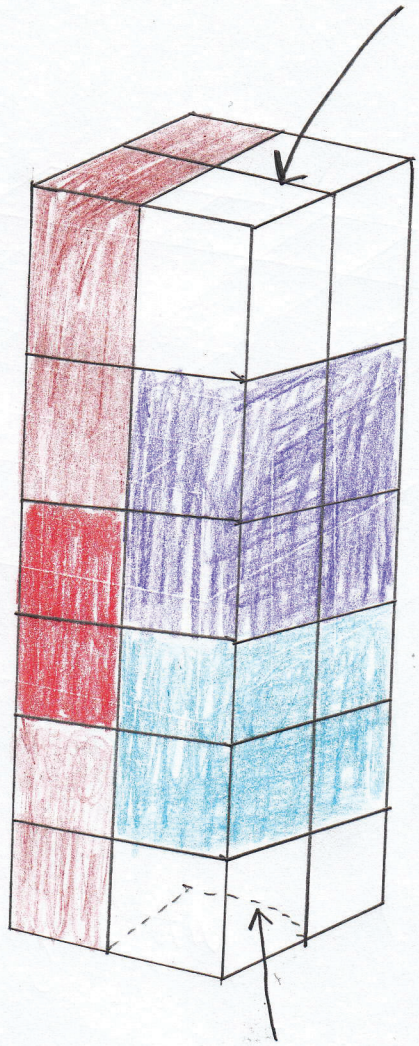


Fig. 5: Tiling the $2 \times 2 \times n$ tower for $n = 6$.

The removal of the arrowed matchsticks will break the boundary of the remaining four $1 \times 1 \times 1$ cubes and of the outer $n \times n \times n$ cube

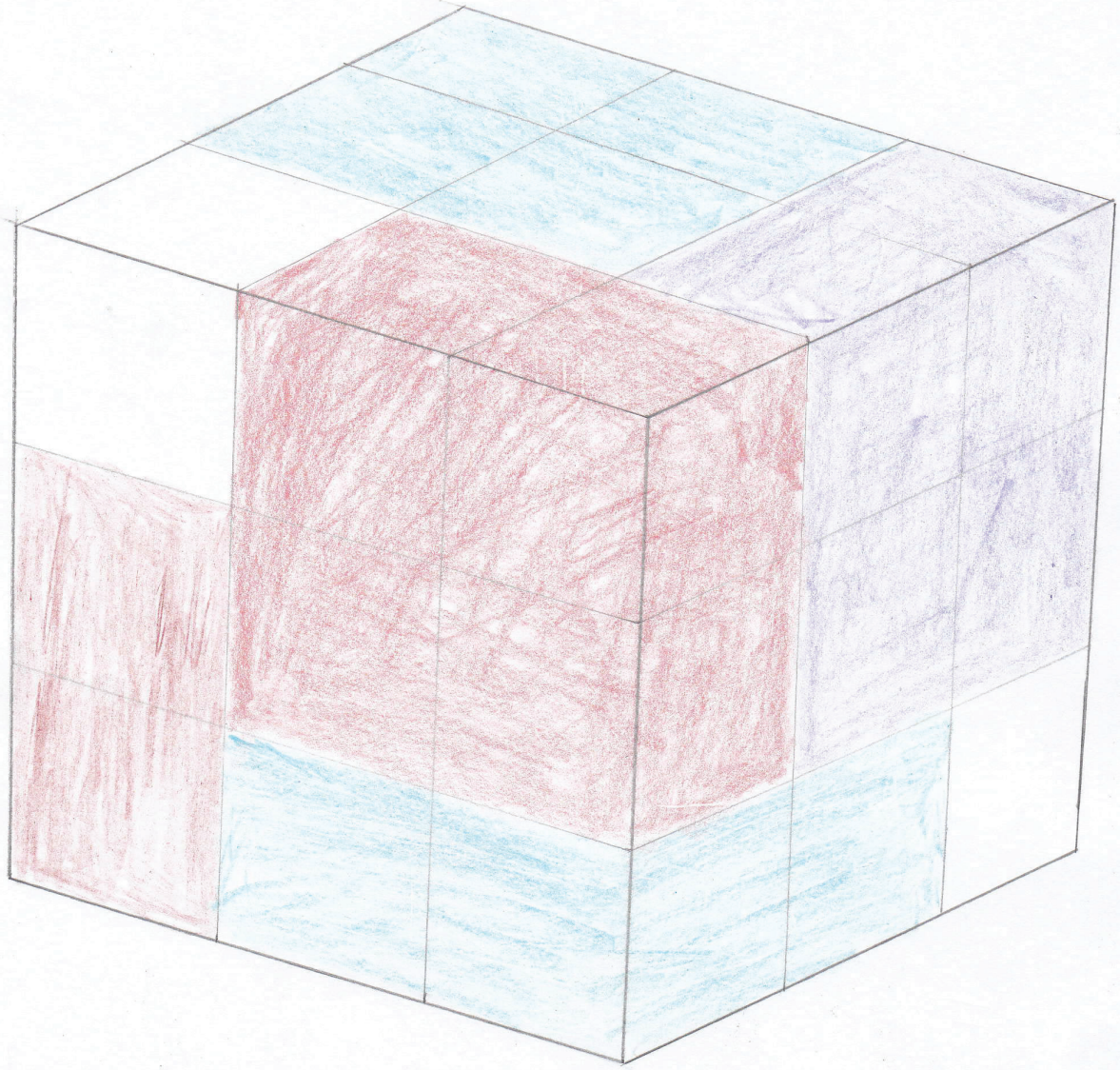


Fig. 6: Tiling the 3x3x3 cube with dominoes before breaking the centre 1x1x1 cube and the corner 1x1x1 cubes (white). One domino at the back of the cube is not visible in the figure.

The Composer

Glenn Hurlbert*

Abstract

A composer attempts to create a musical language from which to build a new form of compositional structure.

1 Rimsakovic Triads

Nick Rimsakov decides one day to see if he can write pieces of music based on chord progressions involving only those triads (triples of notes) whose notes are well separated. He settles on disallowing half and whole steps, meaning that each pair of notes is at least a minor third apart. He considers, for example, G and A to be too close, regardless of how high or low each note is played. That is, while particular instances of the two notes may be separated by several octaves, they still differ by a whole step on the 12-tone scale. This reduces the choices to diminished (gaps of two minor thirds), minor (minor third gap then major third gap), major (major then minor third gaps), and augmented (two major thirds) triads in each of the twelve keys. Since these are fairly consonant chords, he should be able to create nice sounds from various sequences of them. There are $3 \cdot 12 + 4 = 40$ such *separated* triads in all, reduced from the 220 triads that exist without restriction.

Furthermore, Nick would like to choose a special collection from this set of 40 that would have the extra property that any one triad can be followed by any other triad in a chord progression, thus dispensing with more complicated music theory. He feels that this, in some way, generalizes the pentatonic scale, a collection of 5 of the 12 notes (typically thought of as the black keys on a piano) that can be played in any order with pleasing results. Maybe his set of *Rimsakovic* triads can be played in any order nearly as pleasantly.

He reasons that to accomplish this he should require that every pair of Rimsakovic triads share a note. One way to realize this is to choose every separated triad that contains a C, for instance. This produces the following 10 triads.

inversion	diminished	minor	major	augmented
root	$C\ E^\flat\ G^\flat$	$C\ E^\flat\ G$	$C\ E\ G$	$C\ E\ G^\sharp$
first	$C\ E^\flat\ A$	$C\ E\ A$	$C\ E^\flat\ A^\flat$	
second	$C\ F^\sharp\ A$	$C\ F\ A^\flat$	$C\ F\ A$	

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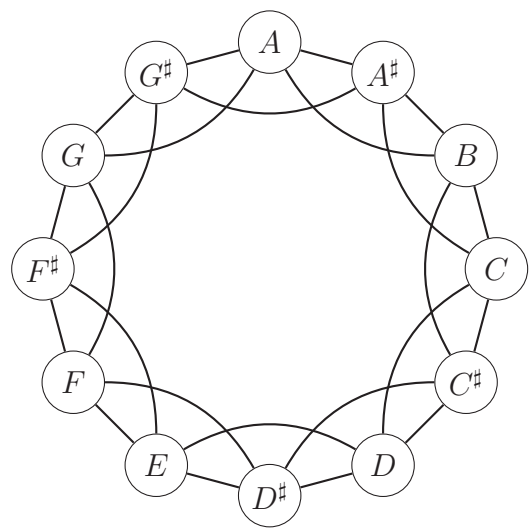
To Nick, this collection doesn't seem reasonable because every single composition will have the C droning throughout. Nothing against drones, but this appears to be a fairly restricted vocabulary. So he wonders if it is possible to conceive of Rimsakovic triads that don't all contain some fixed note. If so, how many triads could there be in such a collection? More than 10?

2 A Graph Model

To represent the situation visually we can refer to the *graph* (set of *vertices* [points] with *edges* [lines] joining some pairs of them) below. Its vertices are the notes of the 12-tone scale, and two vertices are joined by an edge when they are too close together, as discussed above.



The forbidden pairs in Rimsakovic triads.

In graph theoretic terms, the 12 vertices with just the outer 12 edges is called a *cycle*, and is denoted $\mathcal{C}_{12}$ to indicate its number of vertices. For any graph $\mathcal{G}$ we can create a new graph called its *square* (written $\mathcal{G}^{(2)}$) by simply adding edges between pairs of vertices that are otherwise two edges apart from each other. For example, if we start with just the outer edges in the above graph, we see that B and $C^\sharp$ are two edges apart (through C), and so we would add the inner edge between B and $C^\sharp$. Thus the Rimsakovic graph above is $\mathcal{C}_{12}^{(2)}$.

Notice that the separated triads are precisely those triples of vertices that have no edges among them; e.g. the D minor triad $D\ F\ A$. Graph theorists call such things *independent sets*. The Rimsakov condition that each pair of triads share a note means that each pair of independent sets *intersects*. So a Rimsakovic collection of triads is an intersecting collection of independent sets of $\mathcal{C}_{12}^{(2)}$.

The intersecting collection above (with all Cs) is an example of what is referred to as a *star* — a collection of sets for which some element is in each of them. What Nick is wondering is whether there is an intersecting non-star collection of more than 10 independent triples of $\mathcal{C}_{12}^{(2)}$.

3 Some Background

In 1961 Paul Erdős, Chao Ko, and Richard Rado introduced the study of intersecting collections of sets, all of which have the same size r (the situation above has $r = 3$). Without any restrictions between the members of sets, they discovered that no intersecting collection is larger than the

biggest star, provided $r \leq n/2$, where n represents the number of elements available ($n = 12$ above).[1] If $r < n/2$ then the biggest star is the unique maximum collection, but when $r = n/2$ there is one other, namely any collection formed by picking exactly one set from each pair $\{X, \overline{X}\}$, where $\overline{X}$ denotes the set of elements not in X (its *complement*). An example with $n = 6$ and $r = 3$ is below, the choices on the top, their complements on the bottom; both of the resulting collections are intersecting. Note that if instead we always choose the set that contains a 1, we construct a star.

123	124	346	126	134	246	245	236	146	156
456	356	125	345	256	135	136	145	235	234

Because the star is the unique best when $r < n/2$ it is interesting to wonder, then, what is second best. One idea is to fix a set of size 3 and always choose at least two from it. An example with $n = 7$ and $r = 3$ is below — the triples correspond to the columns and the 'x's identify their elements (i.e. the first triple is 123 and the last is 237); the lines merely highlight the structure. This *2-of-3* collection has 13 sets, compared to 15 for the biggest star.

1	x	x	x	x	x	x	x	x	x				
2	x	x	x	x	x					x	x	x	x
3	x					x	x	x	x	x	x	x	x
4		x				x				x			
5			x				x				x		
6				x				x				x	
7					x				x				x

One can imagine constructions such as *3-of-5*, *4-of-7*, and so on. If you know about the binomial coefficients $\binom{m}{k}$ (the number of ways to choose k objects from a set of m objects), then it is not so tricky to count the number of sets in such collections: for sets of size r in a t-*of*-$(2t-1)$ construction with n elements, there are $\sum_{j \geq t} \binom{2t-1}{j} \cdot \binom{n-2t+1}{r-j}$ sets. It can be a challenge to determine which of these is largest for various values of n, r, and t.

But it turns out that there is another construction, due to Anthony Hilton and Eric Milner, which is at least as big as (and usually bigger than) all of these. To describe it, let's use the elements $\{1, 2, \ldots, n\}$. We start with the set $12 \cdots r$ and then include every r-element set that contains n and at least one of 1, 2, ..., or r (see below for the case $n = 7$ and $r = 3$).

1	x	x	x		x	x	x						
2	x	x		x				x	x	x			
3	x		x	x							x	x	x
4					x			x			x		
5						x			x			x	
6							x			x			x
7		x	x	x	x	x	x	x	x	x	x	x	x

In this instance, the Hilton-Milner collection has the same size as the 2-of-3 collection above it. But in general, it has $\binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1$ sets, which is not only bigger than the above constructions but also bigger than every other possible non-star construction.

[1] If $r > n/2$ then every pair of sets intersect, so the entire collection is intersecting. For example, when $n = 5$ and $r = 3$ the collection of all ten triples of $\{1, 2, 3, 4, 5\}$ is 123, 124, 125, 134, 135, 145, 234, 235, 245, and 345.

4 A New Paradigm

Because of the various instances in which there might be prohibitive relations that prevent some pairs of elements from being in the same set (such as making committees that avoid antagonistic pairs of people), we can model more general situations by graphs whose edges represent such prohibitions, as we've done for separated triads. In such graphs, we're looking for the largest intersecting collection of independent sets of size r.

Fred Holroyd first introduced these general investigations, which gave rise to a very challenging conjecture that is still unresolved today. The *independence number* of a graph $\mathcal{G}$ is the size of the largest independent set of $\mathcal{G}$ and is denoted $\alpha(\mathcal{G})$. For example, $\alpha(\mathcal{C}_{12}) = 6$ (one cannot do better than choosing every other vertex of a cycle) and $\alpha(\mathcal{C}_{12}^{(2)}) = 4$ (consider $A\ C\ D^\sharp\ F^\sharp$). Holroyd proved that if $r \leq \alpha(\mathcal{C}_n)$ then the biggest star is the largest intersecting collection of independent sets of size r. John Talbot then proved the same result for $\mathcal{C}_n^{(2)}$ when $r \leq \alpha(\mathcal{C}_n^{(2)})$. (He actually proved this for $\mathcal{C}_n^{(k)}$, the graph that joins each vertex to its k closest neighbors on the cycle.)

More subtle is the *minimax independence number* of $\mathcal{G}$, denoted $\mu(\mathcal{G})$. An independent set is *maximal* if it is not contained in some larger independent set. For example, $A\ C^\sharp\ F$ is maximal, though not maximum, in $\mathcal{C}_{12}^{(2)}$. Then $\mu(\mathcal{G})$ is the size of the smallest *maximal* independent set. Holroyd and Talbot conjectured that, for any graph $\mathcal{G}$, if $r \leq \mu(\mathcal{G})/2$ then the size of the largest intersecting collection of independent sets of size r is achieved by a star. This is a subject of very active current research. It is almost completely unknown, however, what kind of structure the largest non-star has; that is, there is nothing known about a Hilton-Milner type version for independent sets in graphs.

5 Rimsakovic Solution

Because of Talbot's result, we now know that the largest collection of Rimsakovic triads is the star collection in the opening chart (or other like it containing a fixed note different from C). But since Nick wants to avoid drones, maybe we should consider some of the other constructions we've learned.

For Hilton-Milner, we start with some independent triad. Since the construction needs a fourth independent element, the triad must be diminished, say $A\ C\ D^\sharp$, which makes the fourth element $F^\sharp$. Thus the collection consists of the 4 triads $A\ C\ D^\sharp$, $A\ C\ F^\sharp$, $A\ D^\sharp\ F^\sharp$, and $C\ D^\sharp\ F^\sharp$.

For the 2-of-3 construction we again start with the diminished triad $A\ C\ D^\sharp$ (other triads take up too much space). This yields the 8 triads $A\ C\ D^\sharp$, $A\ C\ E$, $A\ C\ F$, $A\ C\ F^\sharp$, $A\ D^\sharp\ F^\sharp$, $C\ D^\sharp\ F^\sharp$, $C\ D^\sharp\ G$, and $C\ D^\sharp\ G^\sharp$.

Now, this doesn't mean that there isn't some yet unthought of construction of 9 Rinsakovic triads. Is there? While you're pondering that question, Nick will start to experiment with various sequences of those 8 triads on his piano. Some are kind of haunting.

References

[1] A.J.W. Hilton and C.L. Spencer, A generalization of Talbot's theorem about King Arthur and his Knights of the Round Table, *J. Combin. Theory*, Ser. A **116** (2009), no. 5, 1023–1033.

The Doodle theorem, and beyond ...

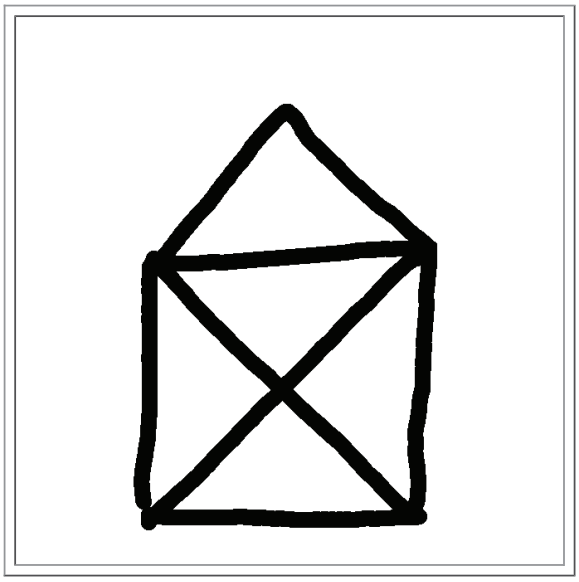
Colin Wright, Solipsys Ltd

One of the things I like about recreational maths; is how we can start with a simple game, play around a bit, poke in the corners, and suddenly fall down a deep hole into some serious mathematics. In this article we start with some well-trodden ground, which some readers will find familiar. However, we quickly find that all is not as it seems, and we soon stumble over a veritable pot of gold. To see how, read on ...

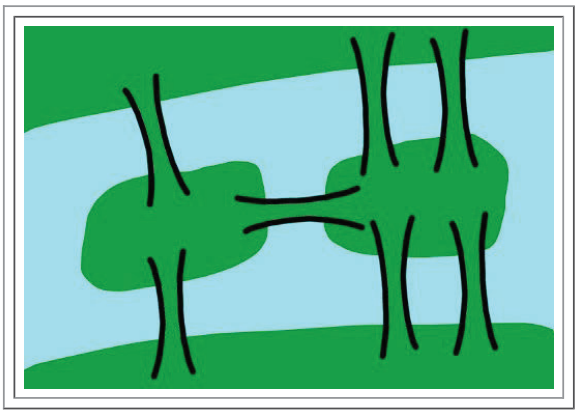
A simple game

There's a game many children are introduced to, one way or another. It takes several forms - one is this:

Here's a drawing: see if you can reproduce it without going over any lines you've already drawn, and without lifting your pencil from the paper.



As it happens, this was exactly the challenge that faced the residents of Königsberg centuries ago: could they take their Sunday afternoon stroll, crossing each of the seven bridges in the city exactly once?



Some have it that the puzzle required that they return to their starting point, while others didn't add that extra condition, but over time all the residents came to agree that it was impossible.

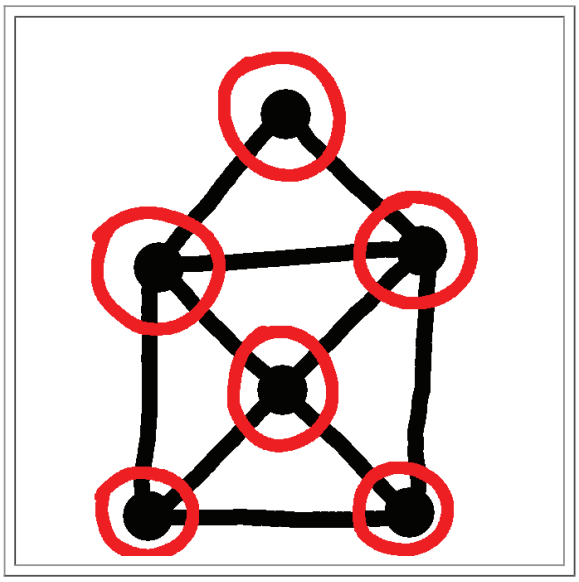
And that's where normal people leave this sort of puzzle. They try for ages, decide they can't do it, and put it down. On the other hand, mathematicians then take up the challenge: is it really impossible?

Can we prove it to be impossible?

That was done, as many readers will know, by Euler in 1736, and is marked by some as the birthplace (birthtime?) of Graph Theory. Euler reasoned that if it was possible to walk about and cross every bridge exactly once, then every piece of land you visited has to have an even number of bridges - one to enter and one to exit for every visit. The plan of Königsberg, however, clearly shows that every piece of land has an odd number of bridges. So it is clearly impossible.

And so it is when trying to reproduce a drawing. To be able to draw the diagram above (or any other diagram) without retracing lines, and in a single continuous stroke, every vertex, every meeting place, must have an even number of lines entering it. If not, we will enter and depart some number of times, and finally enter and have no way out.

Of course, it's OK for the starting point and ending point to have an odd number of lines, but those are the only two. Returning to our original puzzle, we can see that we must start at one of the bottom two points and end at the other, but since every other vertex (meeting place) has an even number of lines, it must be doable.



Have you seen the trap? I'll give you a moment. It's subtle until you see it, and some people still don't really understand, even when it's pointed out.

Have you seen it? Take a moment.

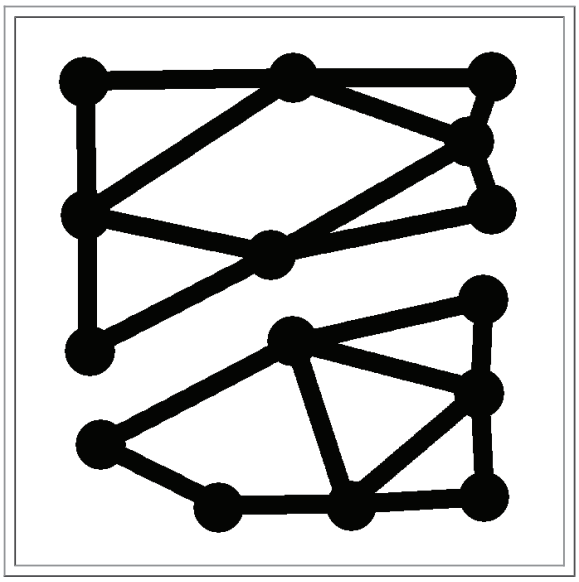
So what is the trap? I have followed in Euler's steps and shown that if a diagram can be drawn then all but at most two of the vertices must have an even number of lines.

I then asserted - without proof - that the converse was also true. I said that if every vertex has an even number of lines meeting it, then it will be drawable.

"Well", say many people. "It's obvious, isn't it? With an even number at every vertex, every time you enter you can leave again, so nothing can go wrong. Only stands to reason."

Ah, but it's *not* true. Even though most people get left with the impression that this is true (and most of the time when this is shown to children the implication is obvious) it actually isn't.

At this point some of you will be spluttering, but others will be nodding along. Here is an example of a diagram where all the vertices have even degree (an even number of lines) and yet it cannot be drawn in a single stroke:



Now some people complain that I've tricked them, and of course they were only thinking of *connected* examples. Well, fair enough, if you were only thinking of connected examples then maybe it is true. But is it? Really? How do you know?

And that's where we need to have a proof. So here's a conjecture:

- Every connected network in which each vertex has even degree can be completely traversed in a single journey that returns to its starting point and does not retrace any paths.

Now you'll notice here that I've added the condition of returning to where we started, because that now means that we can't have the alternate condition of two vertices having odd degree. It just makes life simpler, and it's not hard to prove the more general case from this one.

So how do we prove this? There are several proofs, and people differ as to which they think is the easiest and cleanest. The one I like is the following. Take a walk, and return to where you started. You never get stuck when you do this, because at each vertex you use lines in pairs, so every time you come in, there's always a way out. If you covered every path, you're done.

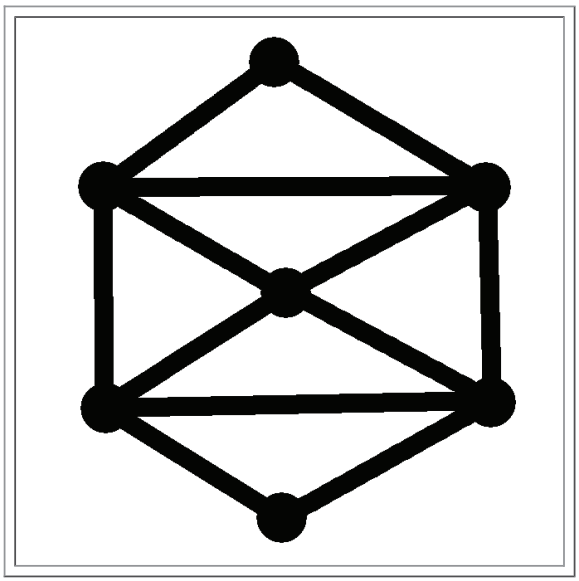
If not, do it again, but when you come across one of the paths you didn't take last time, take it now as a diversion, and wander about on paths you didn't cover the first time until you return to that branch point, and then complete your original journey. You've now covered more of the network. Keep doing this, keep augmenting your journey, until finally you've covered everything.

That's not a formal proof, but it gives the right idea. You need to check that the things I've told you to do are always possible, but that's not too hard.

So in this case the converse is true:

You can draw the diagram if and only if every vertex has even degree.

It's interesting to note that neither the statement nor the proof actually require that the network be drawn on a plane as a diagram. It's true for networks in general, which is nice. But now, let's consider the case where it *is* drawn on a plane. Here's an example:

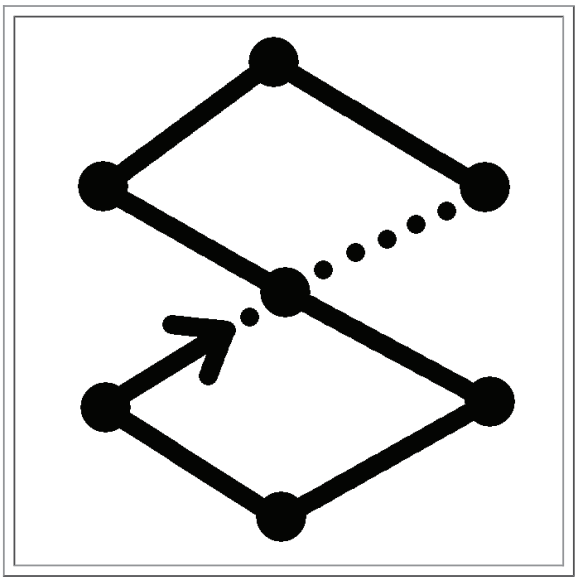


Every vertex has even degree, so we can draw the network, returning to our starting point, without retracing steps, and without retracing a line we've already drawn.

Adding a twist

So far so good.

Claim: We can draw this in a single pencil line without crossing over a line we've already drawn.



Try it. Here's an example where it's gone wrong:

In this attempt we've made a good start, but got ourselves into a position where as we approach the next vertex we can see that we will have to cross over a line we've already drawn. So that's not allowed.

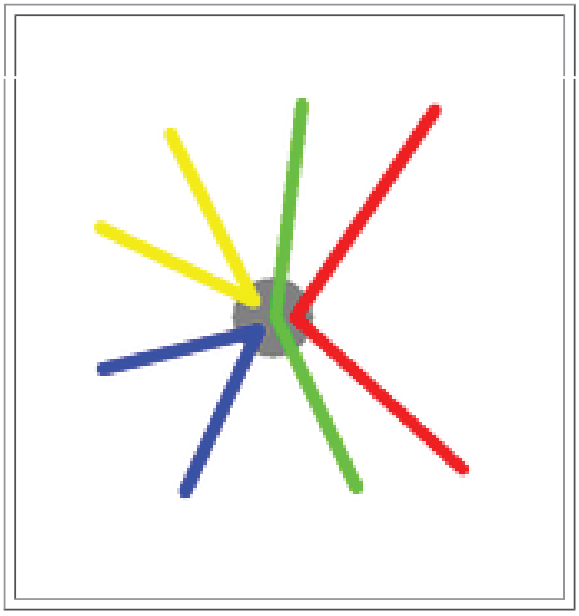
Is it possible? Try for yourself ...

Again, if any vertex has an odd degree then we won't be able to do this, but now the claim is that the converse holds, and it's no longer quite so obvious. A few examples and you might convince yourself that it works, but just because it works on a few examples, that doesn't mean it's *always* true. So is it always true?

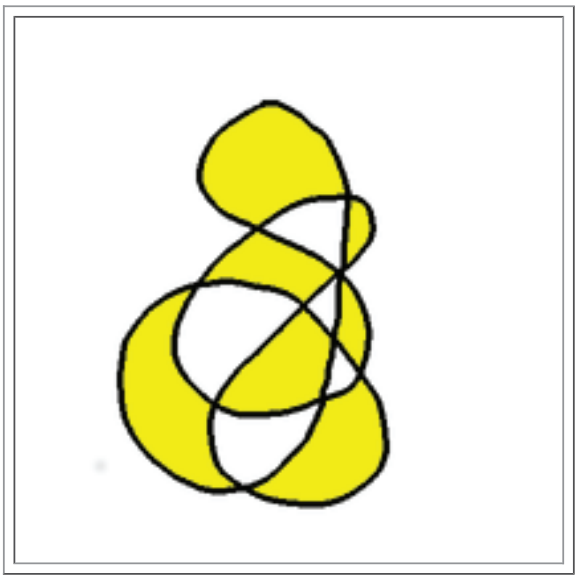
Can we make every vertex look something like this?

Really?

Well, yes, we can make every vertex look something like that, and the proof is quite similar to the one we gave above for the original problem. We won't go into it here, we'll leave that as an exploration for the interested reader.



A different challenge



Instead, let's make another observation about a network drawn on the plane, where all the vertices have even degree. Because of our first theorem we know that it can be drawn in a single line. So take any such "doodle", and try to colour it in, checkerboard style. Here's an example:

You will find, I claim, that every doodle can be coloured like this using just two colours. Try it. Again, try a few examples. Get someone else to draw a doodle, and try to colour that. Try to draw one that *can't* be coloured.

You'll find that you can't, no matter how hard you try.

Claim: Every doodle can be two-coloured.

Again, at this point normal people will say "huh, that's curious" and then move on. Mathematicians, on the other hand, will wonder *why* this is true. Indeed, they will start by wondering *whether* it is true. Sure, it worked for all the small examples they tried, but what if there's a doodle with a million billion regions, will it still work for that?

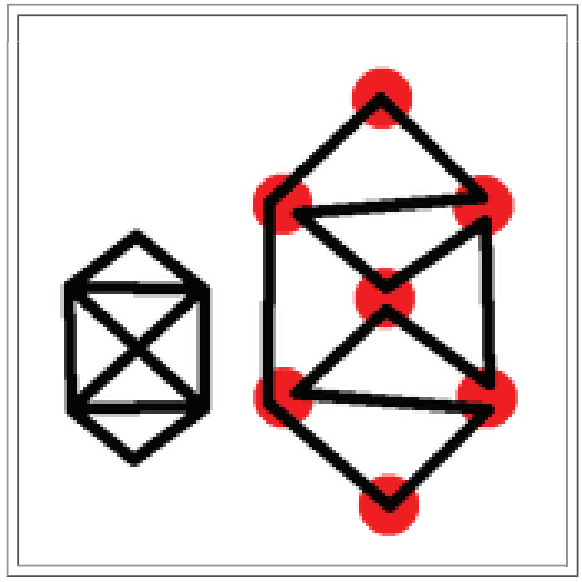
Good question.

Yes.

Here's a proof. We've already seen (although not proved here) that any doodle can be drawn in a single, continuous pencil stroke such that:

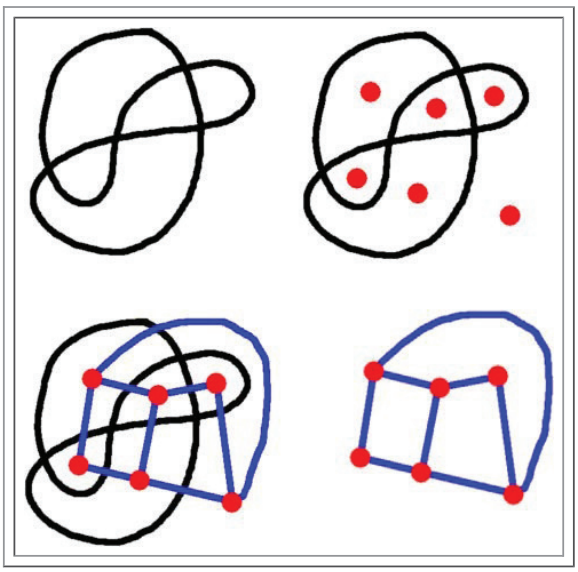
- No line is retraced
- We finish where we started
- We never cross over a line we've already drawn.

That means the vertices can be sort of "exploded" into small regions of their own, and the path enters and exits the region without ever touching the other parts of the path that visit that vertex. Now the doodle is simply a distorted circle, and that means it has an inside, and an outside. We can shade the



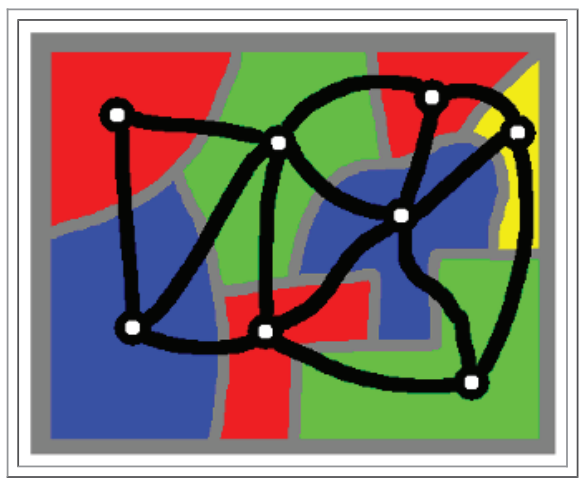
inside, and now we have a two-colouring of the original doodle.

Into the unknown



From doodles to networks

Of course, colouring regions like this restricts us to living in the plane, but there's a way to release ourselves from this limitation. What we do is think of the regions on the doodle as countries, and put a capital city in each country. Then if two countries share a border, we join their respective capitals with a road. In this way we end up with a new network, a network of roads. Colouring the regions corresponds to colouring the capitals, and the rule is now that if two capitals are joined by a road then they must get different colours. Here we can see a general example, not just one that's made from a doodle:



So now instead of a map we have a network, and instead of colouring regions, we are colouring vertices. The reason this is useful is that networks are an abstract concept, not limited to living in the plane. With a map in the plane, for example, it's impossible to have five regions that all border each other, but with a network we can simply declare that we have five vertices and edges between them all. We have gained flexibility and generality, but we have now lost our proof that certain networks/doodles are two-colourable.

But we can rescue that. In our new network, imagine moving from vertex to vertex, travelling around the network and returning to where we started. If the network vertices can be two-coloured (and remember, they correspond to the regions in the original doodle) then we must alternate colours: black, white, black, white, and so on. So any wandering around the new network must consist of an even number of steps.

So now we have a simple way of deciding whether or not a network can have its vertices two-coloured: check to see if

every circuit is of even length. That sounds like a huge job, but actually there is a really easy way of doing it. Colour one vertex red, then all its neighbours green, then all their neighbours red, and so on. If you succeed then you've proved that all the circuits are of even length, because when you traverse a circuit you keep alternating colour.

But if that process goes wrong it's easy enough to show that there must be an odd length circuit. We'll leave that as an exercise for the dedicated individual.

So where is this pot of gold?

So we have shown that there is an easy way to see if a network can be drawn without lifting the pencil off the paper. The same test, to check whether every vertex is of even degree, even works for networks not necessarily limited to the plane.

We've also seen that there is an easy way to discover whether a network can have its vertices two-coloured: it's possible if and only if every circuit is of even length, and we can test that just by trying it! If it fails, we find an odd length circuit. If it succeeds then, well, it's succeeded.

So there is a simple test to see if we can visit every edge of a network, and there is a simple test to see if we can two-colour the vertices of a network.

What next?

Well, we can ask if there is a simple test to see if we can create a cycle that visits every *vertex* of a network exactly once. Or we can ask if there is a simple test to see if it's possible to *three*-colour the vertices of a network.

And here is the fun part: no one knows.

The first question - finding a cycle that visits every vertex - is called finding a Hamilton cycle, and we know of no easy way either to find one, or to prove that there isn't one. The second question is simply called three colouring, and there is currently no known efficient way of knowing whether an arbitrary network can be three-coloured or not. We just don't know.

People have investigated these questions for a long time and they have discovered something really cool. If you can solve one of these problems, you can use it to solve the other. So in some sense these two problems, even though they look *totally* different, are kind of the same.

And there's more. These two problems are not an isolated pair. There are hundreds of problems that are all basically equivalent. And because they are equivalent, solving any one of them will thereby solve them all, and similarly, showing that any one of them has no solution will as effectively show that none of them have solutions.

This question, do these problems have efficient solutions or not, is known as the P vs NP problem, one of the Millennium Prize Problems published by the Clay Mathematics Institute, and for which they have offered a one million dollar prize.

Well, that escalated quickly.

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FIBONACCI PAVILION

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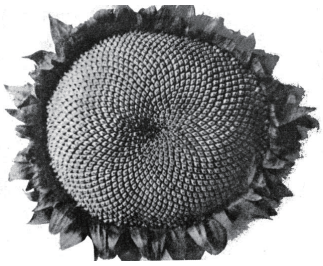
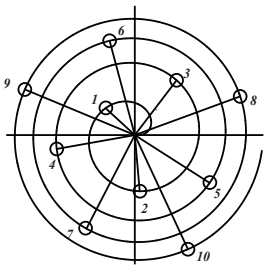


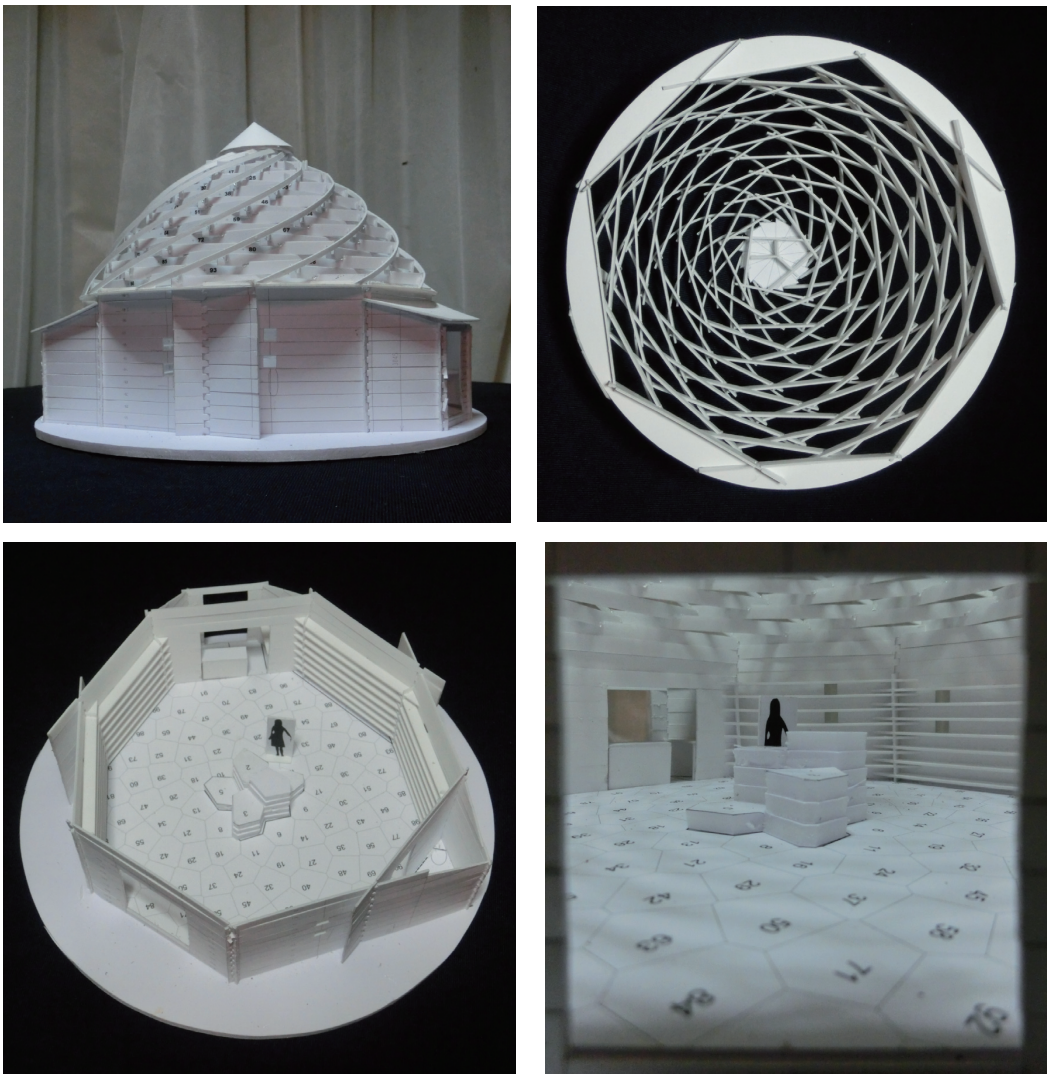
H. Weyl "Symmetry" (1952)

The Phyllotaxis is not only an old subject but also a new subject of science. Most plants have used the golden ratio for several tens of millions of years. Since they can get maximum sunlight and stand stably, they are very successful today. It is the most ecological structure on the earth.
We should learn from plants to build bright, well-ventilated, and stable architecture.

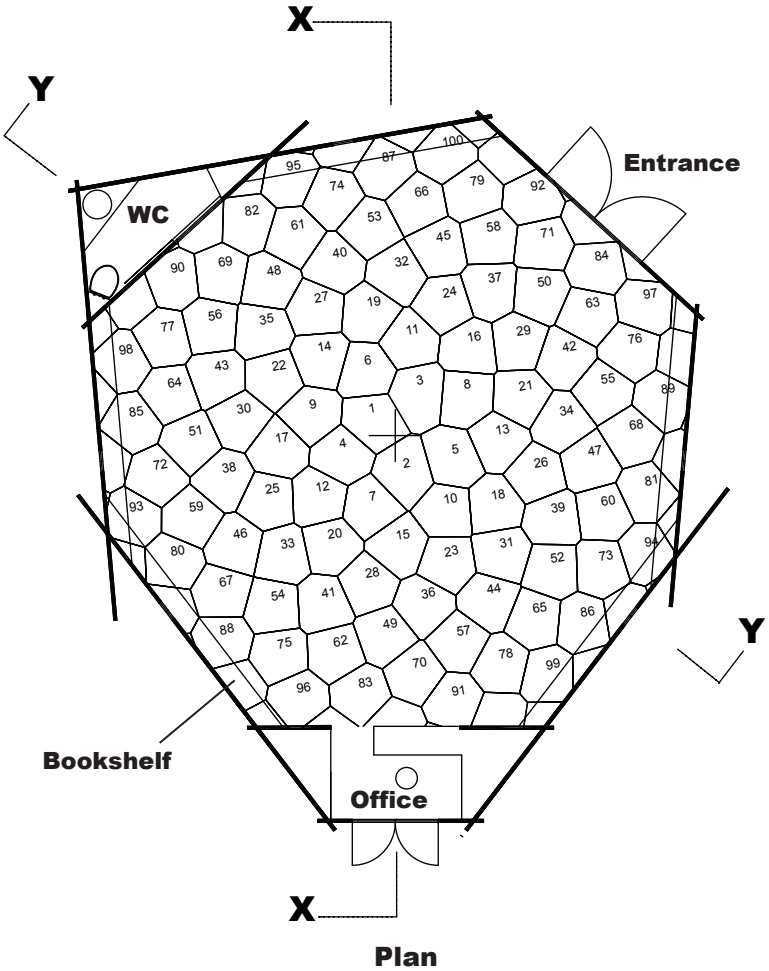
This proposal is just a sample as a prototype.
However, this principle should become a general architectural form in the future.
It will be built anywhere in the world as a house, pavilion, temple, museum, library, theater, or stadium using any materials (lumber, log, bamboo, metal, etc.) on any scale that we want.
In this sample, the main structure consists only of two-by-twelve pieces of lumber for simplicity.

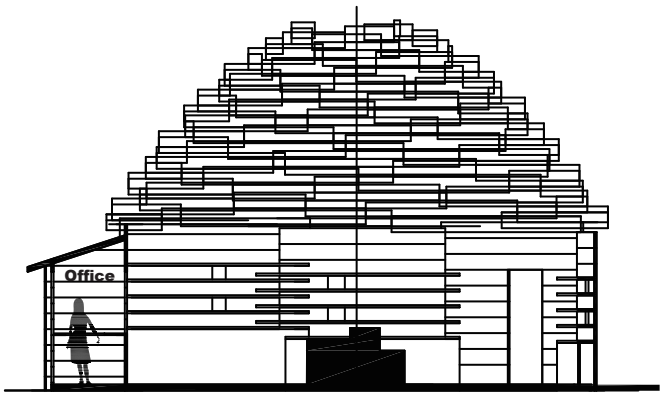


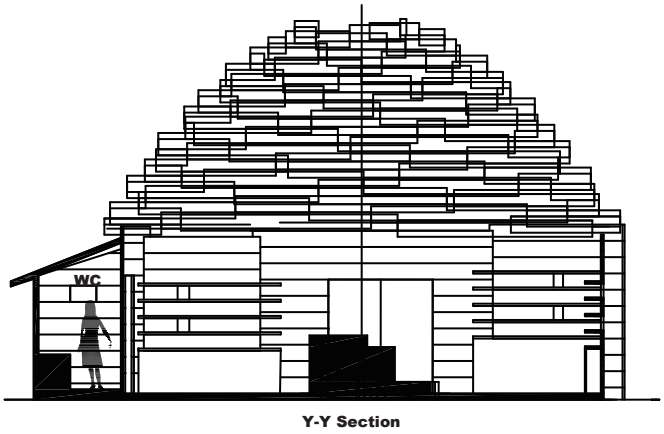
$r_n^2 = a\theta_n$
$\theta_n = 2\pi(2-\tau)n$
$n = 1,2,3,4,5,\cdots$
(where τ denotes the Golden Ratio)



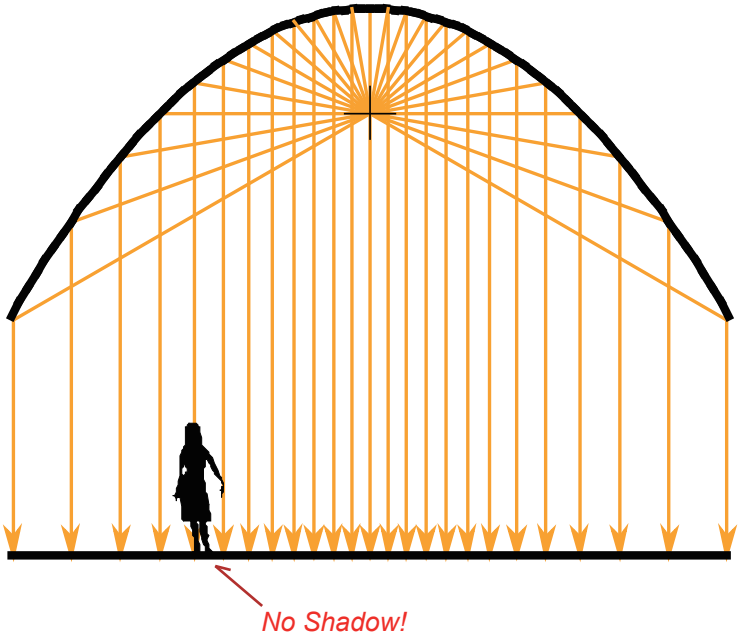
Scale Model 1/40







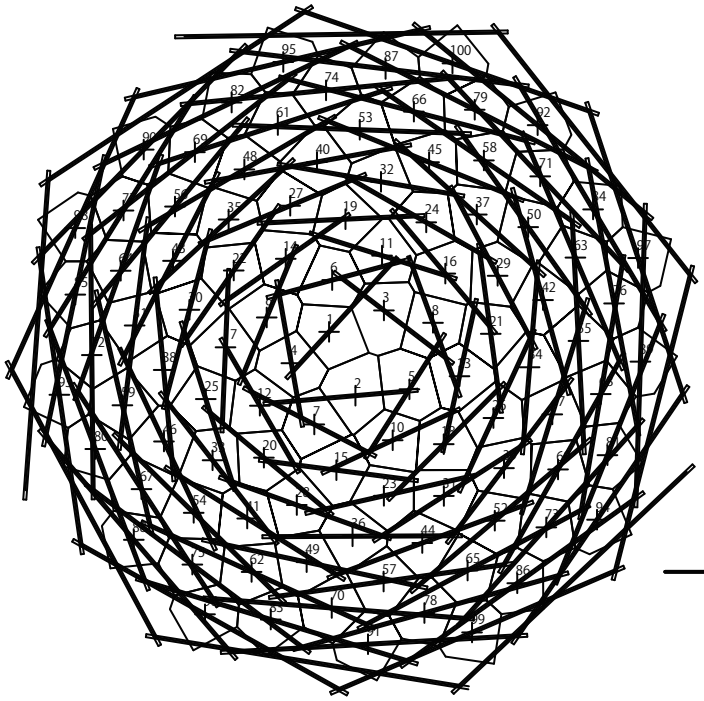
The floor is tiled by the Voronoi tessellation based on the Phyllotaxis.



Wall and Roof Structure

You can find many spirals of phyllotaxis. It is the most earthquake-proof structure. In general, the phyllotaxis (Golden Ratio) diffuses most effectively any kind of wave, that is, light and sound.

The roof structure forms the paraboloid. It is the most elegant way to construct the parabolic antenna.
Only one lamp is enough in a room. For example, if the lamp is at the focus of the paraboloid, all reflected lights fall exactly vertically. There will be no shadow!
I believe that it is special acoustic space.

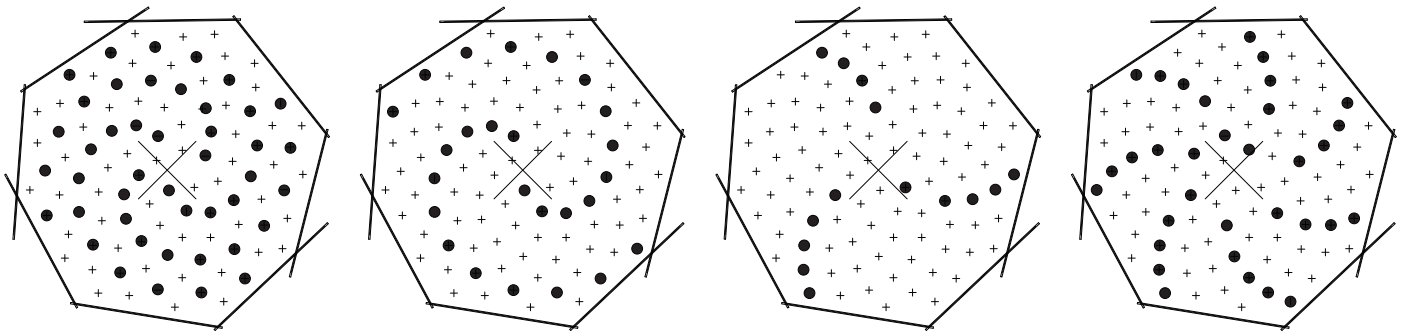


matching of beams and Voronoi cells

suspended lamps

Another Lighting Effect

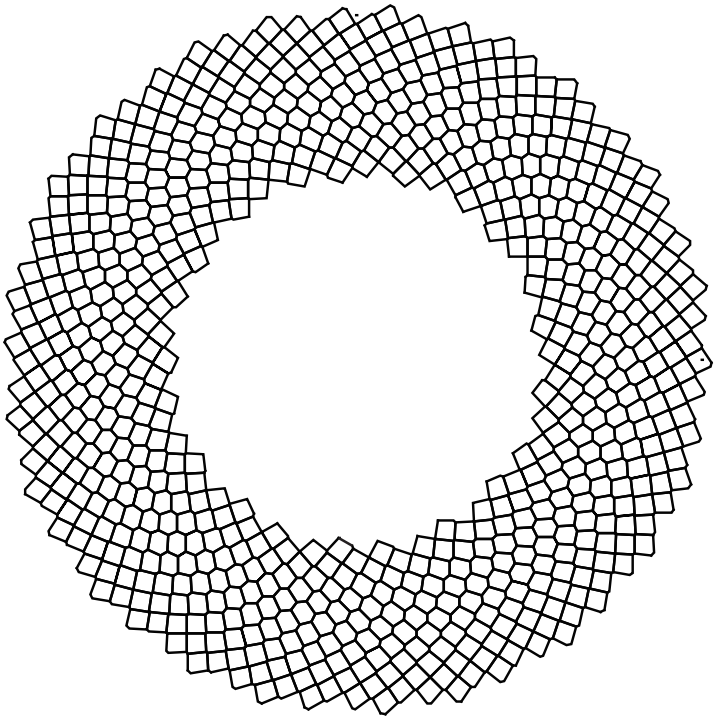
The center of each cell is matched to the center of each beam precisely.
This fact is very useful for lighting effect.
One hundred pendants will be hung from the beams. They light the center of each cell from above as a one-to-one correspondence.
We can use such various spiral patterns for lighting and display. The figures below show some examples of light arrangements. There are various flexibilities.



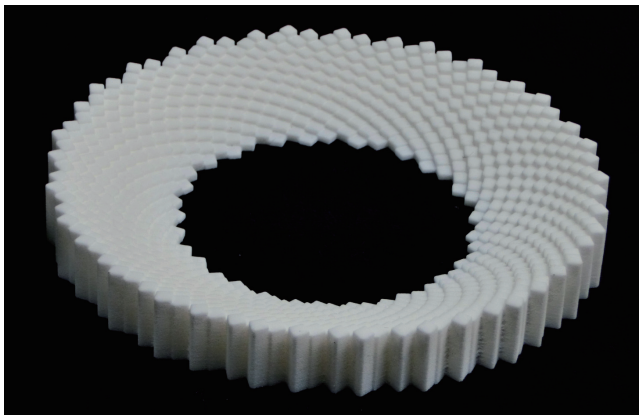
Variations of display and lighting pattern.

EXTENSION

We can extend the principle of this structure.
We can change the height of each Voronoi cell at will.
For example, the following figures are a proposal for an ideal stadium.
In such a stadium, every member of the audience can view the center stage.
Of course, we can build the roof structure in the same way as the Fibonacci Pavilion.
I named this stadium "FIBONACCI COLOSSEO".

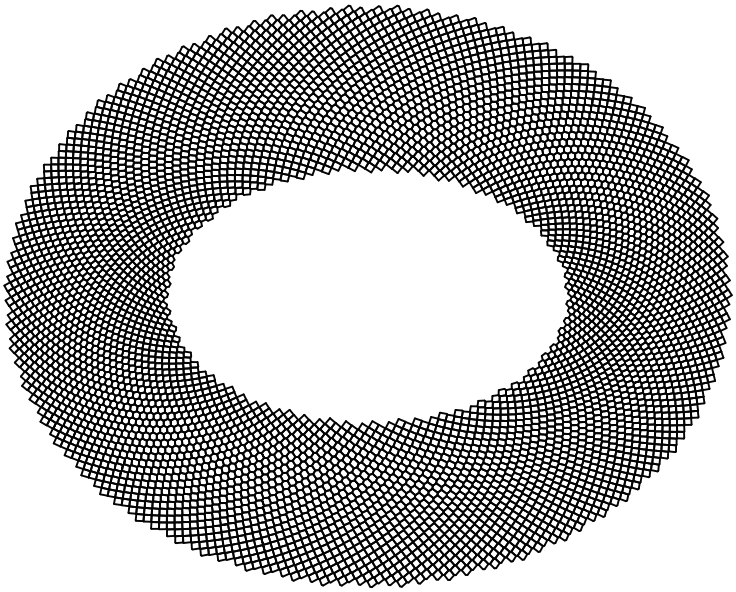


Plan

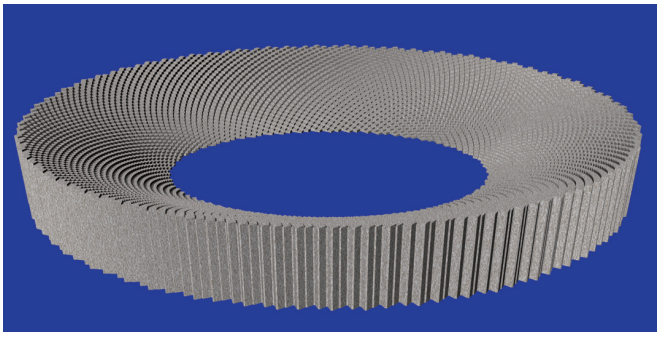


Scale Model

Round Stadium



Plan



Perspective

Oval Stadium

Hexaflexagons and the Other Feynman Diagrams

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Abstract

In my G4G12 gift, I describe what I think must be (or *should* be) well-understood properties of the hexaflexagon structure diagrams that did *not* appear in the *Scientific American* article that spawned Martin Gardner's "Mathematical Games" column. I also share the embarrassing story of how I came to know who originated these diagrams.

A Mortifying Incident

There are some embarrassments that are hard to forget.

I grew up reading Martin Gardner's collected *Scientific American* columns, and like so many before me, I developed a passionate interest in hexaflexagons. By the time I was an undergraduate, I had worked out the classification of possible hexaflexagon structures, made dozens of color-coded hexaflexagons, and even organized a January-term course on hexaflexagons for some of my classmates. So, as an enthusiastic but—in retrospect—young, naïve, and awkward graduate student in the mid-nineties, I decided to give a hexaflexagon talk in our graduate student seminar.

The seminar in question was freewheeling, though the topics were usually less recreational and more connected to our graduate work. The broad focus and lighthearted ethos stemmed from the seminar's unusual origin story. This was in the Harvard Mathematics Department, where at some point in the murky past, the faculty had established a seminar series for the benefit of the grad students called "Basic Notions." In theory, this was a lovely educational opportunity. In *practice*, visitors would accept invitations to speak and, in an effort to impress the world-famous Harvard math professors, give talks that were completely incomprehensible to students. By way of revenge, the graduate students started their own seminar, called "Trivial Notions."

When I was getting my Ph.D., Trivial Notions was well established as a friendly series of talks by and for graduate students in which we would share things we were working on or had stumbled across that we thought our classmates would enjoy. It was understood that the faculty were *not* to attend, although from time to time a visiting faculty member, unaware of the unwritten rule, would sit in on our meetings. What was unprecedented was for someone *else* to attend the seminar. But when I gave my hexaflexagon talk, there was an elderly couple in the audience that none of us recognized.

I began my talk with a brief account of the discovery of hexaflexagons by a group of graduate students: Tuckerman, Tukey, Feynman, and this other guy whose name I could never remember. Then I showed the trihexaflexagon and the standard hexahexaflexagon, as in the original Martin Gardner article, and ran through the general theory that produces arbitrary hexaflexagons as I (and countless predecessors) had worked it out.

After my talk, the man of the unknown couple came up and started commenting on what I had written and drawn on the blackboard. It quickly became apparent from his remarks that he must have known at least some of the original Flexagon Committee, making me even more confused about his mysterious presence. I took in what he said in bemused wonder, but it was one of my lingering classmates who had the presence of mind to ask who he was.

"Arthur Stone."

"The Arthur Stone who invented hexaflexagons?"

"Yes."

I don't recall much after that, though apparently I managed not to drop dead of embarrassment. I can't even remember if I had the good grace to apologize to Stone, who had seen the talk listed as one of the events in the Harvard Mathematics Department and decided he had to check it out, for not crediting him for his discovery, though given my social skills at the time I'm inclined to doubt it. But I have never forgotten Arthur Stone's name since.

Mapping Hexaflexagon Structure

What Stone remarked upon in particular was a set of diagrams I had drawn on the board, the first few of which are shown in Figure 1. These diagrams represent the structures of the tri-, tetra-, and pentahexaflexagons in the first row, and the three different hexahexaflexagons in the second row.

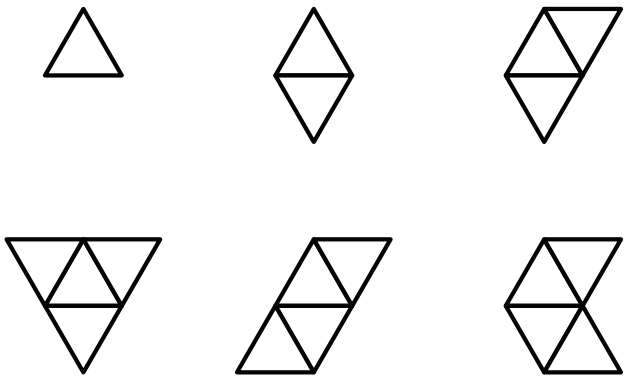

Figure 1: *Structure diagrams for the six smallest hexaflexagons as I prefer to draw them.*

Stone told us that Richard Feynman had diagrammed hexaflexagons in the same way, except that he had drawn the diagrams as polygons (I presumed regular ones) triangulated by diagonals, as

in Figure 2. This connects the question of counting the number of different hexaflexagons with *n* faces to combinatorial questions about triangulations of polygons.

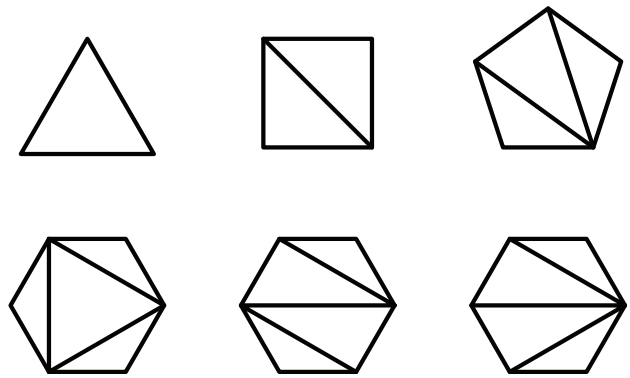

Figure 2: *Richard Feynman's hexaflexagon structure diagrams.*

Readers who are familiar with Martin Gardner's original hexaflexagon article [1] will note that diagrams of the types in Figures 1 and 2 do not appear there. There is a structure diagram for the standard hexahexaflexagon, but it is of a different type pegged to the Tuckerman traverse method of sequentially reaching every face in the hexahexaflexagon. Figure 3 shows a simplified version of this diagram, in which a Tuckerman traverse corresponds to traveling counterclockwise (or clockwise) around the outside of the diagram.

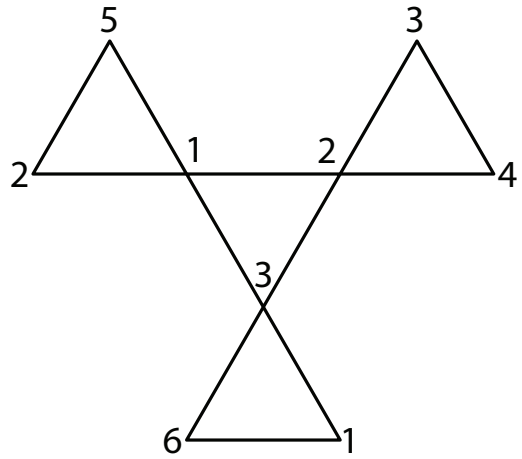

Figure 3: *A sketch of the Tuckerman traverse diagram of the hexahexaflexagon from Martin Gardner's* Scientific American *article [1].*

I am sharing this account with Gathering for Gardner for two reasons: first, because I think the Arthur Stone story is pretty funny, and second, because there are wonderful properties of the

Feynman-style[1] structure maps that I am sure must have been discovered many times over, but that I have not found written up in any concise and unified way. I will now briefly describe those properties in the hope that they may be of interest, but more importantly because if there is a nice account of these readily available, surely one of you can point me to it. If there is none, I might consider expanding these remarks so that the ideas are more widely accessible.

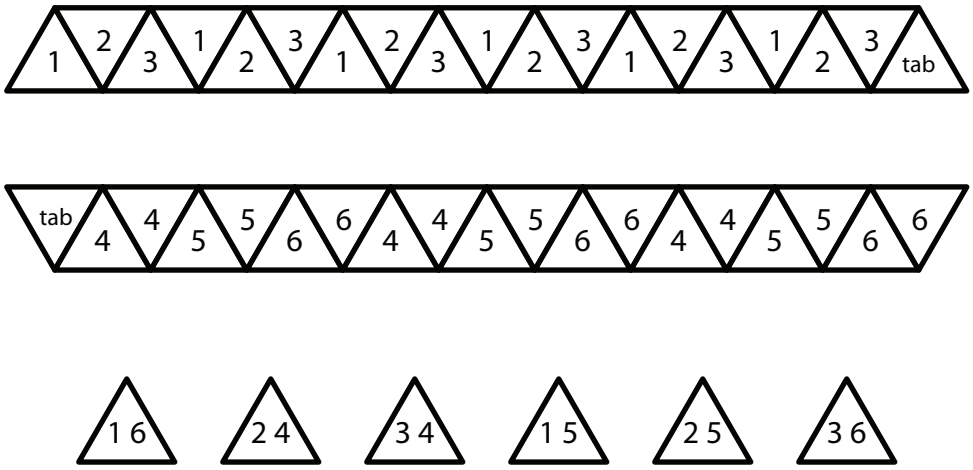

Figure 4: *The net for the hexahexaflexagon, and the six panels that comprise it.*

The net for the standard hexahexaflexagon is shown in the top and middle rows of Figure 4. When the net is folded and glued, the resulting hexaflexagon contains 18 triangles of paper, each with a number on either side. We use the terminology *panel* to describe each distinct numbered triangle that appears; in this case, the panels (pictured at the bottom of Figure 4) bear the number pairs 1 6, 2 4, 3 4, 1 5, 2 5, and 3 6. We denote these panels as (1 6), (2 4), and so on. In the hexaflexagon, these panels are stacked into what Oakley and Wisner [2] termed *pats*; in each possible position, there are two distinct pats that alternate around the hexaflexagon, each appearing three times.

Figure 5 shows the numbered diagram for this hexahexaflexagon. The six vertices naturally represent the six numbered faces of the hexaflexagon. But the key to reading these maps is in the edges. The position of a hexaflexagon—and the positions accessible via a single flex—are determined by which faces are currently on the top and on the bottom, which is essentially a directed edge of the diagram. For instance, at the top of Figure 3, the marked edge indicates the position (1,2) in which 1 is the bottom face and 2 is the top face, represented by an arrow pointing from 1 (the bottom face) to 2 (the top face). A single flex will move the arrow to another edge of one of the triangles containing (1,2); moreover, since the flex pushes face 2 from the top to the bottom, this arrow will point from 2 to another face. Therefore, as shown in the bottom row of Figure 5, the position after a flex will be either (2,3) or (2,5).

[1] Admittedly, Feynman may also have used the type of diagram in Figure 3, but if so, Stone did not mention it. As I hadn't drawn any such diagrams, there's not much to be deduced from this either way.

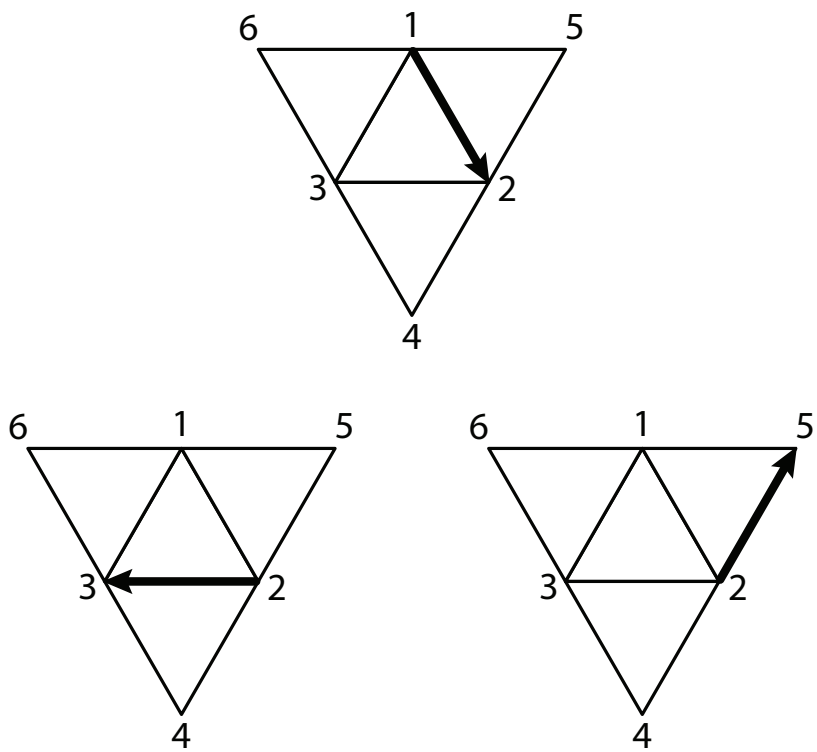

Figure 5: *The diagram for the standard hexahexaflexagon, and the two possible flexes from position (1,2)*

We can further deduce from the diagram that there are two positions accessible from (2,3), namely (3,1) and (3,4), but that there is only one position accessible from (2,5), namely (5,1). We call positions such as (5,1) from which only one flex is possible *terminal positions*.

The essential observations about how the "Feynman" map for any hexaflexagon reflects its structure are as follows.

1. A hexaflexagon with n faces has exactly n terminal positions, not counting which of the faces is on the bottom or the top. (If you count the order, there are $2n$ terminal positions.)

2. Each of the n terminal positions corresponds to one of the n panels in the hexaflexagon. In particular, up to the order of the faces, the 6 terminal positions in the hexaflexagon in Figures 4 and 5 are (1,6), (2,4), (3,4), (1,5), (2,5), and (3,6). Thus, we can read the panels off of the exterior edges of the diagram.

3. In each position, the sets of exterior edges on either side of the edge marking that position correspond to the panels in each of the two distinct pats. For instance, when the hexaflexagon in Figures 4 and 5 is in the position (1,2), one pat contains panels (1 5) and (2 5), and the other contains panels (1 6), (3 6), (3 4), and (2 4). When we flex to position (2,5), the panel (1 5) moves from the thinner pat to the thicker one.

Notice that this final rule explains why the external edges correspond to terminal positions: a flex passes one or more panels from one pat to another, and in the terminal position there is a pat with

only one panel in it that cannot be subdivided. With a little thought and experimentation, we can use these observations to determine the effects of "deleting" a face that is in two terminal positions (effectively gluing it shut by collapsing the net), and to discover a method to reverse this process, grafting a new face into any terminal position of a hexaflexagon.

If you know of a particularly nice source of this theory (or, for that matter, any source less technical than [2], which is so dense and sparsely diagrammed that I find it difficult to discern if the authors were aware of these structure maps), or of a straightforward account of how to construct an arbitrary hexaflexagon, please email me at sgoldstine@smcm.edu and tell me about it. And if you don't, then I hope you have learned something new that will encourage you to pull out paper, scissors, and glue and start playing!

References

[1] M. Gardner, Hexaflexagons, *Probability Paradoxes, and the Tower of Hanoi: Martin Gardner's First Book of Mathematical Puzzles and Games*, Cambridge University Press, 2008.

[2] C. O. Oakley and R. J. Wisner, *Flexagons*, The American Mathematical Monthly, Vol. 64, No. 3 (Mar., 1957), pp. 143-154.

Photo credit William Morgan

Hinge-Elastegrity's Shape-Shifting into all 5 Platonic Solids and the tesseract

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Chiral Icosahedral Hinge Elastegrity's Shape-shifting

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Introduction of Definitions and Origins of the Chiral Icosahedral Hinge Elastegrity

The presentation at G4G12 was collaboration between an architect interested in mathematics and a mathematician who frequently combines mathematics and art. In response to two Bauhaus design exercises assigned in graduate architecture school, the architect had developed over four decades a structure with interesting physical properties[1] (Fig. 1) that, led with the help of the co-author mathematician, to the discovery of mathematical objects examined here.

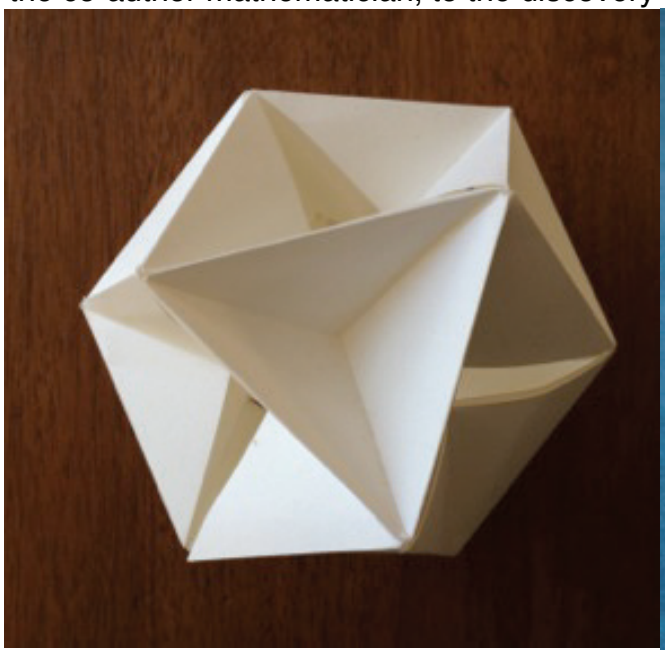


Fig. 1 Chiral icosahedral hinge-elastegrity Pavlides 1982

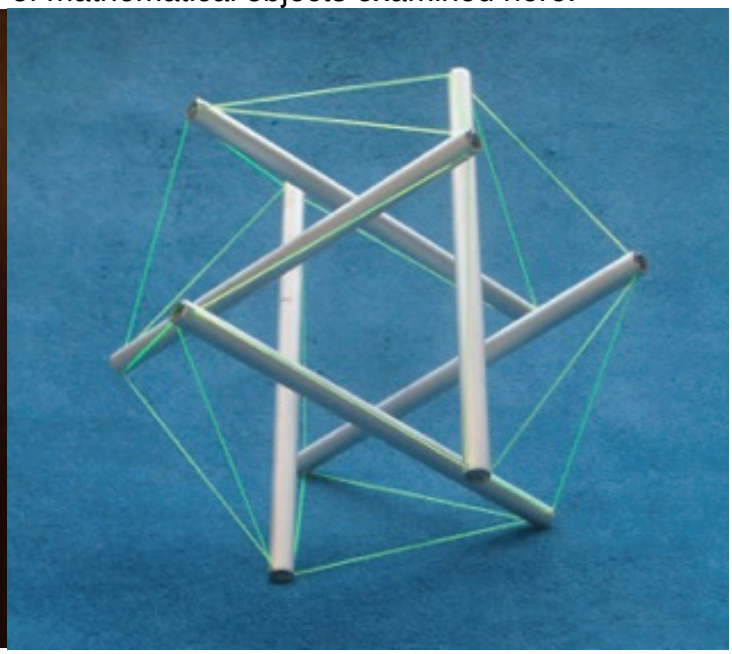
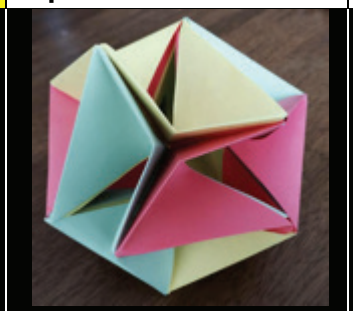
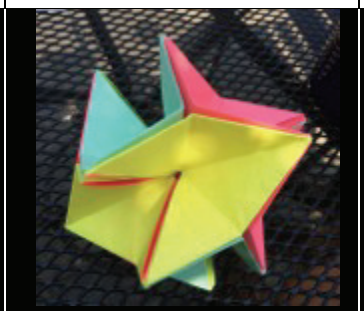
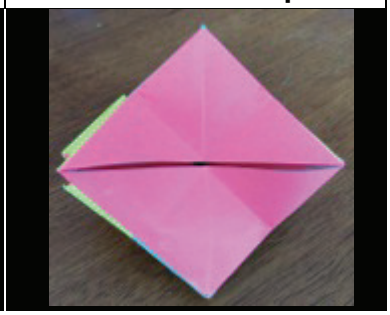


Fig. 2 6-strut-tensegrity Fuller 1948[2]

Named by analogy to tensegrity**,** (Fig. 2) (maintaining form integrity through tension alone), hinge-elastegrity (Fig. 1) (maintaining form integrity with elastic hinges resisting deformation) is illustrated above created with shape-memory membrane, for example paper, folded and woven into a network of rigid members and elastic hinges. The chiral icosahedron hinge elastegrity was created in 1982 but it was not until 1994 the connection to 6-strut tensegrity, that has the same symmetry, was made. It was named chiral icosahedron hinge-elastegrity in 2010. The Bauhaus exercises assigned at the Yale School of Architecture were followed by decades of play exploring structure, symmetry, and material. This investigation has been termed dactylognosis[3] as the fingertips crease, fold, and weave discovering what is possible using mostly paper with no preconceptions or intention to simulate. This led to chiral icosahedron hinge elastegrity shape-shiftings through folding, into all five regular solids (yellow highlight):

Chiral icosahedron	Squeezed cube-corner	12 vertices-structure	Flattened into square
			
Fig 3 Chiral icosahdron structure with 6-strut tensegrity symmetry	**Fig 4** right triangles of cube-corner squeezed into 3 hinges	**Fig 5** squeeze all 8 cube-corners to create a 12-vetices structure	**Fig 6** 12-vertice structure of Fig 5 flattened into a square

Cube	“Jitterbugging” cube	Monododecahedron	Regular dodecahedron

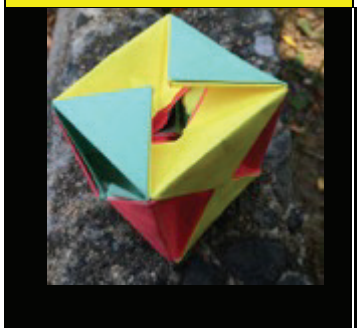
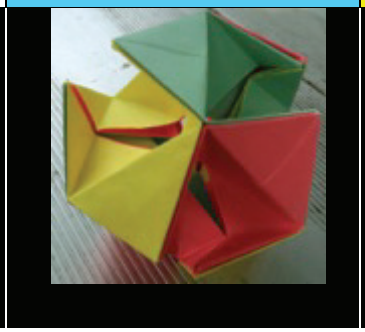
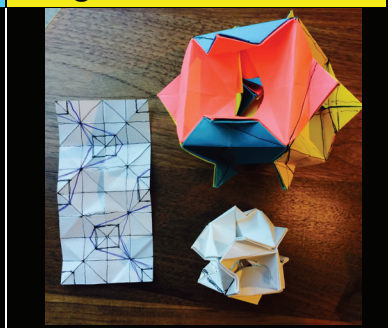
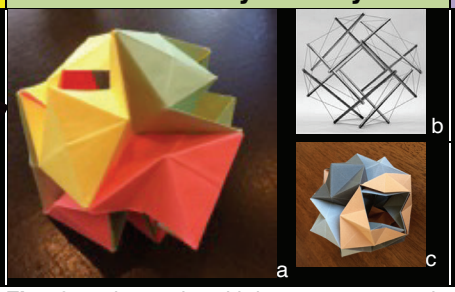
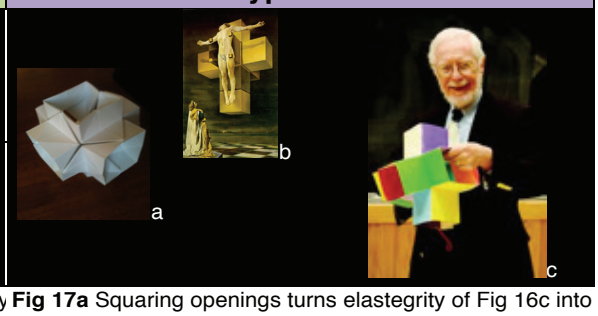

			

Fig 7 squeezed centers of Fig 5 become vertices of a cube	**Fig 8** The cube in Fig 7 gyrates in two directions into a flat square	**Fig 9** Rotating up 12 triangles by 38.17° morphs into dodecahedron	**Fig 10** Folding the rotated triangles of Fig 9 creates regular pentagons

Chiral icosahedron	3 intersecting squares	Semi-stellated 8-hedron	Octahedron
			
Fig 11 Chiral icosahdron structure with symmetry of 6-strut tensegrity	**Fig 12** Opening up Fig 11 cube-corners into 3 intersecting squares	**Fig 13** Bending 12 right tirangles into semi-stelated octahedron	**Fig 14** Pushing down the right triangle of Fig 13 into octahedron

Tetrahedron	12-strut symmetry	Hypercube
		
Fig 15 Right triangles of fig 13 pivot up into a tetrahedron	**Fig 16** a. elastegrity with b.12-strut tensegrity symmetry c. another view or a.	**Fig 17a** Squaring openings turns elastegrity of Fig 16c into hypercube similar to Dali’s “Corpus Hypercubus” (fig 17b) and Professor Banchoff’s model in Fig 17c.

As it shape-shifts, the chiral icosahedron hinge elastegrity adds creases fractally transforming the membrane (fig 18; fig 19), which then weaves into an elastegrity (fig 16a; fig 16c) with same symmetry as a 12-strut tensegtity (fig 16b) (green highlight above).

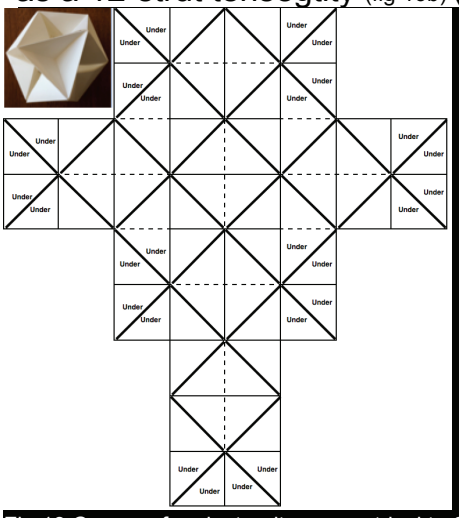


Fig 18 Creases for elastegrity symmetrical to 6 strut

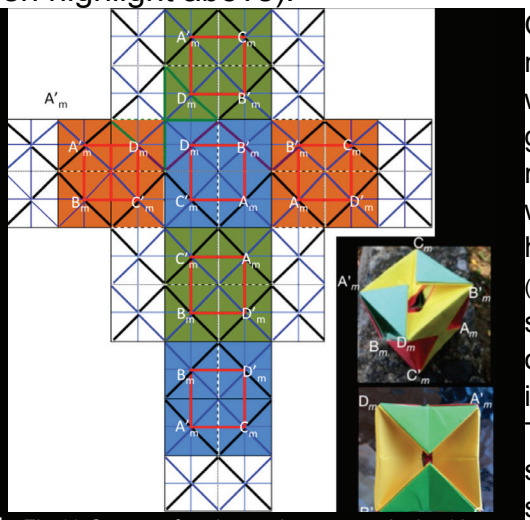
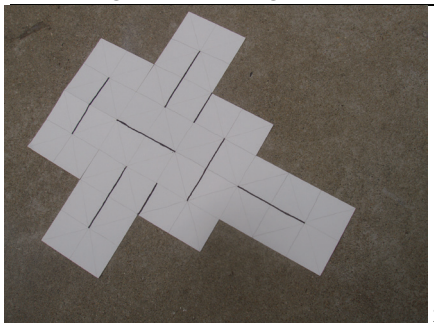


Fig 19 Creases for elastegrity symmetrical to12-strut

On the left (Fig 18) see a 6 slit-membrane. Slits are indicated with dotted line on 8X9 square grid. This flat shape-memory membrane (such as paper) weaves into a chiral icosahedral hinge elastegrity. On the right (Fig 19) see creases on a 16X18 square grid (fractal of the original), that were added as the icosahedron folded into a cube. The new creases weave into a structure with 12-strut tensegrity symmetry (Fig 16).

Folding & Weaving a Chiral Icosahedron Hinge Elastegrity out of single shape memory membrane[4]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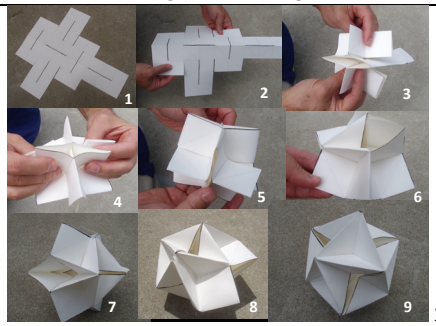

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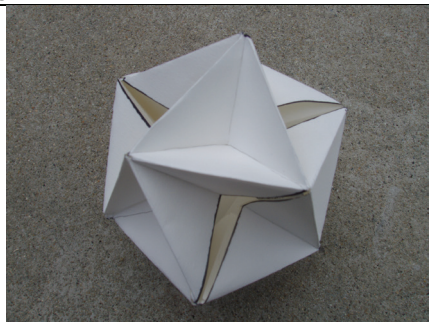
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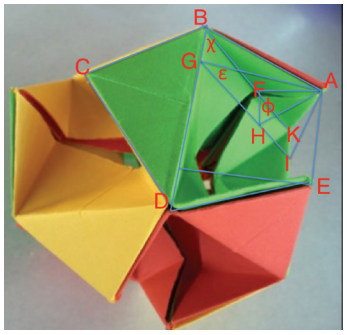
Fig 20 Flat paper with slits, folds and weaves into a hinge elastegrity (Fig 21-1 to 9) by weaving squares with slits over closed squares; this results in 3 intersecting squares (Fig 21-5); inverting a vertex into a 2 right triangle creates an elastic hinge as in Fig 21-6. Fig 22 When all 12 vertices of the 3 intersecting squares in 21-5 are inverted inside out, they form 12 hinges that suport the 8 isosceles trirectangular tetrahedra, now "floating" suspended by the elastic hinges, and a chiral icosahedron hinge elastegrity is created (Fig 21-9 & Fig 22).

The remarkable properties of the elastegrity models relate to many other structural challenges. The shape-shifting qualities of the fundamental model give new insights into structures that appear in classical contexts and some new configurations. In addition to the five regular polyhedra exhibited in specific positions of this single model, there are other shapes that appear during the transitions between different structures, showing the versatility of the original elastegrity figure. This leads, among other things, to a module for the construction of an unfolded two-dimensional polyhedral subobject in the six-dimensional hypercube, a key example constructed by the second-named author that is so closely related to the central figure of Salvador Dali's 1954 painting "Corpus Hypercubus" (Fig. 17b), currently on display at the Salvador Dali Museum in Figures, in Catalonia. The elastegrity symmetrical to a 12-strut turns into a model of the hypercube (Fig 17a) (lavender highlight above) similar to the model held by Professor Banchoff in Fig 17c.

In a café, the paper-folding architect met and asked the mathematician to verify his trig calculations about whether a non-regular dodecahedron that had resulted from shape shifting the elastegrity (highlighted in cyan) was a true monododecahedron i.e. a polyhedron with twelve congruent pentagonal faces, not necessarily equilateral, arranged three at each vertex.



The architect had reasoned that since BCD is a right triangle by construction and ABDE is trapezoid (AE || BD) ⇒

AF = BF = GH $\Rightarrow \frac{AH}{AF} = \frac{AH}{GH} \Rightarrow \tan \varepsilon = \sin \phi$

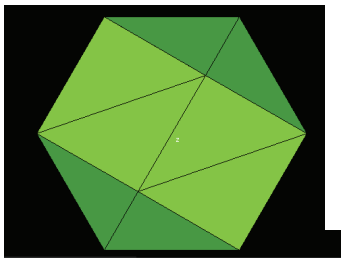
Since BCD and ABDE lie on the same plane when $\phi + \varepsilon = 90°$

or $\sin(90 - \varepsilon) = \cos \varepsilon \Rightarrow \tan \varepsilon = \cos \varepsilon$

which is true for $\varepsilon = \sim 38.1727°$..

so pentagon ABCDE is flat and the structure is a monododehedron.

Fig 21 Monododecahedron with one right angle congruent pentagons



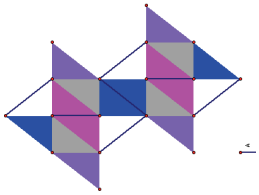
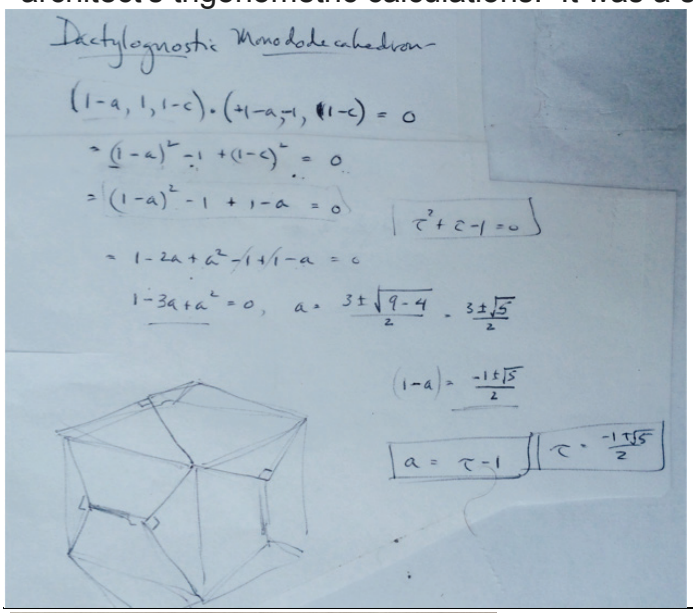


Fig 22a, 22b Monoicosahedron

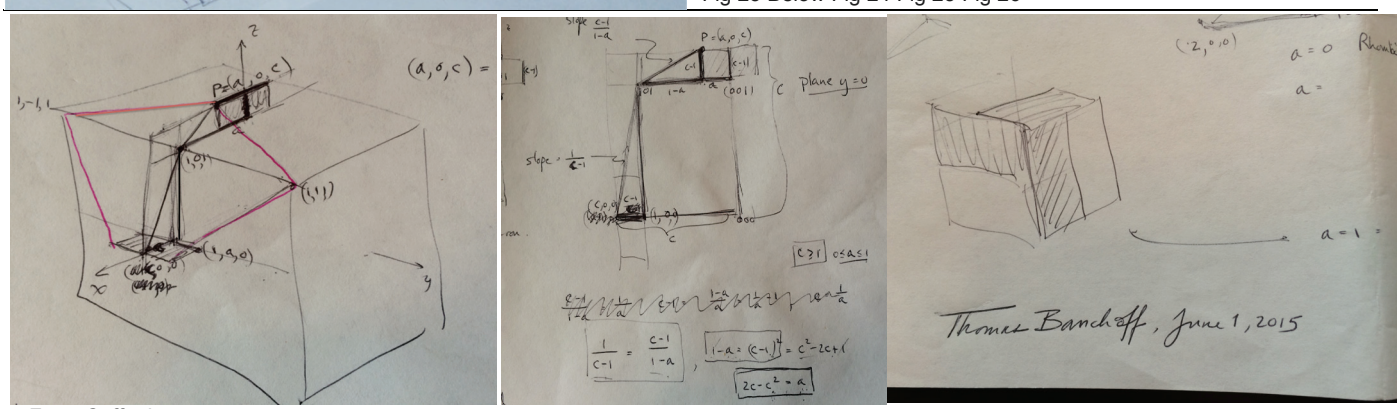
The mathematician had already given two presentations at Gatherings for Gardner concerning monoicosahedra. In G4G7, he displayed the unique monoicosahedron with right-angled triangular faces and no additional folding, with sides of lengths 2, square root 3, and square root 7. In G4G11 he described the unique monoicosahedron with creases so that there are exactly eight planar faces, where the right-angled triangular faces have sides of length 2, square root 7, and square root 11 (Fig 22a, 22b).

In the coffeehouse, the mathematician attached coordinates to the vertices and confirmed the architect's trigonometric calculations. It was a surprise to see a new monododecahedron with pentagonal faces with one right angle realized as an elastegrity configuration.



Mathematicians had realized that there was a progression of monododecahedra all containing the edges of a cube, starting from the regular dodecahedron and ending at a rhombic dodecahedron (where one edge of the pentagon shrinks to a point). It was remarkable that this sequence also contained the new monododecahedron with a right angle at one vertex of each pentagon, and the parameter that gave this new example turned out to be the golden section, a calculation first worked out on coffeehouse napkins."

Fig 23 Below Fig 24 Fig 25 Fig 26



From Coffeehouse notes

This completes the set of connections exhibited for the first time in the Providence RI coffeehouse "Seven Stars" and featured in a double presentation by the two authors at the twelfth Gathering for Gardner in 2016. A more complete and detailed description of the properties of elastegrity shape-shifting polyhedra will appear in the Journal of Mathematics and Art

Endnotes

[1] The Chiral Icosahedral Hinge Elastegrity's that came from the world of art gave rise to some interesting math. In addition, the math has interesting physical properties with possible engineering and science applications. The first author invites engineering and science collaborations in return to signing over exclusive rights: 1) rapidly decreasing volume in response to linear displacement along any of four axes as the network of 36 hinges acts cooperatively as a three dimensional lever suggest a pump for increased thrust. Akin to stepping on a rake's short end, short linear translation increases thrust on the rapidly contracting volume of the elastegrity. It also suggests the design of a pump for variable pressure with fine control for non-Newtonian fluids; 2) a matrix of Chiral Icosahedral Hinge Elastegrities has Negative Poisson's Ratio (-1) suggesting the creation of metamaterials through folding and weaving at any scale using shape memory materials, from nano-scale to architectural scale by introducing mechanical springs to the 36 hinges, for various application from absorbing energy to sensors; 3) tensegrities have been proposed by Donald Ingber, director of the Wyss Institute for Biologically Inspired Engineering at Harvard as

models for all biological structures because their form resilience uniquely accounts for mechanotranduction. Robust Negative Poison's Ratio, ability to pump non-Newtonian Fluids, simple assembly by folding and weaving a flat membrane, and being able to scale up creating a hierarchy of matrices of large elastegrities that can be constructed from smaller ones, expand the tensegrity's properties for modeling biological form. Tensegrities are a special case of the broader family of structures nodal and hinge elastegrities; 4) in addition providing a tool of investigation for studying biological structure, a matrix of elastegrities may play a role in prosthetic devices. For example, it may create an improved exoskeleton better simulating actual muscles that have Negative Poisson's Ratios according to recent publication: Negative Poisson's ratios in tendons: An unexpected mechanical response (Ruben Gatt et al 2015 Published by Elsevier Ltd. on behalf of Acta Materialia Inc.) By varying the elasticity of the hinges, the matrix that may grow into linear (A-B) or central (A-B-C) overall structures, may create members that are overall more elastic such as muscles as well as members that are overall more rigid like bones.

[2] According to Snelson in 1948, soon after showing Fuller the first tensegrity, Fuller created a 6 strut similar to the one shown here.

[3] Dactylognosis is a term introduced here from dactyl= finger and gnosis= revealed knowledge. It means insight leading to innovation derived from manipulation by contrast to innovation derived through cognition. This term encompasses all somatic modalities including art. It is introduced to differentiate the Bauhaus approach to paper folding, which unlike origami is not driven by the intention of simulation.

[4] Readers are encouraged to fold and weave their own chiral icosahedral hinge elastegrity and shape-shift as they read. If the illustration does not suffice you may email to epavlides@gmail.com and request links to videos with instructions for how to fold, weave, and shape-shift.

Larger Golomb Rulers

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Abstract

We present the construction of possibly-optimal Golomb rulers through a size of 40,000 marks, which provides support for the existence of subquadratic Golomb rulers for all sizes. Then we explore the subquadratic question at more length, finding a first gap at 492,116 through 492,118 at which sizes subquadratic Golomb rulers may not exist. Finally, we explore some even larger rulers, looking for those with a maximum quality (a measure of how much better than quadratic they are). Our results are strictly computational in nature, but derive from performance improvements to the known algorithms implementing construction methods based on finite fields.

1 Introduction

A *Golomb ruler* is a set of nonnegative integer values (marks) that includes zero and whose pairwise differences are all unique. That is, for every pair of distinct numbers (a, b) in the set, there is no other pair (c, d) such that $a - b = c - d$ (unless $a = c$ and $b = d$). The values in the set are called *marks*, like marks on a ruler. The *length* of the ruler is equal to the maximum value in the set.

Finding the shortest Golomb ruler with a given number of marks (n) may be a difficult task. Indeed, a massive distributed project is under way to prove that the known 28-mark ruler (with a length of 585) is the shortest possible[4]; it has been running for nearly two years and is about 12% complete.

Finding good (near-optimal) Golomb rulers using normal search techniques is difficult; we are not aware of any effective approach at this point in time, despite much interest and many investigations But a little bit of mathematics originated by James Springer in 1938[11] (with some follow-up help from R. C. Bose and S. Chowla[1][2]) gives construction techniques for rulers that appear to be optimal (and are guaranteed to be near-optimal). Indeed, with the exception of six small sizes, no better Golomb rulers have been found than those generated by these construction methods. All the computational effort expended over the past two decades to prove the optimality of rulers with 18 through 27 marks has supported the assertion that rulers generated by these 75-year-old ideas are the best we can do.

In addition to these two construction methods, there is another method by Ruzsa[9] that is also asymptotically optimal; we consider this as well, though

for the range of mark counts we tried, it was never superior to the other two methods. A good overview of all three construction methods and much more background are given in [5].

James B. Shearer implemented the construction methods in a set of Fortran programs around 1986, and generated the best set of Golomb rulers through 160 marks possible with these methods; his information is published on IBM's research site[10]. He also implemented a brute-force search program that verified or found the optimal ruler lengths through 16 marks.

We have reimplemented Singer, Bose, and Chowla's ideas and run their constructions on modern machines, extending the set of possibly optimal rulers up through 40,000 marks. This has required some new ideas and improvements to the existing algorithms, which we describe below.

We believe these rulers are probably optimal, and are offering a $250 reward if anyone can beat any of these ruler lengths (for 36 through 40,000 marks) using any technique[8].

2 Modular Rulers

Every proper subset of a Golomb ruler is also a Golomb ruler (but one with fewer marks). (If the zero value is omitted you can subtract the smallest value from each value in the set to obtain a valid ruler.) Also, every Golomb ruler can be flipped—take the maximum value in the set, and subtract from this every value in the set to generate a new set that measures the same distances.

All of the constructions we use generate *modular* rulers. A modular ruler is one such that the marks can be placed on the circumference of a circle (of a specific size, the modulus), and all the distances measured are unique. While a standard Golomb ruler measures $n(n-1)/2$ distinct distances, a modular ruler measures $n(n-1)$ distinct distances. Every subset of a modular ruler is also a modular ruler (with the same modulus). Every modular ruler is also a Golomb ruler.

Thus, every ruler we claim to be optimal actually satisfies more constraints than strictly required; in addition to satisfying the normal Golomb requirements, it also is a modular ruler. While any Golomb ruler is also a modular ruler if you make the modulus large enough, it turns out the modulus for our rulers is typically only a little more than the total length of the ruler itself.

A perfect Golomb ruler with n marks measures exactly the distances 1 through $n(n-1)/2$. This is only possible through 4 marks (can you prove it is not possible for 5 or more marks?) A perfect modular ruler with n marks measures exactly the distances 1 through $n(n-1)$ and it is easy to show that its modulus must be $n(n-1)+1$. The 1938 Singer construction, which generates most of the rulers we suggest are optimal, generates perfect modular rulers, every time, but only for a number of marks that is one more than a prime power. Indeed, it generates many, many different perfect modular rulers. For instance, the Singer construction generates the following perfect modular ruler with 12 marks (and with modulus 133):

$$(0, 1, 12, 14, 22, 29, 54, 60, 63, 90, 110, 129)$$

Pick any number from 1 to 133, and you can find two values that differ by that number (modulo 133) from the above set. For instance, for a difference of 65, you can pick 90 and 22; $(22 - 90) \equiv 65 \bmod 133$. The best length 11 ruler that we can extract from directly this ruler is of length 94 (from the 129 wrapping around to the 90). But we can generate a new modular ruler from this one by multiplying each value by 20, which is relatively prime to 133, and sorting the result, giving us

$$(0, 3, 14, 16, 20, 41, 48, 53, 63, 71, 72, 107)$$

From this modular Golomb ruler we can extract a length-11 Golomb ruler of length only 72 (from the 0 on the left to the 72 towards the end); this is one of only two optimal length-11 Golomb rulers.

The other two constructions we consider, the Bose-Chowla construction and the Ruzsa construction, similarly generate near-optimal modular rulers.

3 The Methods

All the methods take a single input g, which is a prime power (or for Ruzsa's construction, just a prime) and require roughly the following steps:

PrimPoly Find a primitive polynomial for a specific field generated by that prime. This provides a numbering of the elements of the field.

Select From that field, select members that satisfy a specific criteria specific to the construction. This will generate the modular ruler.

Multiply From the original modular ruler, generate a lot of additional modular rulers by multiplying that ruler by numbers relatively prime to the modulus.

Sort Since the values of the multiplied modular ruler are not generated in numerical order, sort the modular ruler.

Scan For each of these modular rulers, try all contiguous sub-rulers as candidate Golomb rulers.

The Fortran programs written by James B. Shearer include all of these steps in a clear and straightforward manner. In order to extend the results as far as we did, the algorithms for each step needed improvement.

There have been a number of papers written on finding primitive polynomials in Galois fields, and implementations of these ideas are available on the web. The ideas in these papers are nontrivial and critical in order to generate primitive

polynomials of Galois fields with millions or billions of elements. Both of us independently authored primitive polynomial generation code.

The selection criteria for each of the distinct constructions is fairly complex (especially for the Singer construction). One of us essentially copied Shearer's code into C for this; the other reimplemented everything from scratch. The performance of the Shearer code here was adequate for our purposes.

There were a number of improvements to the multiply step. Both the Singer and the Bose-Chowla construction generated $O(g^2)$ distinct modular rulers (where g is the prime power used for generation). Since we will be sorting and scanning for every multiplicand we consider, it is important to eliminate redundant multiplicands. All of the constructions exhibited symmetry based on a particular subgroup, so we were able to exclude 5/6 of the multiplicands for the Singer construction and 3/4 of the multiplicands for the Bose-Chowla construction. The Ruzsa construction required consideration of many fewer multicands ($O(g)$) so symmetry reduction was not needed here.

In addition, the multiply step itself can be slow, with a multiply and remainder calculated for each value. We improved this by maintaining a cache of small multiples of the input ruler modulo the modulus. When evaluating a new modulus m, if that was small compared to the previous modulus considered, we would just add-and-test the relevant multiple from the cache. The compiler turned this into efficient parallel SSE instructions.

Most of the time in our program was spent in the sort phase. We tried numerous variations of radix sort and finally adopted one that was able to sort more than 100 million elements per second.

In the scan phase, we want to consider every possible sub-ruler from the current modular ruler for every possible Golomb ruler length. A straightforward implementation (as Shearer used) would take time $O(n^2)$ for every modular ruler. We were able to improve this as follows. First, since optimal rulers with 27 marks and fewer are already known, we did not scan for shorter rulers; we started our set of scans at 27. (We only did this optimization for values of g of 1000 or more, so we could pick up the known values as well, all of which are generated with small values of g.) So let us say the shortest ruler found for 27 marks for the current modular ruler turned out to be 950. For each mark count, we decided we did not care about rulers of lengths greater than n^2. Further, from the known lower bound on the lengths of a Golomb ruler $(n^2 - 2n^{1.5} + \sqrt{n} - 2)$. When considering whether to scan at length $27 + x$, we would add our best at 27 (950) to the lower bound on the length of a ruler with x marks, and if that was greater than either $(27 + x)^2$ or our best found ruler at length $27 + x$, we would skip that scan since it could never improve or match our best. This simple idea let us skip almost all scans with little effort. For instance, for 14,533 marks, we considered 22,680,000 moduli. Without this optimization we would have had to do roughly 14,000 scans per moduli; with it, we ended up doing only an average of 14.7 scans per moduli, a reduction by a factor of nearly 1000.

These techniques permitted us to fully explore all three finite-field construction methods through a mark count of 40,000. In no case did the Ruzsa construc-

tion method yield the best result; approximately 2/3 of the time the projective plane method gave the best ruler, and the affine plane method gave the best results in the remaining cases. The final length of the best rulers found were in general very close and slightly less than the number of marks squared. Rulers with n marks and of length less than n^2 are called subquadratic rulers. For each mark count we were able to find a subquadratic ruler.

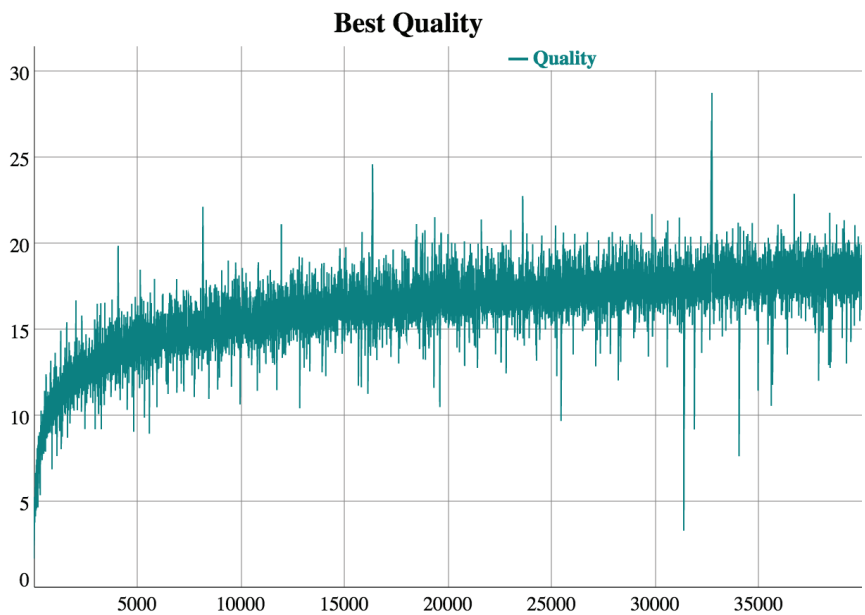

Figure 1: The best quality obtained for mark counts of 7 through 40,000 using the finite field construction methods for Golomb rulers.

The *quality* $b(R)$ of a Golomb ruler R is the difference between the number of marks in the ruler and the square root of its length. A positive quality indicates a subquadratic ruler. For a given number of marks, the quality of the best ruler found by our computations varied from about 1 to about 29, as shown in Figure 1. For a given small range of mark counts, the best quality generally lies within a small range, with occasional spikes up and down. In addition, the quality trends up with increasing mark count and clearly in a sublinear fashion.

The full data, including all modular rulers and the necessary multipliers and moduli to generate the best rulers, is available at [8].

4 Golomb Rulers: Pushing the Limits

The previous section extended the calculation of near-optimal rulers based on affine and projective planes through 40,000 marks. These constructions generate the best known rulers for all sizes in excess of 16 marks. To our knowledge,

despite significant interest, no one has been able to find any Golomb ruler with more than 16 marks with length less than the best of that generated by the affine or projective plane construction. This leads to hypothesis A:

Hypothesis A: For all $n > 16$, the optimal Golomb ruler can be generated by an affine plane or projective plane construction.

No counterexample to hypothesis A is known despite a great deal of CPU expended over the past several decades, not the least of which was the distributed.net's OGR-24 through OGR-28 efforts which have consumed thousands of CPU years over the past 16 years. Also, the search for Golomb rulers has been a central problem for constraint-based search for some time, with no counterexamples found.

Our results, and previous work by other authors [3] have found that subquadratic rulers exist at least through 65,000 marks. This leads to hypothesis B:

Hypothesis B: subquadratic rulers exist for all n.

If OGR(n) is the optimal Golomb ruler of length n, then assuming hypothesis B is true, $b(\mathrm{OGR}(n))$ is positive for all n. Another question of interest is, how large can the quality $b(\mathrm{OGR}(n))$ be?

Hypothesis C: $b(\mathrm{OGR}(n))$ is unbounded.

This hypothesis was conjectured by Erdös and Turán in 1941[6].

5 Finding Subquadratic Rulers

Using the programs created for our previous work, we set out to explore these hypotheses for larger values of n. First, we tried to extend the work of Dimitromanolakis past 65,000. There are three known asymptotically-optimal algebraic constructions: the Ruzsa construction, the affine plane construction, and the projective plane construction. The first two construction methods have very fast (linear time) modular ruler construction methods. The Ruzsa is trivially linear-time; the affine construction is linear from ideas in [7], but for the projective plane ruler construction only a quadratic-time algorithm is known. The affine plane construction usually generates better rulers than the Ruzsa construction, so we use that as our main workhorse. We did not consider all possible derived Golomb rulers but only scanned a modular Golomb ruler until we showed the existence of a subquadratic ruler for all sizes in the range between the generating prime power and the previous prime power. This very quickly covered the entire range from 40,000 through to 500,000 except for eight gaps between prime powers.

For the larger gaps that caused more problems, we started with the projective plane construction rather than the affine plane construction; even though

generating the modular ruler takes longer for the projective plane construction, the actual scanning of the derived rulers, which dominates the run time ($O(n^3)$), can be done more rapidly.

6 Multiplicative Groups for Faster Scanning

When generating derived modular rulers from a given modular ruler, you multiply the original ruler by a value m that is relatively prime to the modulus of the rulers (and the multiplication is done modulo the modulus) and then sort the resulting values. For instance, for the projective plane construction, with a prime of 13, the modulus is $13^2 + 13 + 1 = 183$, which is factored into 3*61. You would consider rulers generated by multiplying the original rulers by the values 2, 4, 5, 7, 8, 10, 11, and so on. The multiplication, remainder, and the sort operations are the most costly operations in the scan when you are focused on a small range of final Golomb rule sizes; we were focused on only the small range for which we had not already found subquadratic rulers.

Rather than considering the distinct multipliers sequentially, you can instead consider them in a sequence that emphasizes multiplication by small integers, especially 2 or 3. Given an already sorted set, you can perform multiplication, modulo, and sorting by such a small integer much more rapidly than the naive implementation. For the multiplication and modulo steps, you can instead use addition and subtraction, where the value to be subtracted changes only a small number of times throughout the range. The sort step can be done by merging a small number of already-sorted subranges. For instance, given the projective-plane-derived modular ruler

$$(0, 1, 8, 24, 37, 41, 59, 107, 119, 128, 134, 139, 153, 181)$$

with the modulus 183, multiplication by two gives

$$(0, 2, 16, 48, 74, 82, 118, 214, 238, 256, 268, 278, 306, 362).$$

The value to subtract to keep it in range is 0 for the first half (through 118) and then 183 for the rest, giving us

$$(0, 2, 16, 48, 74, 82, 118, 31, 55, 73, 85, 95, 123, 179).$$

The sort step is just a simple merge of the two sorted subsequences, giving

$$(0, 2, 16, 31, 48, 55, 73, 74, 82, 85, 95, 118, 123, 179).$$

The multipliers used must be relatively prime to the modulus. For projective-plane-derived rulers, the modulus for a prime power g is $g^2 + g + 1$, which is never divisible by 2, so most of the time the generation of the next derived ruler can use the multiplier 2. For affine-plane-derived rulers, the modulus for a prime power g is $g^2 - 1$, which is usually divisible by 2 (except for when g is

a power of 2) and usually divisible by 3 (since $g^2 - 1$ is $(g-1)(g+1)$ and since g is usually not divisible by 3, one of $g-1$ or $g+1$ usually is) so the smallest multiplier we can frequently use is 5. The time required for the merging of the sorted sublists increases with the logarithm of the multiplier; this may not seem like much, but for a multiplier of 5 we will spend more than twice as much time merging as we would for a multiplier of 2.

For the example above, with $g = 13$ and a modulus of 183, we only ever need use a multiplier of 2, as the first 20 powers of 2 mod 183 generate all necessary multipliers:

$$(1, 2, 4, 8, 16, 32, 64, 128, 73, 146, 109, 35, 70, 140, 97, 11, 22, 44, 88, 176)$$

(The other 100 multipliers that are relatively prime to 183 generate equivalent rulers by symmetry; we omit the details.)

Therefore, even though generating the original modular Golomb ruler is slower with the projective plane construction, if we plan to scan a significant fraction of the possible generated rulers, the projective plane construction ends up being faster. Another benefit was that the projective plane construction typically (about 2/3rds of the time) generated slightly shorter rulers.

7 The Challenge: Large Prime Gaps

For the gaps that turned out to be difficult, we used the projective plane construction. With this, we were able to cover all but one gap in the range through 500,000. The remaining gap was 492,116 through 492,118. In order to fully explore the projective plane construction, we had to construct all 39 billion possible different modular rulers and scan each of them for potential solutions. Next, we did the same for the affine plane construction, constructing and scanning all 10 billion possible different modular rulers. None of them generated subquadratic rulers for the three target mark counts given above.

This result is not completely surprising; the gap between the consecutive prime powers 492,113 and 492,227 is 114, which is a maximal gap (the first gap of that size or greater in the sequence of prime powers). The typical prime gap for primes around this size is ln(p) or about 14. (The projective plane construction for $g = 492,113$ found subquadratic Golomb rulers for 492,113 and 492,114 marks.) Empirically, we see that the projective plane and affine plane constructions typically create subquadratic rulers only for values close to the prime power used to generate them (and for some trivially small values); for this maximal prime gap, the size of the prime gap was too large to span.

We did not evaluate the next prime power up (492,251) for this range because of the computational time required and because we believe it is extremely unlikely to generate subquadratic rulers that small, based on the relationship between the size of the range of subquadratic rulers generated and the prime power used to generate them.

So we have found that for all sizes through 492,115 marks a subquadratic Golomb ruler exists; for all sizes through 500,000 marks, only three values

(492,116 through 492,118) do not have a known subquadratic ruler. In addition, one of the following must be true:

Hypothesis A is false; there is some subquadratic ruler for 492,116 that is not generated by the projective plane or affine plane construction;

Hypothesis B is false; there is no subquadratic ruler for 492,116;

A subquadratic ruler for 492,116 can be constructed by the projective plane or affine plane construction for a prime power of 492,251 or greater.

Some bug exists in our programs. To avoid this we each independently implemented the ideas in this paper and compared the results thoroughly.

We believe one of hypothesis A or B is false.

There has been much attention in the literature to using general optimization and search techniques to find Golomb rulers. So far these techniques have been shown to be effective only for a very small number of marks (through 16 to 20 marks), far short of the 492,116 marks we are interested in. Alternative algebraic or algorithmic construction methods are significantly inferior to the projective plane or affine plane construction (with the exception of one method, the Ruzsa method, which is only somewhat inferior and which we have also explored for the range in question without finding a subquadratic ruler).

Despite great interest, no construction techniques of comparable effectiveness have been found (the projective plane and affine plane construction techniques date back to 1938 and 1941, respectively).

8 Rulers of the Highest Quality

We now turn our attention to hypothesis C. Empirically we have found that powers of two greater than 2^{10} generate rulers with the largest $b(R)$. We do not yet have an explanation on why powers of two generate better Golomb rulers. The symmetry of the set of derived rulers for $g = p^n$ has order $4n$ (for the affine plane construction) and $6n$ (for the projective plane construction), so the higher n for powers of two at roughly equivalent sizes would yield fewer derived rulers to explore, and thus we would expect slightly worse rulers to be found. Yet, consistently, powers of two yield Golomb rulers of higher quality than other prime powers; every record quality (by increasing n) between 4,000 and 40,000 was found by the affine or projective plane constructions starting from a modular ruler derived from a power of two.

We assume this continues for values of n greater than 40,000. We focused on these rulers to try to extend the known best $b(R)$. We explored modular rulers of size $2^{16} = 65,536$ through $2^{21} = 2,097,152$. Since our purpose was to opportunistically look for rulers of high quality, rather than to prove an optimal quality, we only considered derived rulers (sub-rulers) whose length is close to the power of two (within 110). Further, for rulers larger than 32,768, we only considered the projective plane and affine plane constructions (though we have

not yet run the affine plane construction for sizes 2^{20} or 2^{21}). The number of rulers searched grows exponentially with n by approximately a factor of four. For 2^n, we have searched the full range of inequivalent possibilities for a modular ruler, which is $\phi(2^{2n} + 2^n + 1)/6n$ in the projective case, and $\phi(2^{2n} - 1)/4n$ in the affine case (ϕ is Euler's totient function).

For the largest size, we were able to find a Golomb ruler at $n = 2,097,125$ marks of length 4,397,762,317,463, for a $b(R)$ of 40.758. Based on this result, and how much larger this is than the $b(R)$ values for rulers smaller than 40,000, we suspect that $b(\mathrm{OGR}(n))$ is indeed unbounded. Table 1 summarizes our results.

g	n	d	$b(R)$	type	# rulers
2,048	2,046	4,124,805	15.04	aff	60,016
2,048	2,046	4,118,063	16.70	proj	54,498
4,096	4,084	16,517,156	19.87	aff	138,240
4,096	4,091	16,583,409	18.73	proj	139,968
8,192	8,182	66,601,079	21.05	aff	859,950
8,192	8,177	66,501,869	22.13	proj	728,208
16,384	16,371	267,316,415	21.19	aff	2,370,816
16,384	16,367	267,074,038	24.60	proj	1,820,448
32,768	32,750	1,070,915,788	25.15	aff	8,910,000
32,768	32,756	1,071,073,269	28.74	proj	11,748,240
65,536	65,515	4,288,124,720	31.23	aff	33,554,432
65,536	65,523	4,289,307,076	30.20	proj	23,224,320
131,072	131,042	17,162,400,535	36.65	aff	168,424,950
131,072	131,033	17,163,471,622	33.57	proj	142,888,536
262,144	262,125	68,691,708,709	33.97	aff	362,797,056
262,144	262,122	68,685,856,466	32.13	proj	424,189,440
524,288	524,252	274,802,524,432	35.90	aff	2,411,191,314
524,288	524,242	274,803,720,226	34.75	proj	2,066,689,584
1,048,576				aff	5,921,280,000
1,048,576	1,048,529	1,099,332,551,684	38.39	proj	4,704,480,000
2,097,152				aff	28,901,432,448
2,097,152	2,097,125	4,397,762,317,463	40.76	proj	34,426,570,752

Table 1: Record qualities by powers of two. The two missing rows are values we did not fully explore.

9 Discussion

We have extended the known results on Golomb rulers in three significant ways. First, we have extended the computation of the likely-optimal rulers from 160[10] to 40,000. Second, we have extended the search for subquadratic rulers from 65,000[3] to 500,000, except for a gap of three values 492,116 through 492,118 for which we've proved the known constructions do not suffice. Third, we have

found Golomb rulers with the best known qualities, including one with a quality of 40.76.

Decades of search support the assumption that the known finite-field methods (which are over 70 years old at this point) generate the best rulers larger than 16. In hopes of spurring additional investigation, we are offering a prize of $250 to anyone who can find any Golomb ruler in the range 30 through 40,000 that is shorter than the ones we have computed.

At the same time, the existence of subquadratic rulers through nearly half a million support the assumption that subquadratic rulers always exist. Our computational results indicate that one of these two suppositions is likely false, because the finite field construction methods only work well near prime powers, but the prime powers exhibit increasing gaps as they get larger. If there is a subquadratic ruler of size 492,116, it is well beyond our current techniques to find it. At the same time, we have little hope of proving its nonexistence.

10 Acknowledgements

We thank Martin Gardner for introducing us to Golomb rulers decades ago, and Al Zimmermann for reawakening our interest in them through a recent programming contest[12].

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The Leaning Tower of Pingala

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July 25, 2016

As Leibniz has told us, from 0 and 1 we can get everything:

$$0$$

$$1$$

Multiply the previous line by a
and add b times the line before that

$$a + b$$

$$a^2 + 2ab$$

$$a^3 + 3a^2b + b^2$$

$$a^4 + 4a^3b + 3ab^2$$

$$a^5 + 5a^4b + 6a^2b^2 + b^3$$

$$a^6 + 6a^5b + 10a^3b^2 + 4ab^3$$

$$a^7 + 7a^6b + 15a^4b^2 + 10a^2b^3 + b^4$$

$$a^8 + 8a^7b + 21a^5b^2 + 20a^3b^3 + 5ab^4$$

$$a^9 + 9a^8b + 28a^6b^2 + 35a^4b^2 + 15a^2b^3 + b^5$$

$$a^{10} + 10a^9b + 36a^7b^2 + 56a^5b^3 + 35a^3b^4 + 6ab^5$$

$$a^{11} + 10a^9b + 36a^7b^2 + 56a^5b^3 + 35a^3b^4 + 6ab^5$$

If you tip your head on one side

											1
										1	
									1		1
								1		2	
							1		3		1
						1		4		3	
					1		5		6		1
				1		6		10		4	
			1		15		10		1		
		1		20		5					
	1		15		1						

you'll see that the coefficients (A011973)
form the Leaning Tower of Pascal, 1653AD /
Omar Khayyam, 1100AD
Al Karaji, 1000AD
Pingala, 200BC

There are infinitely many particular cases:

$$
\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & & a & \\
 & & & & a^2 & + & b \\
 & & & a^3 & + & 2ab & \\
 & & a^4 & + & 3a^2b & + & b^2 \\
 & a^5 & + & 4a^3b & + & 3ab^2 & \\
a^6 & + & 5a^4b & + & 6a^2b^2 & + & b^3
\end{array}
$$

For example, $a = 2$, $b = -1$ gives the natural numbers (A000027)

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots$$

$$
\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & & a & \\
 & & & & a^2 & + & b \\
 & & & a^3 & + & 2ab & \\
 & & a^4 & + & 3a^2b & + & b^2 \\
 & a^5 & + & 4a^3b & + & 3ab^2 & \\
a^6 & + & 5a^4b & + & 6a^2b^2 & + & b^3
\end{array}
$$

$a = 3$, $b = -2$ gives the Mersenne numbers (A000225)

$$0,\ 1,\ 3,\ 7,\ 15,\ 31,\ 63,\ 127,\ \ldots 2^n - 1$$

Everyone believes that infinitely many of them ($n = 2,3,5,7,\ldots$) are prime

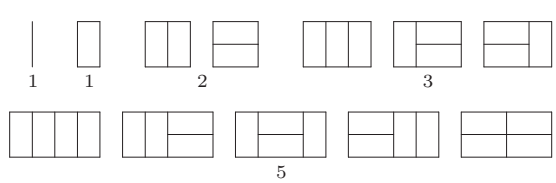
…but no-one can prove that!

$$
\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & & a & \\
 & & & & a^2 & + & b \\
 & & & a^3 & + & 2ab & \\
 & & a^4 & + & 3a^2b & + & b^2 \\
 & a^5 & + & 4a^3b & + & 3ab^2 & \\
a^6 & + & 5a^4b & + & 6a^2b^2 & + & b^3
\end{array}
$$

$a = 1$, $b = 1$ gives the Fibonacci numbers (A000045)

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ \ldots$$

about which whole books have been written.
They are the numbers of ways of packing dominoes in a $2 \times (n-1)$ box:



$a = 2$, $b = 1$ gives the Brahmagupta-Pell numbers (A000129)

$$0,\ 1,\ 2,\ 5,\ 12,\ 29,\ 70,\ 169,\ 408,\ \ldots$$

These were probably known to the Babylonians nearly 4000 years ago.

They are the denominators of good approximations

$$\frac{1}{1} \quad \frac{3}{2} \quad \frac{7}{5} \quad \frac{17}{12} \quad \frac{41}{29} \quad \frac{99}{70} \quad \frac{239}{169} \quad \frac{577}{408}$$

(convergents to the continued fraction) to the square root of 2

Suppose that you want to know if there's a number whose square is 2.

1 is too small, and 2 is too big.

So take the average, $\frac{3}{2}$.

Divide it into 2, giving $\frac{4}{3}$.

$\frac{3}{2}$ is too big, $\frac{4}{3}$ is too small.

We've already learned that the arithmetic mean is greater than the geometric mean!

Take the average of $\frac{3}{2}$ and $\frac{4}{3}$: $\frac{17}{12}$.

Then the average of $\frac{17}{12}$ and $\frac{24}{17}$: $\frac{577}{408}$.

$1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}$, are the 1st, 2nd, 4th, 8th of the convergents;

and $3^2 = 2 \cdot 2^2 + 1$, $17^2 = 2 \cdot 12^2 + 1$, $577^2 = 2 \cdot 408^2 + 1$, ...

The process doesn't stop!! $\sqrt{2}$ **is irrational!!**

In Babylonian $\sqrt{2} = 1; 24, 51, 10, 7, 46, 6, 4, \ldots$

compared with $577/408 = 1; 24, 51, 10, 35, 17, 38, 4, \ldots$

They knew that 1; 24, 51, 10 is better than 1; 24, 51, 11
and, that if they had enough clay tablets, they could get as close as they liked.

$a = 1$, $b = 2$ gives the Jacobsthal numbers (A001045)

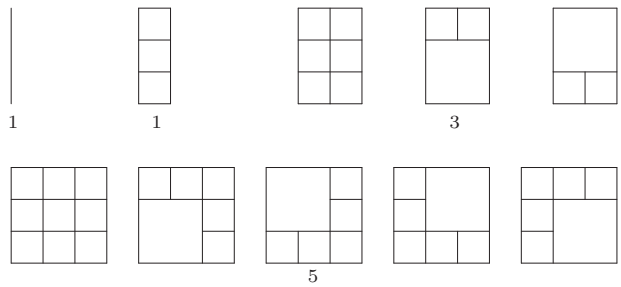
$$0,\ 1,\ 1,\ 3,\ 5,\ 11,\ 21,\ 43,\ 85,\ 171,\ 341,\ \ldots$$

which, apart from the zeroth, are all odd. In fact

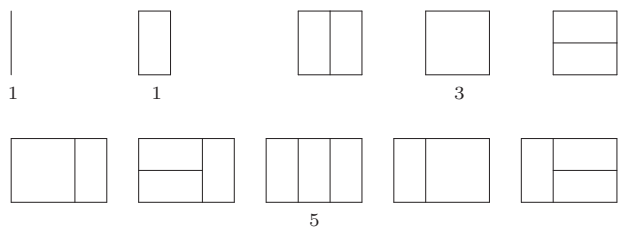
$$J_{n+1} = 2J_n + (-1)^n$$

They were useful to us when we analyzed Conway's "subprime Fibonacci sequences" [*Math. Mag.*, Dec. 2014.]

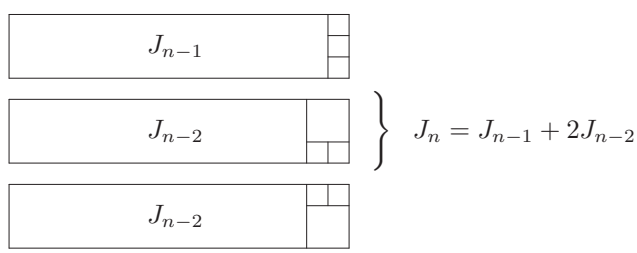
They are also the number of ways of tiling a $3 \times n - 1$ rectangle with 1×1 and 2×2 square tiles.

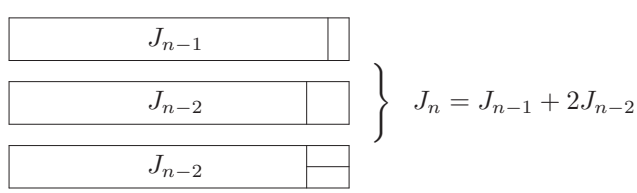


Or the number of ways of tiling a $2 \times n - 1$ rectangle with 2×1 dominoes and 2×2 squares.



These facts follow from the following diagrams:





$a = x$, $b = -1$ gives the Chebyshev polynomials of the first kind, $T_n(x)$,

$a=2x$, $b = -1$ gives Chebyshev polynomials of the second kind, $U_n(x)$; (A049310 and A039599),

$$U_0(x) = 1$$
$$U_1(x) = 2x$$
$$U_2(x) = 4x^2 - 1$$
$$U_3(x) = 8x^3 - 4x$$
$$U_4(x) = 16x^4 - 12x^2 + 1$$
$$U_5(x) = 32x^5 - 32x^3 + 6x$$
$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1$$
$$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x$$
$$U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$$
$$U_9(x) = 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x$$
$$U_{10}(x) = 1024x^{10} - 2304x^8 + 1792x^6 - 560x^4 + 60x^2 - 1$$
$$U_{11}(x) = 2048x^{11} - 5120x^9 + +4608x^7 - 1792x^5 + 280x^3 - 12x$$

which satisfy the following formulas:

$$(1 - x^2)U_n'' - 3xyU_n' + n(n + 2) = 0, \qquad U_n(\cos\theta) = \frac{\sin(n + 1)\theta}{\sin\theta}$$

Let's factor our original polynomials:

$$P_2 = a$$
$$P_3 = a^2 + b$$
$$P_4 = a(a^2 + 2b)$$
$$P_5 = a^4 + 3a^2b + b^2$$
$$P_6 = a(a^2 + b)(a^2 + 3b)$$
$$P_7 = a^6 + 5a^4b + 6a^2b^2 + b^3$$
$$P_8 = a(a^2 + 2b)(a^4 + 4a^2b + 4b^2)$$
$$P_9 = (a^2 + b)(a^6 + 6a^4b + 9a^2b^2 + b^3)$$
$$P_{10} = a(a^4 + 3a^2b + b^2)(a^4 + 5a^2b + 5b^2)$$
$$P_{11} = a^{10} + 9a^8b + 28a^6b^2 + 35a^4b^3 + 15a^2b^4 + b^5$$
$$P_{12} = a(a^2 + b)(a^2 + 2b)(a^2 + 3b)(a^4 + 4a^2b + b^2)$$

The underwaved polynomials are "primitive parts", analogous to the **cyclotomic polynomials**.

This illustrates that our sequences are **divisibility sequences**, that is:

$$m \mid n \text{ implies that } u_m \mid u_n$$

Here's how to see that: $u_n = au_{n-1} + bu_{n-2}$. Guess that $u_n = Ax^n$.

$$Ax^n = aAx^{n-1} + bAx^{n-2}$$

$$x^2 = ax + b$$

$$x = \frac{-a \pm \sqrt{D}}{2} \quad \text{where} \quad D = a^2 - 4b$$

say $x = \alpha$ or β, so that $u_n = A\alpha^n + B\beta^n$ and $u_0 = 0$ and $u_1 = 1$ give

$$0 = A + B, \quad 1 = A\alpha + B\beta, \quad A = -B = 1/(\alpha - \beta)$$

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and the divisibility is clear.

The Lucas-Lehmer theory tells us that a prime p divides

$$u_{p-\left(\frac{D}{p}\right)}$$

where $\left(\frac{D}{p}\right)$ is the Legendre symbol: ± 1 according as D is, or is not, a quadratic residue (square) mod p (or is zero if $p \mid D$).

For example, for the Fibonacci numbers the discriminant $D = 5$.
So u_{p-1} is divisible by p if p is of shape $10k \pm 1$,
and u_{p+1} is divisible by p if p is of shape $10k \pm 3$,
and u_{5n} is divisible by 5.

But this is not "only if" !!

For example $13 \mid u_{14}$ $(= 377)$ and hence $13 \mid u_{14k}$ for all k,
but in fact $13 \mid U_{7k}$ for all k.
We know that the "rank of apparition" of p is a divisor of $p - \left(\frac{D}{p}\right)$,

but we don't know which!

Here's something else we don't know!
A member, u_n, of one of these sequences can be prime only if n is prime;
since u_{pq} is divisible by u_p and by u_q.
But if p is prime, then u_p is not necessarily prime!
Among the Fibonacci numbers
$u_3 = 2$, $u_5 = 5$, $u_7 = 13$, $u_{11} = 89$, $u_{13} = 233$, $u_{17} = 1597$ are all prime,
but $u_{19} = 4181 = 37 \times 113$ is not!
We do not even know if there are infinitely many Fibonacci primes, ...
or infinitely many Mersenne primes, ...
or infinitely many Brahmagupta-Pell primes, ...
or infinitely many Jacobsthal primes, ...
...
There are infinitely many things we don't know !!

but there are infinitely many things WE DO KNOW !!

That's the beauty of Mathematics !!

On the Border Between Recreational and "Serious" Mathematics: Rectangle Free Coloring of Grids

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1 Introduction

Martin Gardner wrote about *recreational mathematics*. However, if you browse through the *Gathering for Gardner proceedings* you will see some *serious mathematics* inspired by him. The line between recreational and serious mathematics is thin indeed. We give an example of a *serious* theorem in Ramsey Theory and some work in *recreational* mathematics that it inspired. We do not rigorously define the terms *recreational* or *serious* but leave that serious problem in sociology to the reader.

Notation 1.1 If $n \in \mathbb{N}$ then $[n] = \{1, \ldots, n\}$. If $n, m \in \mathbb{N}$ then $G_{n,m}$ is the grid $[n] \times [m]$.

The Gallai-Witt theorem[1] (also called the multi-dimensional Van Der Waerden theorem) has the following corollary: *For all c, there exists $W = W(c)$ such that, for all c-colorings of $[W] \times [W]$ there exists a monochromatic square.* The classical proof of the theorem gives very large upper bounds on $W(c)$ and is somewhat difficult. It is in the *serious* camp. Despite some improvements [1] the known bounds on $W(c)$ are still quite large.

What if we relax the problem to seeking a *monochromatic rectangle*? Then we can obtain far smaller bounds. Of more importance, the proofs are easily understood and fun. I would call the area *recreational* at least in the case of two colors.

Def 1.2 A *rectangle* of $G_{n,m}$ is a subset of the form $\{(a, b), (a + c_1, b), (a + c_1, b + c_2), (a, b + c_2)\}$ for some $a, b, c_1, c_2 \in \mathbb{N}$. A grid $G_{n,m}$ is *c-colorable* if there is a c-coloring of $G_{n,m}$ with no monochromatic rectangles.

By two applications of the the pigeonhole principle $G_{c+1,c^{c+1}+1}$ does not have a c-coloring. We leave that *fun* exercise to the reader.

[1] It was attributed to Gallai in [7] and [8]; Witt proved the theorem in [12].

Our Main Question:

Fix c. For which values of n and m is $G_{n,m}$ c-colorable?

Def 1.3 Let $n, m, n', m' \in \mathbb{N}$. $G_{m,n}$ *contains* $G_{n',m'}$ if $n' \le n$ and $m' \le m$. $G_{m,n}$ *is contained in* $G_{n',m'}$ if $n \le n'$ and $m \le m'$. Proper containment means that at least one of the inequalities is strict.

Clearly, if $G_{n,m}$ is c-colorable, then all grids that it contains are c-colorable. Likewise, if $G_{n,m}$ is not c-colorable then all grids that contain it are not c-colorable.

Def 1.4 Fix $c \in \mathbb{N}$. OBS_c is the set of all grids $G_{n,m}$ such that $G_{n,m}$ is not c-colorable but all grids properly contained in $G_{m,n}$ are c-colorable. OBS_c stands for *Obstruction Sets*.

We leave the proof of the following theorem to the reader.

Theorem 1.5 *Fix $c \in \mathbb{N}$. A grid $G_{n,m}$ is c-colorable iff it does not contain any element of* OBS_c.

By Theorem 1.5 we can rephrase our main question as:

What is OBS_c*?*

In the remainder of this paper we state tools used to obtain colorings and proof of lack of colorings, and describe what we know. There are no proofs. For the full version of the paper see

`https://arXiv.org/abs/1005.3750`

or Google *arXiv* and search for *Gasarch*.

Cooper, Fenner, and Purewal [2] generalize our problem to multiple dimensions. Molina, Oza, and Puttagunta [4] look at some variants of our questions.

2 Tools to Show Grids are Not c-colorable

A *rectangle-free subset* $A \subseteq G_{n,m}$ is a subset that does not contain a rectangle.

Theorem 2.1 *If $G_{n,m}$ is c-colorable, then it contains a rectangle-free subset of size $\lceil \frac{nm}{c} \rceil$.*

Theorem 2.2 *Let $a, n, m \in \mathbb{N}$. Let $q, r \in \mathbb{N}$ be such that $a = qn + r$ with $0 \le r \le n$. Assume that there exists $A \subseteq G_{n,m}$ such that $|A| = a$ and A is rectangle-free.*

1. *If $q \ge 2$ then*
$$n \le \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$
2. *If $q = 1$ then*
$$r \le \frac{m(m-1)}{2}.$$

We leave it to the reader to show that, for all $c \ge 2$, G_{c^2,c^2+c+1} is not c-colorable. Use Theorems 2.1 and 2.2.

3 Tools for Finding c-colorings

Def 3.1 Let $c, n, m \in \mathbb{N}$ and let $\chi : G_{n,m} \to [c]$. A *half-mono rectangle with respect to* χ is a rectangle where the left corners are the same color and the right corners are the same color. χ is a *strong c-coloring* if there are no half-mono rectangles.

Table 1 is a strong $(4, 2)$-coloring of $G_{6,15}$.

1	1	1	1	1	3	3	3	2	3	3	2	2	2	2
1	2	2	2	2	1	1	1	1	4	4	3	3	3	2
2	1	3	3	2	1	2	2	2	1	1	1	4	4	3
2	2	1	4	3	2	1	4	3	1	2	2	1	1	4
3	3	2	1	4	2	2	1	4	2	1	4	1	2	1
4	4	4	2	1	4	4	2	1	2	2	1	2	1	1

Table 1: Strong $(4, 2)$-coloring of $G_{6,15}$

Theorem 3.2 *Let $c, n, m \in \mathbb{N}$. If $G_{n,m}$ is strongly c-colorable then $G_{n,cm}$ is c-colorable.*

In the full paper we generalize strong c-colorings to strong (c, c')-coloring and then use use finite fields, tournaments, and combinatorics, to obtain many strong (c, c')-colorings which leads to many c-colorings.

4 Which Grids Can be 2-Colored?

Using our tools we can show the following:

Theorem 4.1 $\mathrm{OBS}_2 = \{G_{7,3}, G_{5,5}, G_{3,7}\}$.

5 Which Grids Can be 3-Colored?

We state a theorem about which grids can be 3-colored and which ones cannot. If we do not have a comment on the grid then the result follows from our tools. If we mention $G_{a,b}$ we omit saying the same holds for $G_{b,a}$.

Theorem 5.1 *The following are not 3-colorable:* $G_{19,4}$ $G_{16,5}$, $G_{13,7}$, $G_{11,10}$ *(special case). The following are 3-colorable:* $G_{19,3}$, $G_{18,4}$, $G_{15,6}$, $G_{12,9}$, $G_{10,10}$ *(program).*

We found the 3-coloring of $G_{10,10}$ by first finding a size 34 rectangle free subset of $G_{10,10}$ (by hand) and then wrote a program to find a coloring where that set was RED. Frankly we were trying to prove there was no such rectangle free set and hence $G_{10,10}$ would not be 3-colorable.

Theorem 5.2 $\mathrm{OBS}_3 = \{G_{19,4}, G_{16,5}, G_{13,7}, G_{11,10}\}$ *and their reversals.*

6 Which Grids Can be 4-Colored?

We state a theorem about which grids can be 4-colored and which ones cannot. If we do not have a comment on the grid then the result follows from our tools. If we mention $G_{a,b}$ we omit saying the same holds for $G_{b,a}$.

Theorem 6.1 *The following are not 4-colorable:* $G_{41,5}$, $G_{31,6}$, $G_{29,7}$, $G_{25,9}$, $G_{23,10}$, $G_{22,11}$, $G_{21,13}$, $G_{19,17}$ *(special case). The following are 4-colorable:* $G_{41,4}$, $G_{40,5}$, $G_{30,6}$, $G_{28,8}$, $G_{24,9}$ *(used a strong 4-coloring of* $G_{9,6}$*),* $G_{22,10}$ *(Brad Larsen program),* $G_{21,12}$ *(Bernd Steinback and Christian Posthoff program, and Tom Sirgedas program),* $G_{20,16}$, $G_{18,18}$ *(Bernd Steinbach and Christian Posthoff program).*

Theorem 6.2 $\mathrm{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\}$ *and their reversals.*

For a long time we didn't know OBS_4. We had a rectangle free set of $G_{18,18}$ of size $18 \times 18/4 = 81$ so we thought that 18×18 was 4-colorable. But we didn't even have a 4-coloring of $G_{17,17}$. On November 30, 2009 Bill Gasarch posted on his blog [3] *The* 17×17 *challenge:* the first person to email him a 4-coloring of $G_{17,17}$ gets \$289.00. Brian Hayes, a popular science writer, put the problem on his blog [5] thus exposing the problem to many more people, including Brad Larsen and Tom Sirgedas who contributed as can be seen in Theorem 6.1. In February of 2012, Bernd Steinbach and Christian Posthoff emailed Bill their solution. They used sophisticated SAT-solvers and also wrote three brilliant papers [9, 11, 10] about their method. Bill happily paid them the \$289.00. They then obtained a 4-coloring of 18×18 and emailed it to Bill free of charge.

7 Open Questions

The next step is to obtain OBS_5. It is likely that our tools will take of most of the way, but not all, and that the problems left are beyond today's computers. So the real open question is to develop better tools.

There is also an obstruction set for the set of grids that are 2-colorable without a monochromatic square. Obtaining this seems rather difficult.

8 Acknowledgments

We thank Brad Larsen for providing us with a 4-colorings of $G_{22,10}$; Bernd Steinbach and Christian Posthoff for providing us with 4-colorings of $G_{21,12}$ and $G_{18,18}$; Tom Sirgedas for providing us with another 4-colorings of $G_{21,12}$.

We thank Ken Berg and Quimey Vivas for providing us with proofs that, for c a prime power, G_{c^2,c^2} is c-colorable. We would also like to thank Ken Berg for the proof that, for c a prime power, G_{c^2,c^2+c} is c-colorable.

We thank Michelle Burke, Brett Jefferson, and Krystal Knight who worked with the second and third authors over the Summer of 2006 on this problem. Brett Jefferson has his own paper on this subject [6].

We thank Nils Molina, Anand Oza, and Rohan Puttagunta [4] who worked with the second author in Fall 2008 on variants of the problems presented here. They won the Yau prize for their work.

Stephen Fenner thanks NSF grant CCF-0515269.

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Philatelic 12s for G4G12

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Written for the occasion of Gathering for Gardner 12
March 31 – April 3, 2016
Atlanta, GA, USA.

In *Martin Gardner's Sixth Book of Mathematical Games from Scientific American* (W.H. Freeman and Company, 1971), Martin Gardner introduced his readers to Patrick O'Gara, the (fictitious) mathematical mailman. One of O'Gara's subcategories of his recreational mathematics passion was mathematical philately (stamp collecting), and he shared with Mr. Gardner some of the stamps from his collection, including a Greek postage stamp from 1955 illustrating the Pythagorean theorem, and stamps from a variety of countries like France, Russia, and Ireland honoring mathematicians such as Pascal, Laplace, Euler, Chebyshev, and Hamilton. Through O'Gara, Mr. Gardner introduced his readers to the concept of topical philately, and supplied several mathematical postage stamp references for readers who may have wanted to explore this area further. In honor of G4G12, the twelfth gathering in celebration of the life of Martin Gardner, and in Mr. Gardner's memory, this paper presents philatelic items — postage stamps, postmarks, postcards, etc. — of mathematical content related specifically to the number twelve.

Of course, the theme of twelve appears on any postage stamp whose illustration contains a standard analog clock. A fun example of this appears on a Swiss stamp from 2012 [**Figure 1**], on which a clock is "deconstructed" into its elements by Swiss comedian and "deconstruction artist" Ursus Wehrli.

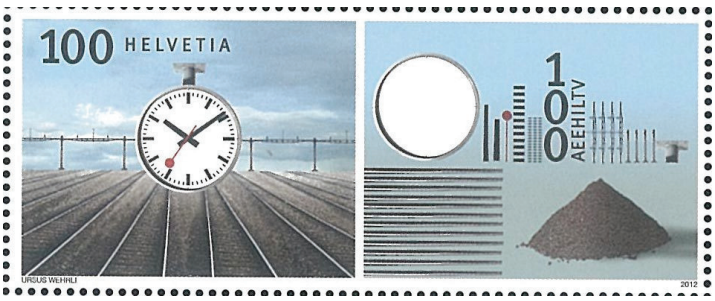

Figure 1
Clock on a Swiss stamp deconstructed into its basic elements, including its **12** *major (hour) tick marks*

The twelve tribes of Israel appear on several stamp issues by the Israeli government. Many countries have issued stamps featuring the twelve signs of the zodiac. The twelve atoms on a benzene molecule appear on German and East German stamps from 1964 and 1979, and a Belgian stamp from 1966 [**Figure 2**]. Regrettably, however, none of these examples of "12's" on stamps would be of particular interest to Patrick O'Gara. For "twelve" stamps with a more mathematical flavor, we turn to geometric figures, both two- and three-dimensional.

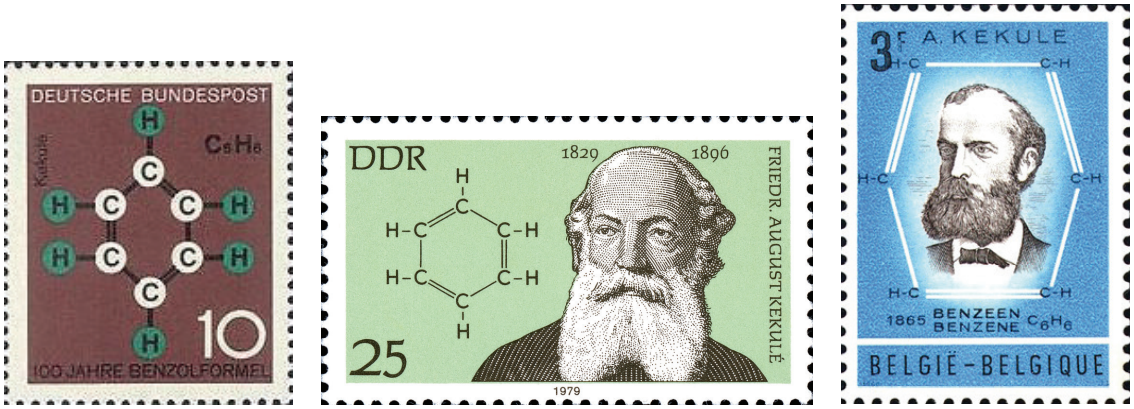

Figure 2

Three stamps illustrating the **12** *atoms in a benzene molecule*

The Chinese mathematician Liu Hui (c. 260 CE) determined upper and lower bounds for π by calculating the areas of concentric 96- and 192-gons. A page from his book, *Jiuzhang Suanshu*, *The Nine Chapters On the Mathematical Art*, appears on a Micronesia stamp from 1999 [**Figure 3**], part of a larger sheet of postage stamps celebrating the science and technology of ancient China during the first millennium. The page illustrates Liu Hui's method of approximating π using the regular dodecagon, the 12-sided regular polygon, as an example. A few hundred years later, Zhu Chongzhi (429–500 CE), also from China, bested Liu Hui's upper and lower bounds for π by approximating a circle with a $2^{13}\times 3$-sided regular polygon. His work, also at the stage of a 12-sided regular polygon, is illustrated on a 2015 stamp from Hong Kong [also **Figure 3**], part of a four-stamp set honoring ancient Chinese scientists.

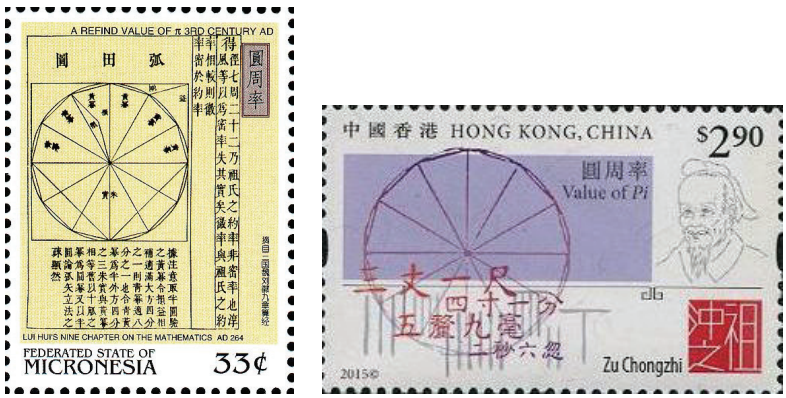

Figure 3

The **12***-sided regular dodecagon, used to approximate* π

Austrian composer and music theorist Josef Matthias Hauer developed a method of composing music using twelve equispaced pitches or tones. All possible combinations of these tones can be enumerated, graphically, using a dodecagon, as on the Austrian stamp from 1983, commemorating Hauer's 100th birthday [**Figure 4**].

Figure 4
The **12***-sided regular dodecagon*
used to illustrate all possible combinations
of **12** *equispaced musical pitches*

While the number of stamps featuring twelve-sided polygons is limited, the offerings increase when we turn to three dimensions and the Platonic (regular) solids (discussed by Martin Gardner in his second collection of "Mathematical Games" columns in *The 2nd Scientific American Book of Mathematical Puzzles & Diversions*, Simon and Schuster, 1961), which, all except for the tetrahedron, are rich in the number twelve!

We may begin with the **cube**, **which has twelve edges**. While many stamps feature cubes (or other twelve-edged rectangular solids), such as the 1970 Child Welfare stamp from the Netherlands [**Figure 5**], issued as part of a set of five cubes of different colors, a particularly entertaining cube is featured on the Austrian stamp from 1981 [also **Figure 5**], issued in honor of the 10th International Austrian Mathematical Congress held that year.

Figure 5
Cubes, whether real or impossible, have **12** *edges.*

Another special depiction of the cube appears on a Chinese postcard from 1990 [**Figure 6**] issued in honor of the 31st International Mathematical Olympiad, hosted by China that year. The illustration on the postcard demonstrates how one may bisect a cube to yield a face that is a regular hexagon. An East German first day cancellation from 1981 uses a cube to illustrate Euler's formula for non-self-intersecting polyhedra, faces + vertices – edges = 2 [**Figure 7**]. (More on that icosahedron, shortly.) A recently issued cube-on-stamp will

Figure 6
A demonstration of the **12***-edged cube revealing the regular hexagon*

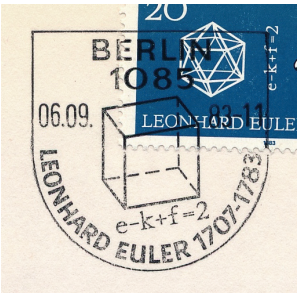

Figure 7
The **12** *edges of the cube*
support Euler's formula.

resonate with recreational mathematicians and mechanical puzzle enthusiasts, alike. "Europa" stamps are issued annually by participating European postal administrations, and have an agreed-upon common theme. The theme for 2015 was "old toys," and Lithuania's entry was a set of two burr puzzles, one of them being in the shape of a cube [**Figure 8**].

Figure 8
Composed of twenty-four sticks, this burr puzzle cube still has **12** *edges.*

The next regular solid is the **octahedron**, **which also has twelve edges**! The city of Nancy in France chose the octahedron as their symbol for the postmark advertising their hosting of the International Trade Fair and Exhibitions in 1968 [**Figure 9**], and Stockholm used the octahedron as a symbol in their commemorative postmark for the International Conference on Coordination Chemistry held in Stockholm in 1962 [also **Figure 9**].

Figure 9
*<u>**12**</u> edges on a regular solid? Not a cube, but an octahedron.*

A set of five stamps issued by Sweden in 2012 illustrate the space-tiling capabilities of the octahedron [**Figure 10**].

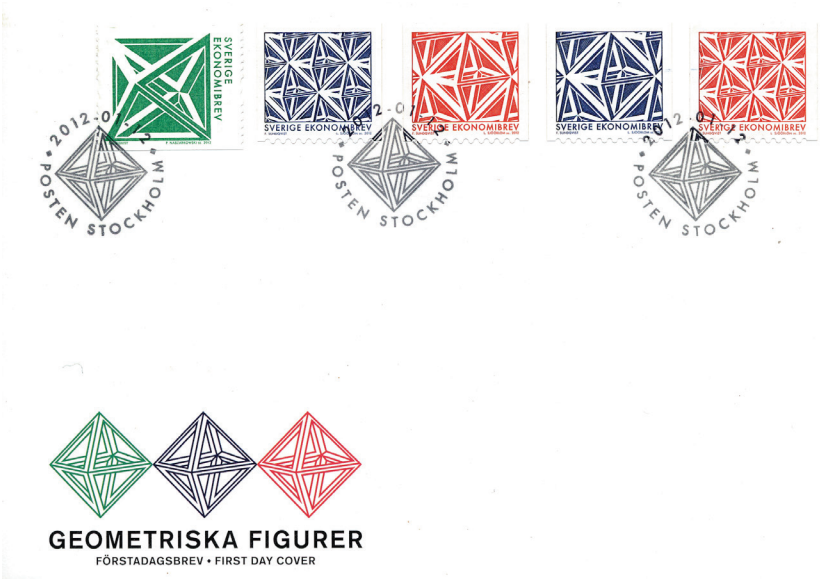

Figure 10
*The <u>**12**</u>-edged regular octahedron has space-tiling capabilities.*

Shifting our focus from edges to faces, the **dodecahedron**, the next Platonic solid, has **twelve faces**. To the ancient Greeks, the tetrahedron, cube, octahedron, and icosahedron represented the four basic elements of fire, earth, air, and water. The dodecahedron, however, was considered a representation of the entire universe, which could explain its prominent place in the Macau cosmology stamps from 2004 [**Figure 11**], part of Macau's ongoing annual "science and technology" series. (All five of the Platonic solids are pictured on the stamp on the mini-sheet of the same issue.) Most interestingly,

some modern cosmology theories are returning to the ancient Greeks' notion that the universe is dodecahedral in shape! (See, for example, Luminet, J.P., et al., "Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background," Nature 425 (6958):593–5, 2003.)

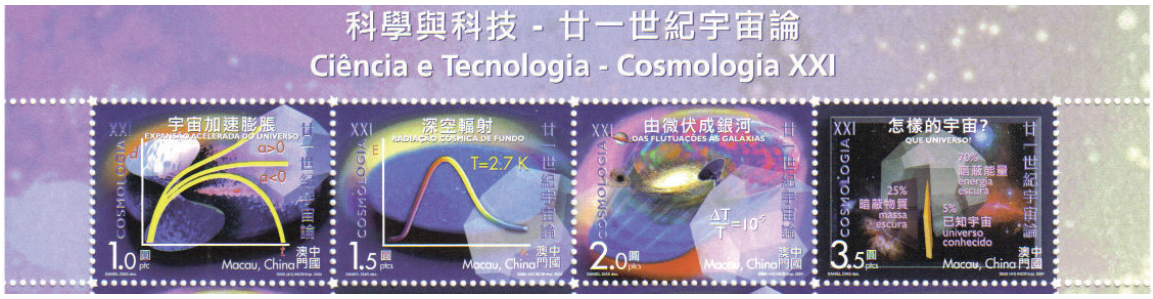

Figure 11
Does the universe exhibit the ***12****-sided symmetry of the dodecahedron?*

For reasons unknown to this author, the Republic of China used the dodecahedron as the vehicle for celebrating the 40th anniversary of its Labor Insurance System, on a stamp issue from 1990 [**Figure 12**]. In 1964, Spain issued a series of fourteen stamps to commemorate twenty-five years of peace. The 1.50 peseta stamp focused on "modern" architecture that could be achieved during peacetime, and featured a dodecahedral building [also **Figure 12**]. And for much more purely mathematical reasons, the World Mathematical Year stamp from Monaco in 2000 prominently featured a dodecahedron among other symbols and figures associated with the golden ratio [also **Figure 12**].

Figure 12
Whether symbolic, applied, or purely mathematical,
dodecahedrons have ***12*** *faces.*

The dodecahedron has been used as a symbol to convey the study of fundamental properties of solids, such as when it was used in a Swedish commemorative cancellation for

the 1952 International Symposium on the Reactivity of Solids [**Figure 13**], or by France to commemorate the 3rd International Congress on Crystallography held at the Sorbonne in Paris in 1954 [also **Figure 13**]. And not surprisingly, the dodecahedron was used as the first day of issue postmark for the Macau cosmology set of stamps [also **Figure 13**], mentioned earlier.

Figure 13
__12__-faced dodecahedrons used in a variety of philatelic applications

Lastly, we turn to the **icosahedron**, which with its twenty faces and thirty edges has only **twelve vertices**. We caught a glimpse of an icosahedron in Figure 7, part of the illustration for an East German stamp from 1983, commemorating the 200th death anniversary of the great Swiss mathematician, Leonard Euler [**Figure 14**], and used as an example with which to demonstrate the aforementioned Euler's formula. The field of virology makes claim that arrangements of virus subunits are governed by icosahedral symmetry. Hence, the icosahedron was used a part of the design of a 1984 Japanese postage stamp issued in honor of the 6th International Virology Congress [also **Figure 14**].

Figure 14
Whether in mathematical analysis or applied to the field of virology, the icosahedron's __12__ vertices are significant.

We may find another philatelic instance of the icosahedron being used to represent the field of virology, but for this, we need to look in the *selvage* of a sheet of German stamps issued in 2010 in honor of the 100th anniversary of the founding of the Friedrich Loeffler Institute, the National Institute of Animal Health in Germany [**Figure 15**]. (The stamp itself

illustrates some virus-like microbe.) Another example of an icosahedron appearing only in the selvage of a stamp issue is for the commemorative issued by Germany in 2008 honoring the 300th birthday of goldsmith and inventor of scientific instruments Wenzel Jamnitzer [also **Figure 15**]. The beautiful gold icosahedron on the corner selvage piece of the stamp sheet is just one of Jamnitzer's many works featuring the Platonic solids and their stellated counterparts.

Figure 15
Elusive philatelic icosahedrons and their ***12*** *vertices*
turn up in unexpected places, including a vertex!

Another "viral" philatelic example of the icosahedron is on the postage meter strip advertising the anti-viral drug Zovirax (generic name acyclovir, or aciclovir in Dutch) [**Figure 16**]. Lastly, borrowing a role from one of the dodecahedrons in Figure 13, the icosahedron was used in postmarks commemorating the 11th and 12th European Crystallographic Meetings in 1988 and 1989 in Vienna and Moscow [**Figure 17**].

Figure 16
The ***12****-pointed virus structure*
in the face of the anti-viral drug

For any reader interested in further pursuits in mathematical philately, the author recommends that they consider membership in the Mathematical Study Unit of the American Topical Association. The study unit's web site is at ***www.mathstamps.org***. The unit's journal, *Philamath*, is published quarterly in January, April, July, and October.

Figure 17
The **12***-cornered icosahedron, like its dodecahedron cousin,*
has been used as a symbol for the study of fundamental properties of solids.

(Note: Stamps, postmarks, and other philatelic items appearing in the figures are not sized to relative scale.)

A far-from-comprehensive list of references on mathematical philately:

Gardner, M., *Martin Gardner's Sixth Book of Mathematical Games from Scientific American*, W.H. Freeman and Company, San Francisco, (1971) [Several references are provided at the end of Chapter 23, "O'Gara, the Mathematical Mailman."]

Miller, J., "Images of Mathematicians on Postage Stamps," http://jeff560.tripod.com/stamps.html

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The "ΦTOP": A Golden Ellipsoid

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Abstract

The golden ratio can be incorporated into an ellipse. Rotated around its major axis, the resulting 3-dimensional figure can be called a "golden ellipsoid". This object can act as a new kind of spinning top I call the "ΦTOP". The golden ellipsoid can be made into elegant sculptures as well. It also demonstrates an unusual visual illusion.

Introduction

Spinning tops and geometrical objects have each been of interest to many cultures spanning at least the past two millennia. (For a review of spinning tops, as well as to the literature of the subject, see [1]). Occasionally the Platonic solids, cones and some other regular three-dimensional geometrical shapes have been incorporated into spinning tops. Here we report the outcomes of a project exploring the physics, mathematics, art, aesthetics and psychophysics of spinning tops that utilize a special prolate ellipsoid.

The Tippe-Top, Kelvin's Pebbles, Jellett's Eggs and Shiva's Lingam Stones

Study of the problem of the rise of the center of mass (COM) of spinning objects is usually said to have begun in the late nineteenth century. These early mathematical treatments aimed to explain the motion of the newly invented and patented "tippe top" (Fig 1a). This semi-spheroidal top will invert when spun on a smooth surface while raising its COM. Because of the importance of friction in its dynamics, such a nonholonomic system is not readily amenable to analytic treatment, or of intuitive understanding. In notes written in 1844 — well before the invention of the tippe top in 1891 — William Thomson (later, Lord Kelvin) appears to have been the first person to analyze the problem of the rising COM of spinning objects. As a young student, he experimented with both oblate and prolate ellipsoidal pebbles (Fig. 1b), but never did a complete analytic theoretical treatment of the problem. J. H. Jellett, in his 1872 book *Theory of Friction*, provided a partial account of the related problem of the rise of the COM for an egg-shaped (ovoid) object (Fig.1c), making use of a new (adiabatic) invariant of the motion that he devised.

Figure 1 (from left to right): *(a) aluminum tippe top; (b) natural pebble; (c) polished agate stone egg; (d) classic Lingam stone; (e) ruby/fuschite Lingam stone; (f) black Lingam stone. Each is about 5 cm tall.*

Naturally occurring prolate ellipsoidal "Lingam stones" (Fig. 1d) from the Narmada River in India exhibit similar counter-intuitive dynamical behavior. When spun around their minor axis on a smooth horizontal plane, some Lingam stones (Fig. 1d and 1e) can stand erect and spin around their major axis in a vertical position. Most of them (such as Fig. 1f), however, will not stand upright. Is there an optimal prolate spheroidal shape that will stand erect if spun? After experimenting with natural pebbles, polished stones and rapid prototyped prolate ellipsoids, I found a range of optimal parameters for the prolate ellipsoids.

Ellipsoids, Ovoids and Related Objects

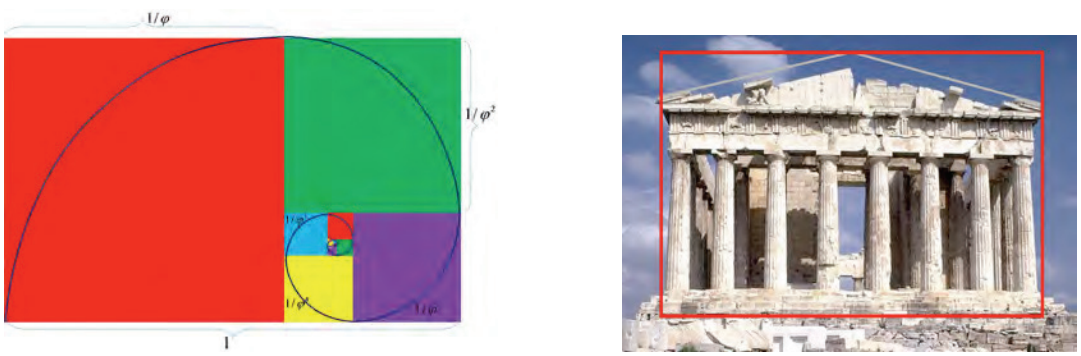
The equation for an ellipse is given by $x^2/a^2 + y^2/b^2 = 1$. A spheroid or ellipsoid has a surface defined by:

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

where a, b and c are constants. Taking a = b, one has an oblate or prolate ellipsoid. Experiments done for this project have found that prolate ellipsoids with ratio c/a in a range from ~ 1.5 - 1.7 optimize the novel dynamics discussed in the previous section. (Note: Generalizing the equation of an ellipse to allow the powers of the co-ordinates to be different from 2 leads to a Lame curve, which can be also extended to three dimensions. Ovoids, or egg shaped objects, do not have such a well-defined mathematical basis.)

Mathematics and the Golden Mean

The "golden mean" — also called the "golden ratio" or the "golden section" — is designated by the Greek letter ϕ that is equal to about 0.61803398… Study of its many properties dates back at least 2500 years (cf. [2] for a good discussion of the subject). It can most easily be derived from the relation $1/\phi = 1 + \phi$. The "golden rectangle" (Fig. 2a) has the ratio of its side lengths equal to this number.



Figures 2a (l.), and b (r.): *(a) geometry of the golden section; (b) Parthenon with golden rectangle.*

Golden Mean, Art and Aesthetics

It has been claimed that the golden mean was included in a variety of art works and architectural monuments dating back to the Greek temples such as the Parthenon (Fig. 2b) — which ϕ does not fit well — and in the paintings of Leonardo da Vinci. Some cases that fit may have been conscious choices by the artists or architects; others may just be mathematical coincidences. If ϕ was consciously included, why? The 19th century physicist Gustav Fechner conducted the first quantitative study of an aesthetic preference for the golden mean by employing a set of rectangles. In the intervening years, many further studies have questioned his conclusion that there is a preference for the golden rectangle. As part of this project, I have conducted an informal preference survey using ellipsoids with c/a from 1.0 - 2.2. I found that the most selected c/a ratios were from 1.5 - 1.7, including what can be called the "golden ellipsoid".

Introducing The "ΦTOP": A Golden Ellipsoid

Taking the equation for an ellipsoid and setting the ratio c/a equal to Φ which is the *inverse* of the often defined "golden mean" (or "golden ratio" or "phi" or "ϕ" - lower case), $c/a = \Phi = 1/\phi = (1 + 5^{1/2})/2 \sim 1.61803398\ldots$) one has what can be called the "golden ellipsoid" - a uniquely shaped solid body. This object exhibits near optimal rotational performance in the physical sense discussed above. It explicitly includes the golden mean in its shape, and looks visually appealing.

Figure 3. *A brass version of the ΦTOP with major axis ~ 5.1 cm, weighing ~ 220 grams.*

I have now designed and had fabricated versions of this unique spinning top and have named it the "ΦTOP". A machined brass version is shown above. In settling on a final design for the top, there were several human and mechanical design considerations. First, it should be of a size such that it could be easily spun using a person's fingers. If it has a length of about 5 cm, by placing a thumb of one hand on one side of the long axis and placing the index finger of the other hand on the opposite side, the top can easily be spun rapidly in a horizontal plane. If the object is much smaller or larger, it is more difficult to spin up. Second, the object should be made of a material of suitable mass, density and frictional coefficient so that it can actually stand up before friction damps the rotation. Metals such as brass, aluminum, stainless steel and copper have been tried, as well as plastics such as Delrin, acrylic and PVC. All work quite well, though the aluminum ΦTOP is the most easily spun up and spins the longest. A few seconds after being spun, the ΦTOP will stand upright. Once standing erect, it spins about twice as fast as when it was spinning horizontally (owing mainly to the difference of the moments of inertia about the major and minor axes). After ~ 60 - 120 seconds, it begins to settle down again owing to friction.

Some further considerations. The ΦTOP has a shape fairly close to that of a standard European rugby ball (c/a ~ 1.5 - 1.6), though the ΦTOP, of course, is much smaller. It is very pleasing to look at. It also feels quite nice in one's hand, producing a sensation something like Chinese "stress relieving" balls do. In some sense the object is reminiscent of Piet Hein's handheld "super-egg" (see Fig. 4a). However, that object is not a good top, though it can stand upright by itself without rotating. The golden ellipsoid, like the super-egg, has a uniquely defined shape – unlike that of an egg (ovoid). Nonetheless, both ovoids and prolate ellipsoids, when spun, can stand upright, provided that the ratio of the major to minor axes lies in the range of about to 1 to 2. What about other values? For a ratio greater than 2, the prolate ellipsoids will not stand erect. What about less than 1? Then one is considering oblate ellipsoids (sort of thick pancakes). These objects can indeed stand upright when spun around their minor axis. What about other physical characteristics of the ΦTOP? If one brings a fairly strong magnet (such as a rare earth neodymium magnet, though an ordinary refrigerator magnet will suffice) near a non-magnetic metal (e.g., aluminum) ΦTOP while it is spinning in the upright position, it will quickly stop spinning. This is a very nice display of induced eddy currents in a moving conductor while in the presence of a magnetic field. It is the inverse of Tesla's "Egg of Columbus" demonstration that spins up a conducting egg using alternating currents.

Super-eggs, Superalls and Golden Ellipsoids

Ellipsoidal forms — including ovoids — have been used as the basis for a number of sculptures over time. Below we show Danish polymath Piet Hein's sculptural "Super-egg" (based on the Lame curve); a "Golden Ellipsoid" made by American artist Randy Rhine (designed with the author); and Canadian artist Gord Smith's "Superall" (which is topologically closely related to the Meissner solid of constant width).

Figures 4a (l.), b (c.) and c (r.): *(a) P. Hein "Super-egg", fiberglass, ~ 4 m tall, 1999; (b) R. Rhine and KB, "Golden Ellipsoid", wood, ~ 13 cm tall, 2015; (c) Gord Smith,"Superall", brass, ~18 cm tall, 1982.*

The Gelatinous Ellipsoid

The great 19th century physicist and philosopher of science Ernst Mach appears to be the first person to have written [3] about a visual phenomenon that can be seen while watching slowly rotating hard boiled eggs: they appear to deform, almost as if they were made of jelly. A two-dimensional version of the effect can be seen on my web site at: http://lite.bu.edu/vision-flash10/applets/Form/Ellipse/Ellipse.html. The ΦTOP exhibits the effect beautifully. The cause of this visual phenomenon is still under investigation.

Summary

This project explored the mathematical, physical, artistic, perceptual and aesthetic aspects of a special prolate ellipsoid that can be called a "golden ellipsoid". Spun rapidly, it is a counter-intuitive top; rotated slowly, it elicits the "gelatinous ellipse illusion"; and it also can be incorporated into elegant sculptures.

Acknowledgments

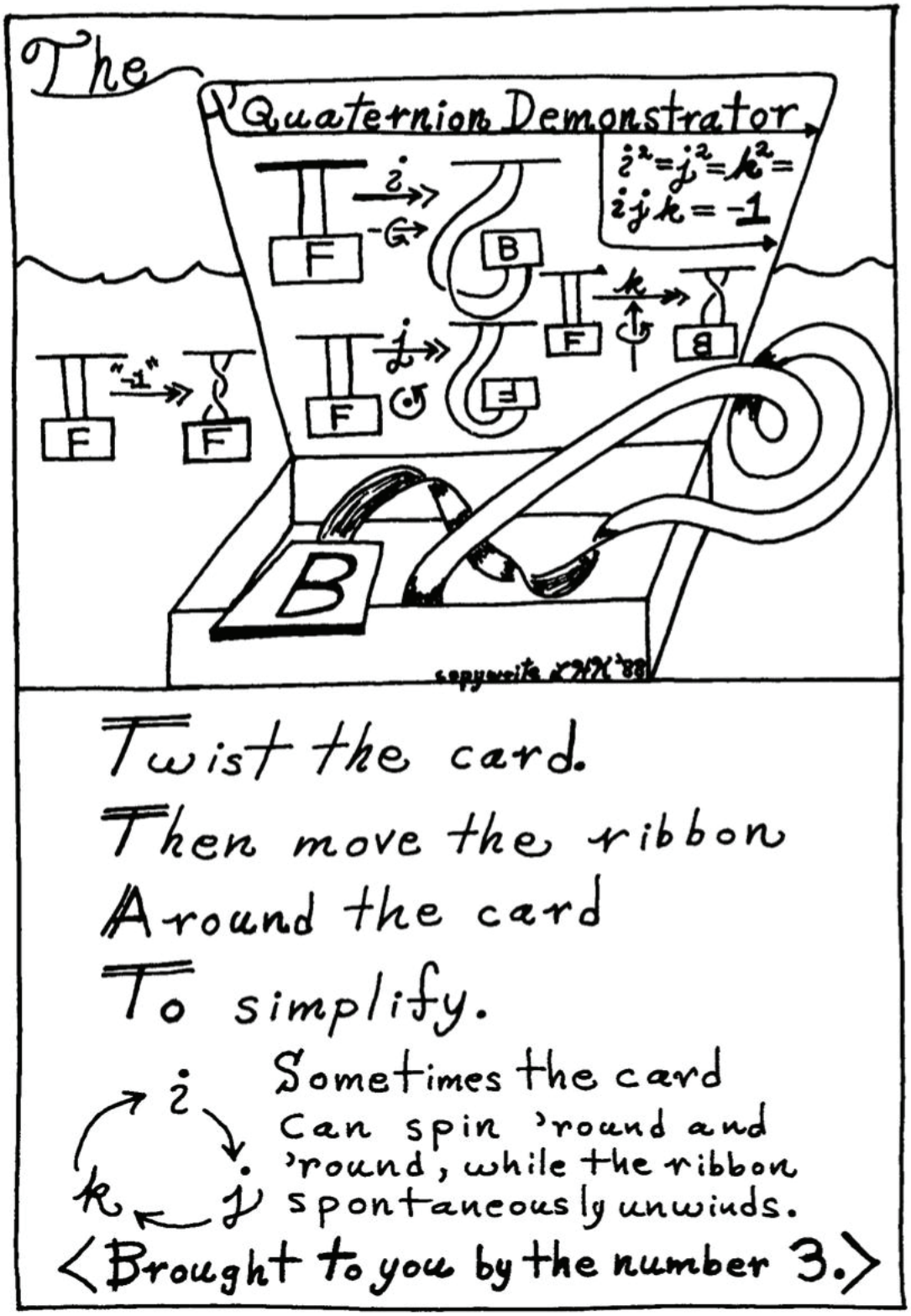
I thank Boston University engineer Robert Sjostrom for fabricating the brass version of the ΦTOP shown here. I also thank BU engineer David Campbell and BU undergraduate Victor Li for help in making rapid prototype plastic versions of the top. This project is part of a larger study of tops, but also gained impetus from an inquiry about the physics of spheroidal objects from the designer and machinist Richard Berner.

References

[1] K. Brecher, "A Torque About Tops", *Bridges Seoul*, p. 51,Tessellations Publishing, 2014.
[2] M. Livio, *The Golden Ratio*, Broadway Books, 2002.
[3] E. Mach, *The Analysis of Sensations*, p. 234, Dover Pubs., 1959.

The Quaternion Demonstrator

by Louis Kauffman



ROOTS OF POLYNOMIALS WITH FIBONACCI COEFFICIENTS

JILL COCHRAN, ERIC MCDOWELL & RON TAYLOR

In our class for humanities majors students complete an activity where they construct a golden rectangle and then consider powers of the golden ratio φ. The primary idea behind the activity is to provide a geometric foundation for the idea of the golden ratio. As an extension, as the students are computing powers of φ they see they can be written as linear combinations of φ and 1 according to the following pattern

$$\varphi^2 = \varphi + 1, \quad \varphi^3 = 2\varphi + 1, \quad \varphi^4 = 3\varphi + 2, \quad \varphi^5 = 5\varphi + 3, \quad \cdots \tag{1}$$

and notice that the coefficients are Fibonacci numbers. This also reinforces the connection between this famous sequence and this equally famous number. (Hooray for math!) An unexpected consequence for the authors was that this activity led to an interesting question about sequences. In particular, one day after completing this activity in class we were discussing the equations in (1) and wondered what real zeros these equations might have besides the golden ratio.

The sequence of polynomials. Taking each equation in (1) and rewriting as a monic polynomial gives us a sequence of polynomials of the form

$$f_n(x) = x^n - F_n x - F_{n-1}.$$

The process by which the students generate the equations in (1) uses the fact that $\varphi^2 = \varphi + 1$, so for $n \geq 2$ each f_n will have $f_2(x) = x^2 - x - 1$ as a factor. This allows us to rewrite f_n as shown below.

$$f_n(x) = f_2(x) \cdot \sum_{i=1}^{n-1} F_i x^{n-1-i}$$

Because we will need it later, we will call the second factor g_{n-2}. For example $f_7(x) = f_2(x)(x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8) = f_2(x) \cdot g_5(x)$. The general form of g_n is given by

$$g_n(x) = \sum_{i=1}^{n+1} F_i \, x^{n+1-i}. \tag{2}$$

Interestingly, each g_n looks like a partial sum of the generating function of the Fibonacci sequence, but in this case the coefficients increase while

Date: 23 April 2013.

the exponents decrease rather than having the coefficients and exponents increasing concomitantly.

Since the roots of f_2 are φ and $-\frac{1}{\varphi}$, we know that each f_n will also have these two numbers as roots. Hence our original question about roots can be answered by looking for the roots of g_n.

Finding roots of $\mathbf{g_n}$. Notice that the first and second derivatives of f_n are given by

$$f_n'(x) = nx^{n-1} - F_n \quad \text{and} \quad f_n''(x) = n(n-1)x^{n-2}.$$

Then we can see that f_n will have one critical value when n is even, and two critical values when n is odd.

The Even Degree Case. In the case where the degree is even we see that the one critical value corresponds to a relative minimum since the second derivative is positive everywhere except $x = 0$. Hence f_{2n} is convex and can have at most two real roots. So the corresponding g_{2n-2} will have no real roots.

The Odd Degree Case. In the case where $n \geq 3$ is odd, the second derivative test tells us that the critical values correspond to a relative minimum at $m_n = (F_n/n)^{1/(n-1)}$ and a relative maximum at $M_n = -(F_n/n)^{1/(n-1)}$. Then, since $f_n'(-1/\varphi) < 0$ and $f_n(-\varphi) < 0$, we know that f_n has a single real root in the interval $\left[-\varphi, -\frac{1}{\varphi}\right]$. That is g_{n-2} has one real root. If we define the sequence (z_n) by letting z_k be the real root of g_{2k-1} then this sequence has an interesting limit.

The limit of $(\mathbf{z_n})$. To calculate the limit of (z_n) we need three basic ideas. The first two of these are well known.

Property 1. $\lim_{n\to\infty} n^{1/(n-1)} = 1$ **Property 2.** $\varphi^{n-2} \leq F_n \leq \varphi^{n-1}$

The third property was one which we knew, but could not find a reference for, so we verify it here. Notice that if we take the $(n-1)$-st root of all three sides of the identity in Property 2 we get the inequality

$$\varphi^{(n-2)/(n-1)} \leq (F_n)^{1/(n-1)} \leq \varphi^{(n-1)/(n-1)}.$$

Then if we take the limit as $n \to \infty$ on all three sides we get

$$\varphi = \lim_{n\to\infty} \varphi^{(n-2)/(n-1)} \leq \lim_{n\to\infty} (F_n)^{1/(n-1)} \leq \lim_{n\to\infty} \varphi = \varphi.$$

This gives us the third property.

Property 3. $\lim_{n\to\infty} (F_n)^{1/(n-1)} = \varphi$

Applying these three ideas to the consequences of the derivative tests from before gives us the limit of (z_n).

Proposition. *For each n let z_n be the real root of the polynomial g_{2n-1} defined in Equation 2. Then $\lim_{n\to\infty} z_n = -\varphi$.*

Proof. Let M_n be the value at which f_{2n+1} has a local maximum. That is $M_n = -\left(\frac{F_{2n+1}}{2n+1}\right)^{1/(2n)}$. Then it follows that $-\varphi \leq z_n \leq M_n$.

By the properties above we have the following limit.

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} -\left(\frac{F_{2n+1}}{2n+1}\right)^{1/(2n)} = -\frac{\lim_{n\to\infty}(F_{2n+1})^{1/(2n)}}{\lim_{n\to\infty}(2n+1)^{1/(2n)}} = -\frac{\varphi}{1}$$

Hence $-\varphi \leq \lim_{n\to\infty} z_n \leq \lim_{n\to\infty} M_n = -\varphi$. That is $\lim_{n\to\infty} z_n = -\varphi$.

□

Laurent polynomials. If we divide both sides of the equation $\varphi^2 = \varphi + 1$ by φ^2 then we get $1 = \varphi^{-1} + \varphi^{-2}$. Then we can use this fact to write negative integer powers of φ as a linear combination of φ^{-1} and 1 in an analogous manner to the way we did in (1) as follows:

$$\varphi^{-2} = -\varphi^{-1} + 1, \quad \varphi^{-3} = 2\varphi^{-1} - 1, \quad \varphi^{-4} = -3\varphi^{-1} + 2, \quad \varphi^{-5} = 5\varphi^{-1} - 3, \ldots$$

Using the same ideas as before, we can write a sequence of Laurent polynomials of the form

$$\ell_n(x) = x^{-n} + (-1)^n F_n x^{-1} + (-1)^{n+1} F_{n-1}.$$

Each of these will have $\ell_2(x) = x^{-2} + x^{-1} - 1$ as a factor and so we can express them as

$$\ell_n(x) = \ell_2(x) \cdot \sum_{i=0}^{n-2} (-1)^i F_{i+1}\, x^{i+2-n}.$$

We call the second factor h_{n-2} whose general form looks like

$$h_n(x) = \sum_{i=0}^{n} (-1)^i F_{i+1}\, x^{i-n}.$$

Then we can connect the Laurent polynomials to the polynomials from before using a function transformation. Notice that we can realize h_n from g_n as follows.

$$(-1)^n g_n\left(-\frac{1}{x}\right) = (-1)^n \sum_{i=1}^{n+1} F_i \left(-\frac{1}{x}\right)^{n+1-i} = \sum_{i=0}^{n} (-1)^i F_{i+1}\, x^{i-n} = h_n(x)$$

So the roots of h_n will correspond to the roots of g_n using the same transformation. Hence h_n will have no roots when n is even. When n is odd, the real root y_n of h_n will correspond to z_n according to $y_n = -1/z_n$. Note that the factor of $(-1)^n$ is not necessary since the function values are zero. Thus

we can see that the sequence (y_n) of real roots of the odd degree Laurent polynomials h_{2n+1} converges to $\varphi - 1$ as follows.

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(-\frac{1}{z_n}\right) = \frac{-1}{\lim_{n\to\infty} z_n} = \frac{-1}{-\varphi} = \varphi - 1.$$

Summary. We define a sequence of polynomials and a sequence of Laurent polynomials by considering powers of the golden ratio φ. A subsequence of each polynomial sequence corresponds to a sequence of real numbers. The numerical sequences have interesting limits.

References

[1] E. Burger & M. Starbird, *The Heart of Mathematics: An Invitation to Effective Thinking*, 3rd. Ed., Wiley, 2009.

[2] S Vajda, *Fibonacci & Lucas numbers, and the Golden Section: Theory and Applications*, John Wiley & Sons, New York, 1989.

SAINT MARY'S MATH CONTEST QUALIFYING PROBLEMS

HIGH SCHOOL DIVISION

INTRODUCTION BY NANCY BLACHMAN AND
PROBLEM SETS PROVIDED BY NICK BAXTER

Solving the Saint Mary's Math Contest Qualifying Problems with my father, Nelson Blachman, inspired me to to found the *Julia Robinson Mathematics Festival* (see jrmf.org) in 2007.

I found the Saint Mary's Math Contest (SMMC) qualifying problem sets problems intriguing and challenging to solve. I have fond memories of my father asking questions that helped me to figure out ways of approaching problems and solving some of them. When I solved a Saint Mary's math problem, I felt a sense of accomplishment.

Students were given a set of 8-10 problems four times a year. We were given about a month or two for each set, ample time for experimentation and research. To get credit for solving a problem, not only did we need to find a reasonable answer, but we also had to write up our logically defensible process for solving the problem.

In 2006, I searched on the Internet for Saint Mary's Math Contest and was disappointed to find little of significance. While attending the East Bay Community Foundation Math/Science Fair at the Chabot Space and Science Center in Oakland, California, I asked Jim Sotiros if he knew anything about SMMC. Jim asked Hugo Rossi, the Deputy Director of the Mathematical Sciences Research Institute (MSRI). Hugo, in turn, asked a half a dozen or so of his math circle buddies. Joshua Zucker informed him about the contest, having learned about it from the book *Saint Mary's College Mathematics Contest Problems* (published by Creative Publications, Inc. in 1972) he won as a prize when he was in high school in Los Angeles. He later came across files of problem sets from the contest when he was a math teacher at Gunn High School in Palo Alto. Unfortunately, the Saint Mary's Math Contest was discontinued in 1985.

Being a great development director at the Mathematical Science Research Institute (MSRI), Jim Sotiros asked me if I wanted to resurrect the Saint Mary's Math Contest. With Joshua Zucker and Jim Sotiros' encouragement, I decided to create a festival rather than a competition and name it after Julia Robinson (1919-1985), who was renowned for her role in solving Hilberts tenth problem, a conundrum that had baffled the worlds finest minds for half a century, and was the first woman president of the American Mathematical Society (1983-1984).

The goal of the festival is to expose not only students, but also educators to different ways of doing math. Festival activities are designed for students to make discoveries. Attendees may work together or individually, whichever they prefer. There is at least one facilitator at each table who strives to provide encouragement and ask questions rather than provide answers. The festival problems offer diverse accessible entry points arithmetic, hands-on puzzles, card tricks, patterns, coloring so that K-12 students hopefully will find one or more activities that grab their attention. Participants are encouraged to seek logical approaches to problem solving, not just answers. The first Julia Robinson Mathematics Festival was hosted by Google in April 2007. Our second Festival was hosted bv Pixar in 2008. There have been over a hundred of

these festivals in the US and abroad. Unlike math competitions, these non-competitive Julia Robinson Mathematics Festivals equally attract girls and boys. For more information about the Festival, visit `jrmf.org` and the Huffington Post article `tinyurl.com/JRMFcolm`.

Saint Mary's Math Contest

In 1975, Saint Mary's College announced that it was discontinuing the competition. Many teachers expressed regret. Lyle Fisher and William Medigovich, teachers at Redwood High School in Larkspur, California, took over the program, renamed it, and compiled the book *Brother Alfred Brousseau Problem-Solving and Mathematics Competition Senior Division* (Dale Seymour Publications, 1984). The following introduction and history is excerpted from that book.

Introduction to the Saint Mary's Math Contest

We believe that problem solving can and should be stimulating, challenging, and fun. Problem solving is an art; it requires a certain feel, or touch.

The problem solvers of the future will be those who can examine information and look for patterns that suggest solutions. The ability to generalize from specific data may be a better preparation for a student than the repetition of arithmetic skills. It is important to make mathematics vital and exciting, and problem solving is the approach.

Though mathematics may be a paper and pencil sport, when approaches to solutions demand long and tedious calculations, we encourage the use of calculators and computers.

History of the Saint Mary's Math Contest Qualifying Problems

In 1959, long before others saw the importance of teaching problem solving, Brother Alfred Brousseau of Saint Mary's College, Moraga, California, recognized the need. He began a problem-solving and mathematics competition for junior and senior high students.

The competition was unique in that the core of the program extends over the entire school year. Students were given a set of eight to ten problems four times during the year and were allowed ample time for experimentation and research. They were instructed not only to find a reasonable answer, they also were encouraged to develop a logically defensible process for solving the problem. Answers weren't accepted unless accompanied by a fully developed solution.

When Brother Alfred Brousseau and his colleague at Saint Mary's, Brother Brendan Kneale, initiated the program, its purpose was to stimulate math achievement in the nine high schools of the Christian Brothers of California. A statement as to the nature of the competition was made that year. "The ratings of problems will take into account the following factors: (1) correctness of the solution; (2) its brevity and elegance; (3) neatness of presentation."

Four problem sets were distributed to participating schools on specific dates and were returned to Saint Mary's and Holy Names' staff members for grading.

At the ninth Gathering for Gardner (G4G9), I learned that Nick Baxter also participated in Saint Mary's Math Contest and that he saved his problem sets. I borrowed them, made copies, and typeset the problems so you won't need to decipher my mimeographed copies.

Problem Set #2, November 15, 1972, Due December 15, 1972

11. A die is thrown until one of the number previously obtained comes up again. What is the average number of throws for which the throws are different?

12. In the Fibonacci sequences, $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$, etc., where each number after the first two is the sum of the two preceding numbers, find pairs of numbers whose product is 9 greater than the square of a Fibonacci number. As a result, make a conjecture regarding an infinity of such pairs.

13. There are four tangent spheres of radius r (each sphere is tangent to the other three externally). Find the radius of the sphere between the four spheres and tangent to each of them.

14. Study the Pascal triangle and determine which rows consist entirely of numbers *not* divisible by 3.

15. An idealized version of the trunk of a tree might be taken as a cone. If the growth of the tree continues to give a cone similar to the original cone and if the amount of substance added yearly is the same, find an expression for r_{n+1}/r_n where r_n is the bottom radius of the tree at the end of n years.

16. Given the following algebraic operations along with the commutative and associative laws of addition and multiplication and the distributive law of multiplication over addition:

$$a + a = a \qquad aa = a \qquad (ab)' = a' + b' \qquad (a+b)' = a'b'$$

$$a + ab = a \qquad b + b' = 1 \qquad bb' = 0$$

Simplify the expression $(a + bc)'(b + b'c)'$
(Prime is an operation of complementation.)

17. Find all the right triangles with integral sides for which the radius of the in-circle is 3. Explain your method of arriving at your results.

18. Find the cube root of $-46 + 9i$ in the form $a + bi$ where a and b are integers an i is the square root of -1. Show the method employed.

19. A police car traveling at 120 ft/sec is following another car going at the same speed at a distance of 1320 ft. The top speed of the police car is 150 ft/sec. At a given time (call this $t = 0$), the police car starts to accelerate. What acceleration should the police car have if it is to overtake the other car just when it reaches maximum speed? Equations of motion:

$$v = v_0 + at$$

$$s = v_0 t + \frac{1}{2}at^2$$

where v_0 is the speed at time $t = 0$, v is the speed at time t, a is the acceleration and s is the distance covered since $t = 0$.

20. A recursion operation is performed as follows. A three-digit integer in base ten abc has its digits operated on as follows:

$$20c + b + a$$

to form a new integer $a'b'c'$. Is there any three digit integer that are respectively equal to a', b', c'? Derive.

PROBLEM SET #3, JANUARY 5, 1973, DUE FEBRUARY 15, 1973

21. Given an $n \times n$ square (a checkerboard would be such with $n = 8$). Determine a formula for the following. We wish to know in how many ways 4 squares can be in sequence either horizontally, vertically, or diagonally (as 45° to the horizontal in either direction). Find a formula that gives this result. Explain.

22. Use a tabular method analogous to the Pascal Triangle to find the coefficients in the expansion of

$$(1 + x + x^2 + x^3 + x^4)^5$$

23. The midpoints of a parallelogram (P_0) are connected in sequence to give a quadrilateral P_1. Then the midpoints of P_1 are connected in sequence to give a quadrilateral P_2. And so on. Express the area of quadrilateral P_n in terms of the area of P_0.

24. Consider the sequence of values formed as follows:

$$(5 + 1)/6 = 1$$

$$(5^3 + 1)/6 = 21$$

$$(5^5 + 1)/6 = 521$$

$$(5^7 + 1)/6 = 13021$$

etc.

Find the recursion relation for this sequence in the form

$$T_{n+1} = AT_n + BT_{n-1}$$

Show that the relationship holds in general.

25. An equilateral triangle has sides s. Each side is divided into three equal parts and on the middle segment an equilateral triangle of side $s/3$ is constructed. Consider now the outside perimeter of the resulting figure. The process is repeated five times in all, at each step each line segment in the figure is divided into three parts and an equilateral triangle constructed on the middle segment. What is the perimeter of the resulting figure in terms of s?

26. The corners of a cube are cut off so that the cuts do not interfere with or touch each other. Then the same process is repeated for the resulting figure. How many space diagonals does the final figure have?

27. The difference of the cubes of two consecutive odd primes is 31106. What are those primes?

28. Objects are in positions $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$. Their positions are changed according to the linear transformations $2k + 1$. For example, 1 goes to 3, 2 goes to 5; 11 goes to 23 reduced by 13 or 10. However 6 goes to 13 (not zero). After the positions have been changed the same operation is performed again. How many such transformations are required to return the objects to their original positions?

29. If in a given interval $|x - x_0| \leq d$ around a fixed point x_0, the maximum of the absolute value of $f(x)$ is M, while in the same interval the maximum of the absolute value of $g(x)$ is M', while
$$|f(x) - f(x_0)| < e$$
and
$$|g(x) - g(x_0)| < e'$$
in the interval, show that
$$|f(x)g(x) - f(x_0)g(x_0)| < Me' + M'e$$

30. Analyze the following equation and plot its graph.
$$|y| + |x - 3| + |x - 7| = 10$$

Problem Set #4, February 25, 1973, Due March 15, 1973

31. Cut a 10×10 square into two pieces that can be fitted together to form a $60/7$ by $35/3$ rectangle. Explain.

32. According to Fermat's last theorem[1], there is no solution in positive integers to the equation
$$a^3 + b^3 = c^3$$
Find the minimum value that may be taken $|a^3 + b^3 - c^3|$ where a, b, c are positive integers, and $a < b < c$. (*Hint:* Look for a numerical example that will give the minimum.)

33. A straight line connects the points $(20, 0)$ and $(0, 30)$. If a line is drawn from $(4, 0)$ to $(14, 9)$ on this line, what is the equation of the reflected line (reflected as if the given line were a mirror)?

34. There are three mutually tangent circles of radius r. External tangents to the circles in pairs form an equilateral triangle. What is the area inside this triangle, but outside the circles?

35. Given an $n \times n$ tic-tac-toe structure in space where each level consists of an $n \times n$ board and there are n such boards. Spaces can be arranged in sequence n at a time horizontally, vertically, and diagonally, including space diagonals. What is the formula for the number of ways n spaces in this structure can be aligned? Explain.

36. Factor $x^{30} - 1$ into eight factors with integer coefficients. Explain.

[1] At the time this problem set was written, Fermat's last theorem was generally believed to be true. It was proved by Euler for cubes and finally proved for all larger exponents by Andrew Wiles in 1994.

37. Given that the sequence $7, 27, 111, 483, 2199, 10347, \ldots$ is the sum of two geometric progressions, namely
$$ar^{n-1} + bs^{n-1}$$
determine the values of a, r, b, and s by means of algebra.

38. A table of numbers
$$\begin{array}{ccccc} 1 & 1 & & & \\ 1 & 3 & 2 & & \\ 1 & 5 & 8 & 4 & \\ 1 & 7 & 18 & 20 & 8 \end{array}$$
is built up as follows. There are multipliers 2 and 1 that operate on successive elements of one row to give an element in the next row. Thus to obtain the latest given row from the preceding row, we multiply 2 and 1 by 0 and 1 respectively, add the results and obtain the first element 1. Then we multiply 2 and 1 and 5 respectively, add the results, and obtain 7. Multiply 2 and 1 by 5 and 8 respectively, add the results, and obtain 18. Multiply 2 and 1 by 8 and 4 respectively, add, giving 20. Multiply 2 and 1 by 4 and 0 respectively, add, giving 8.
Find the sum of the numbers in the fifteenth row.

39. Prove that $x^2 + 3x + 1$ does not divide $x^{24} - 9x^{12} - 1$.

40. Prove that the determinant
$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$
is non-negative if $a, b, c,$ etc. are real.

Problem Set #1, October 1, 1973, Due November 1, 1973

1. A table

1	1	1	1						
1	2	3	7	4	3	2	1		
1	3	6	10	12	12	10	6	3	1

is built up after the manner of a Pascal triangle, but adding four consecutive terms instead of two. What is the sum of all the numbers in the first ten rows of this table?

2. Find the equation whose roots are $2 \pm i$, $4 \pm 2\sqrt{3}$, 5, -6.

3. A bullet of radius $3/16''$ is shot into a block of styrofoam at an angle of $30°$ to the vertical to the surface and comes out at the same angle through a parallel surface, the length of the track being 30 inches. What is the volume of the hole?

4. Two regular hexagons, one of side 6 inches and the other of side 4 inches are similarly placed with their centers 8 inches apart. What is the difference of the non-overlapping areas in the two polygons?

5. Find the equations of the plane through the intersection of the two planes
$$3x - 7y + 4z = 5 \qquad 2x + 5y - 8z = 11$$
and the point $(4, -7, 13)$.

6. If b is 25, find a value of a satisfying $a^b = b^a$ to the nearest thousandth.

7. Find the smallest base in which
$$15^2 + 21^2 = 630$$
(all these numbers are in the required base).

8. Find the forms of all integers in terms of prime factors that have exactly 24 divisors. One example would be:
$$p^5 q^3.$$

9. A square is inscribed in the ellipse
$$x^2/a^2 + y^2/b^2 = 1$$
with its sides parallel to the axes of the ellipse. Find its area.

10. There are ten questions in an examination. A student feels that he knows the answers to five of them perfectly. For two, he thinks he has a 50% chance of being correct and for the other three, a 25% chance of being correct. What is his probability of getting 80% or better in the test?

Problem Set #2, November 15, 1973, Due December 15, 1973

11. There are 100 tickets in a lottery and five prizes are to be given. Prove or disprove the following: Buying two tickets doubles the probability of winning at least one prize.

12. What is the side of a regular tetrahedron for which the volume is numerically equal to the surface area? Show derivation.

13. The expression
$$\frac{1+2x}{1-x-x^2}$$
expanded into a finite series. Find the first ten coefficients in the expansion and the law governing the relation of these coefficients.

14. Find the quadratic equations whose roots are the square of the roots of
$$x^2 + ax + b = 0$$

15. An airplane is traveling at 500 miles an hour at an angle of 7.5° to the horizontal. How long will it take to rise 40,000 ft vertically?

16. Express
$$\frac{40!}{16!24!}$$
as the product of primes of powers beginning with the smallest prime.

17. A number N (positive integer in base ten) is divided by three and the integer quotient found. Then this quotient is divided by three; etc. Find a relation between N, the sum of the quotients (Q), and the digit in the base three representation of the number. Example of the process. $N = 632$. Quotients are $210, 70, 23, 7, 2$. Their sum is 312. The base three representation is 212102. The required relation involves the sum of the digits of the base three representation.

18. For the following equations, find the number of solutions in positive integers.
$$\begin{aligned} x+y+z &= 2 \\ x+y+z &= 4 \\ x+y+z &= 6 \\ x+y+z &= 8 \\ x+y+z &= 10. \end{aligned}$$
For example, for $x+y+z=8$, we could have $x=4, y=2, z=2$. But these quantities may be permuted to give $x=2, y=4, z=2$ and $x=2, y=2, z=4$.
Part Two: Find an expression for the number of solutions of $x+y+z=2n$.

19. Given a line segment $1''$ long. Points above the line are taken and joined to the ends of this line segment so as to form an angle of 60°. What is the measure of the area enclosed by the given line segment and the curve formed by these points?

20. Dice are constructed with two 1's, two 2's, a 3 and a 4. Show the number of ways the various sums can come up with four of these dice.

PROBLEM SET #3, JANUARY 5, 1974, DUE FEBRUARY 15, 1974

21. Amounts are put in the bank as follows: At the end of 1 year, \$2048; at the end of the second year, \$1024; at the end of the third year, \$512; ... this continues with the amount being put in at the end of each year being one-half what was put in at the end of the previous year. The process continues for twelve years ... \$1 being put in the bank at the end of the 12th year. If interest is at 5% compounded annually, how much money is in the bank at the end of the twelfth year? *Note:* The formula for compound amount is $P(1+i)^n$, where P is the principle, i is the interest rate per conversion (compounding) period and n is the number of conversion periods.

22. A car starts from rest accelerating at 5 ft/sec/sec. Two seconds later, another car starts from rest accelerating at 7 ft/sec/sec. How far will each of the cars have traveled by the time the second car catches up with the first? Distance covered from rest at acceleration a in t seconds is $\frac{1}{2}at^2$.

23. Find all the primes (report them in base ten notation) whose reciprocals give a decimal period of length five in base seven.

24. For the sequence $T_1 = 1, T_2 = 3, T_3 = 10, T_4 = 33, T_5 = 109, \ldots$ with the general recursion relation
$$T_{n=1} = 3T_n + T_{n-1}$$
find the continued fraction representation of $T_2/T_1, T_3/T_2$, etc. and conjecture the form of the continued fraction representation of T_{n+1}/T_n.

25. A parabola with its axis parallel to the x-axis has a general form
$$(y-a)^2 = bx + c$$
Find the equation of such a parabola going through $(3,4), (-1,7)$ and $(4,-6)$.

26. Find the value of
$$\sum_{k=76}^{99} k^3$$

27. Find five unit fractions (numerator is 1, denominator is positive integer) whose sum is a unit fraction for which the denominators are in the ratios $3:4:5:6:7$.

28. Two triangles have two sides in proportion, i.e., $a : b < a' : b'$ while their areas are such that
$$A : A' = a^2 : a'^2 = b^2 : b'^2$$
Prove or Disprove: The triangles are similar.

29. There is a series of concentric spherical shells such that the volume inside the first sphere is to the volume between the first two spheres as 1:2 the volume between the first two spheres is to the volume between the second and third spheres as $2:3$; etc. the successive ratios being $1:2:3:4:5:6:\ldots$. Let the radii of the spheres be $r_1, r_2, r_3, \ldots$. Find the ratio:
$$r_1 : r_2 : r_3 : \ldots : r_n$$

30. The opposite sides of a quadrilateral are $120°$ and $80°$. On measuring the diagonals, it was found that the diagonal was 1.093 ft more than the other. By how much (in minutes and hundredths of a minute) do the angles differ from $90°$?

PROBLEM SET #4, FEBRUARY 25, 1974, DUE MARCH 25, 1974

31. Find the value of $\cot 15°$ in closed radical form.

32. Find all the primes whose reciprocal gives a decimal with a period length of five.

33. A line segment is drawn from $(-3, -7, 4)$ to $(18, 42, 18)$ in space. How many points with integral coordinates are on this line segment? Explain.

34. Tangents are drawn to the circle
$$x^2 + y^2 = 25$$
at $(3, 4), (3, -4), (-3, 4)$, and $(-3, -4)$. What is the area enclosed by the quadrilateral formed by these tangents?

35. If the roots of the equation $x^3 - 7x^2 + 5x - 8 = 0$ are each increased by 2, what is the equation with these altered roots?

36. A spiral ramp goes around a cylinder of radius 20 ft with an angle of rise of $10°$. What distance is covered in 20 circuits of the cylinder?

37. In a game you throw two dice, the house betting that you will get a seven in a certain number of throws. What is the minimum number this might be if the house is to make a profit?

38. Two lines are drawn from the centroid of a regular tetrahedron to two of the vertices of the tetrahedron. Find the cosine of the angle between these lines.

39. Form a "Pascal" triangle with two numbers a, b in the first row and the other numbers building up in the usual way a "Pascal" triangle builds. Select a and b so that the sum of the numbers in the upward slanting diagonal (left-justified table) gives the sequence: $1, 4, 5, 9, 14, \ldots$ with each term after the first two the sum of the two preceding terms. Show the first six lines of the table.

40. If $f_1 = x, f_2 = 1/1 + x, f_3 = (1 + x)/(2 + x)$, where at each step x is replaced by $1/(1 + x)$, what is an expression for f_n with coefficients given in terms of a well known sequence of numbers?

Problem Set #1, October 1, 1974, Due November 1, 1974

1. Find the equation of the line passing through the intersection of
$$3x - 7y = 4$$
$$2x + 6y = 5$$
and the point $(7, -12)$.

2. Build an equilateral triangle on each side of a square facing into the square. Connect the four vertices inside the square. Find the ratio of the area of this figure to the area of the square.

3. Form a sequence of squares consisting of six terms such that the sum of the first two is a square, the sum of the first three is a square, etc. Explain your method of arriving at this sequence.

4. Alice in Wonderland came to a hole that was 1/5 of her height. How to get through. The Rabbit began nibbling on a fungus and at each nibble decreased in volume by 10%. Alice made a quick calculation: How many nibbles will it take before I reach 1/5 of my height?
   So she got through the hole. But on the other side people seemed to be enormous. But again the Rabbit found a fungus such that each nibble increased one's volume by 10%. Another quick calculation: How many nibbles to get back to normal height?
   Your answers and calculations, please.

5. Solve for x in degrees and minutes.
$$\sin(x - 12^\circ) = 3\sin(60^\circ - x)$$

6. Prove or disprove: An equilateral triangle can be inscribed in any regular polygon. (Inscribe means that its vertices are on the sides of the polygon.)

7. The circle
$$x^2 + y^2 = 144$$
is drawn on graph paper. How many complete squares on the graph paper are inside the circle? (If the circle cuts through a square, that square is not counted; if the circle goes through the upper corner of a square, the square is counted.)

8. For the equation
$$x^2 - 3x - 5 = 0$$
with roots r and s, find the value of
$$r^7 + s^7.$$

9. What is the probability of getting three or more 1's when throwing six dice?

10. Given a set of n points in a plane (n is even). Is it possible to find a line such that the projection of the n points on the line gives n distinct points? Explain.

Problem Set #2, November 15, 1974, Due December 15, 1974

11. Find the dimensions of a box with integral edges such that the surface area is twice the volume.

12. Prove that for the infinite sequence $1, 2, 4, 8, 16, 32, \ldots$ consisting of the powers of two, no three terms are in arithmetic progression.

13. A tunnel 13 ft. in diameter is discharging water at the rate of 10 ft/sec over the entire area of the tunnel. If a lake is 6 miles long and 3/4 miles wide on the average, how long will it take to raise the level of the water 1 inch? Express the answer in hours, minutes, and seconds.

14. Find a rule of divisibility by 4 in base seven and prove that this rule holds in general. (A rule of divisibility enables one to determine whether the number is divisible by 4 using the digits of the number without actually carrying out the division.)

15. There are three containers with balls as follows:
    A. 5 white, 3 black.
    B. 4 white, 4 black.
    C. 3 white, 7 black.
    What is the probability that on transferring a ball from A to B, then from B to C, and finally from C to A, the contents of the three containers remains unchanged?

16. Given an ellipse with its major diameter in place. Find a ruler and compass construction for determining the foci.

17. n equally spaced points are marked on a circle. Starting with any point, every k^{th} point is connected (k is less than $n/2$). In this way a star is formed. (If k and n have a common factor, it will be necessary to have more than one starting point to cover all the points.) What is the angle at the vertex of the star? Prove your result.

18. Given the first n integers. A particular arrangement of these integers will have a certain number of inversions. An inversion is defined as a situation where a larger number precedes a smaller number. Thus, for the arrangement $3, 1, 4, 2, 5$ of the first five integers, 3 is larger than 1 or 2 and 4 is larger than 2. Hence there are three inversions. Find a rule for determining the number of inversions when the sequence of integers is reversed. Prove this rule.

19. Two men take four days to do a job when working together (call them A and B); B and C take five days when working together; C and A take 6 days when working together. How long does it take for all three working together to do the job?

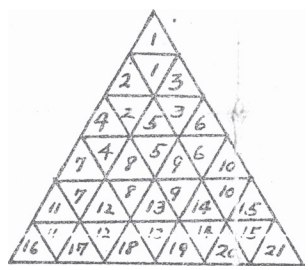
20. Find the value of:
$$\sum_{a=1}^{\infty} 1/(a^2 + 16a + 63)$$

Problem Set #3, January 5, 1975, Due February 15, 1975

21. Two teams A and B are playing a series of five games in a playoff. The team that wins three games first wins. A has sixty percent probability of winning a single game; B has a forty percent probability of winning a single game. What is the overall probability that team A wins the series?

22. The corners of a square of sides s are lopped off and rearranged with the remaining figure to form a square of the same size. What is the length of the sides of the right triangles removed from the corners?

23. In a triangular diagram where the side of the triangle is of length n and the side is divided into n unit parts, the upward pointing triangles are numbered and the downward pointing triangles are numbered as in the figure (Case of $n = 6$ is shown). Find a formula for the sum of the numbers in the triangle. (For $n = 6$, the sum is 351.)



24. The length of the period of 1/7 in base two is 3, of 1/13 is 12 and of 1/17 is 8. What is the length of the period of

$$(1/7)(1/13)(1/17) = 1/1547?$$

Explain.

25. A freight train traveling 45 miles/hour decelerates at the rate of 1/2 ft/sec/sec. How far does it go before stopping?
Equations of motion for uniformly accelerated motion are:

$$d = v_0 t + (1/2)at^2$$

$$v = v_0 + at$$

$$v^2 = v_0^2 + 2ad$$

where v_0 = initial velocity at time taken as $t = 0$, v = final velocity at time t, a = uniform rate of acceleration and d = distance traveled in time t.

26. Find the equation whose roots are the square of the roots of the equation:

$$x^3 + px^2 + qx + r = 0.$$

27. Six dice are thrown. What is the probability that three pairs will show?

28. If the probability that people ar born in a given month is the same for all months, how many people should there be in a group so that the probability that two people are born in the same month is greater than 75%?

29. Find the value of the infinite sequence of cube roots.

$$\sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\dots}}}$$

30. Three circles of radius r are mutually tangent in pairs. What is the radius of the circle that is tangent to all three circles?

Problem Set #4, March 1, 1975, Due April 3, 1975

31. A circle of radius 10 inches has ten equal circles tangent to it and tangent to each other placed around it. What is the radius of this circle?

32. The triangular number $T_n = n(n+1)/2$. Find four instances of three consecutive triangular numbers adding up to a perfect square.

33. Prove that if $A + B + C = 180°$, then

$$\tan(A) + \tan(B) + \tan(C) = \tan(A)\tan(B)\tan(C)$$

and if

$$A + B + C = 90°,$$

then

$$\tan(A)\tan(B) + \tan(B)\tan(C) + \tan(C)\tan(A) = 1.$$

34. Find an expression for the side of a regular polygon of n sides for which the perimeter is numerically equal to the area.

35. In the *Mathematics Magazine* of January 1975, p. 51, the following problem is proposed. Eight points are taken on a circle and connected consecutively to form an inscribed octagon. Prove that if the angles at vertices $1, 3, 5, 7$ are added, the sum is the same as for the angles at vertices $2, 4, 6, 8$ and the sum is 3π.

    Discover and prove the following. If $2n$ points are taken on a circle and connected consecutively to form an inscribed polygon of $2n$ sides, then the sum of the angles at vertices $1, 3, 5, \dots, 2n-1$ is the same as the sum of the angles at the vertices $2, 4, 6, \dots, 2n$ and the sum is equal to $(n-1)\pi$.

36. Given n quantities. Determine a formula for the number of sums that can be formed with these quantities. (A sum involves at least two quantities.)

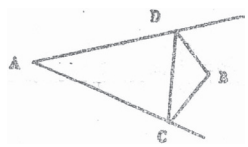
37. An integer of the form $aabb$ has its digits permuted in all possible ways. Prove that no matter in what base the integer is interpreted, it may not represent a prime in any of its permutations.

38. Let z equal the infinite continued fraction

$$1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\ddots}}}$$

where the denominators are alternative 1 and 2. Find the value of z in closed radical form.

39. A mirror consists of two vertical strips, making an angle A between them. If a point source of light at B has the light reflected at C and D and return to B, find the angle CBD in terms of angle A.

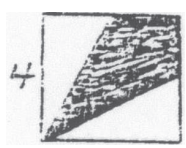


40. There are six cats, each of which is pursuing one of the other cats and each of which is being pursued by one of the other cats. In how many different ways may this be done? (Cats may form several groups.)

Problem Set #4, March 15, 1979, Due April 15, 1979

31. A two-digit number consisting of two different non-zero digits has its digits reversed. The two numbers are then squared and the difference taken. What is the largest integer that always divides this difference?

32. If a vertex angle on a square with side 4 is trisected, the following figure results:



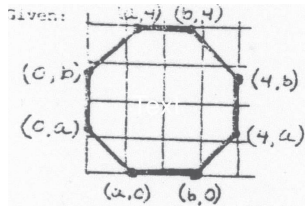
Find the area of the shaded portion.

33. An automobile comes a distance d_1 at speed s_1, then a distance d_2 at a speed s_2, and finally a distance d_3 at speed s_3. Write a formula for its average speed over the entire distance.

34. A spherical balloon 5 inches in diameter has a thickness of .05 inches. What is the thickness when the balloon has a diameter of 9 inches? Find the answer correct to four decimal places.

35. Sue Harris was drilling for water in Death Valley and hit pay sand. She noticed the spot was 10,000 meters from one corner of a rectangle plot, 15,000 meters from the opposite corner, and 6,000 meters from a third corner. How far is it to the fourth corner?

36. Given



What is the value of a and b if the shaded figure is a rectangular octagon? Use

$$\sqrt{3} \approx 1.732$$
$$\sqrt{2} \approx 1.414$$

37. Find a set of whole numbers when the product of each by the sum of all the others is known. You are to find six number a, b, c, d, e, f if you are given that:

$$a(b+c+d+e+f) = 184$$
$$b(a+c+d+e+f) = 225$$
$$c(a+b+d+e+f) = 301$$
$$d(a+b+c+e+f) = 369$$
$$e(a+b+c+d+f) = 400$$
$$f(a+b+c+d+e) = 525$$

38. How many ways could one make \$2.43 with 5-cent and 8-cent stamps? What are the possible combinations?

39. It is possible to select four grid points on a coordinate system so that the joins of all points (six in all) do not contain any other grid points. The points $A(2,3)$ and $B(11,17)$ have a join with no other grid point on it. Find two other points C and D so that no three of the points are on the same line and the joins of any of the four points, A, B, C, D, do not contain any other grid point.

40. In what bases (less than or equal to 12) is 2101 a perfect square?

The Samaritani Formula – more details

Adam Atkinson (ghira@mistral.co.uk)
Presented at G4G12 30th March to 4th April 2016
(Updated after real world events in late 2016 to early 2017)

"Fads and Fallacies" is one of my favourite Gardner books and I think the Samaritani Formula, at least as presented in the modern day, is the sort of thing Gardner would have found amusing and/or maddening.

This article contains more information than my 5 minute talk could possibly hope to. At the time of writing this, if I google "Samaritani Formula" the only hits in English I find relate to me talking about it elsewhere in 2015. If you look for "formula di Samaritani" you should find many hits in Italian, not all of which have anything to do with me. Googling "ritardi" and "Samaritani" and "lotto" in some combination should find quite a lot. And not all of it will be mathematicians arguing with Samaritani boosters.

If terms like "idiot" and "conperson" seem a bit strong, please bear in mind that bankruptcies and suicides are involved in this whole affair. I would regard the victims in this as, say, merely ill-informed or gullible. Alternatively, we could argue that "person who says untrue things without realising they are untrue, having spent some time thinking about them" and "person who says untrue things knowing them to be untrue" are much longer and would make the text unwieldy.

Italy has a lot of people offering or selling advice on how to pick lottery numbers, and while a lot of the stuff one sees is usual abuse of (some form of) the law of large numbers or the central limit theorem, one thing I've only come across in Italy is *quantitative* nonsense in the form of the Samaritani Formula. It's possible that the formula is also used nonsensically elsewhere under the same or a different name and if so I'd love to hear about it.

The amusing part is that actually the S.F. is not as nonsensical as it might seem at first glance: it merely doesn't mean what its modern boosters say it means. It is, of course, of no help whatsoever in playing the lottery.

My encounters with the Samaritani Formula have almost all involved people turning up in Italian newsgroups (typically it.scienza.matematica but also it.scienza.fisica, it.hobby.enigmi and others including the now defunct it.fan.dewdney) asking how many draws in a row a number in the Italian lottery can fail to turn up. If you are very brave, look at it.hobby.lotto. Well, actually they ask what the "theoretical maximum delay" in Lotto is. ("ritardo massimo teorico"). We have always taken this to mean "largest possible value" but at least a few ritardisti say it means something else.

Of course we all say "there is no limit" or "infinity" or similar but some of these people want to be told the answer is 220, or tell us that it is. (It seems likely the number they give must have been updated for reasons which will become clear below.)

If you ask them why 220 is the limit, they say the Samaritani Formula says so.

$$\log_{(17/18)} 1/300000 = 220$$

The first time I and others came across this, when we said it wasn't clear why this should make any sense at all, it was explained to us that 300000 was 50x6000. Which indeed it is. So the clarified version of the formula was:

$$\log_{(17/18)} 1/(50*6000)=220$$

In case this isn't immediately convincing either, now may be the time to explain where all the numbers in this come from.

The lottery we're talking about here is the main one, “Lotto”. It has 11 lottery machines, called “ruote” (wheels), each of which has 90 numbers. At each draw, 5 numbers are picked from each of the 11 wheels. So “Lotto” is 11 parallel “90 pick 5” lotteries. You can bet on various things, right down to the level of betting on a single number from one of the wheels turning up. This is apparently a popular bet, and also one of the least worst deals. It pays about 10 to 1, and of course has a 1 in 18 chance of working. Note that all the bets are fixed odds: there is no jackpot type mechanism. (So “Avoid common combinations if you have to play at all” does not apply here.)

In the formula above, 17/18 is the probability of a given number failing to appear, 6000 is the number of draws that had taken place in the history of Lotto at some point in the past, and 50 is the number of numbers drawn in each draw. Of course, you will have noticed that 50 is not 55. The reason for this is that in the late 1800s there were 6 wheels, then 7 later, 8 when Samaritani was alive, 10 some time later (including the late 1990s), and 11 now. You may see the Samaritani formula quoted as $R_{max}=\log_{(17/18)} 1/d$ where R_{max} is the maximum run of absences being calculated and d is the number of numbers that have ever been drawn. In simulations I use a fixed number of wheels and at least for now will live with fixed numbers of wheels in attempts to model Lotto as well.

Of course when we understood they were asking about the maximum possible run of absences in 6000 draws not in general, we changed our answer from infinity to 6000. In case 6000 draws seems like a large value to use in examples, bear in mind that Lotto has been around for well over a century. The frequency of draws has increased over the years, and is currently at three a week.

Some Italian lottery players (and the media in Italy) refer to the number of draws a number has been absent as its “delay” (“ritardo” in Italian) and people who claim to obtain insight from studying these delays are referred to (perhaps not by themselves) as “ritardisti” (singular: ritardista). I will use the term “delay” in this article because it's shorter than “block of absences”. I have occasionally used the term “delay theorist” when talking about ritardisti in English but I'll use “ritardisti” in this article just in case there actually are people called “delay theorists” who are not idiots. Or conpeople.

Some Samaritani fans claim that Samaritani claimed that the maximum possible block of absences of a number in the Italian lottery was the logarithm (to base 17/18) of one over the number of numbers that had been drawn. Clearly this is a number that increases slowly with time. Equally clearly, if Samaritani said any such thing he was some kind of idiot. Or con artist. (Some other Samaritani fans are less clear about the meaning of the number that comes out of the formula, and may say that it is rarely breached, and only by a little, or that it is “extra information”, without saying what use it might be.)

They then say that clearly you can guarantee winning a bet on a single number on the Italian lottery by

waiting for one to reach a delay of (say) 160, then making progressively larger bets on it every draw until it appears. (Over almost 20 years of this, the 160 and 60 version has been very consistent. They are all getting this from somewhere, I have to assume.) Since single number bets pay about 10 to 1, you only need to increase your bet by about 11% (actually 10.46 but I'm rounding up) each time you fail. And if 220 is a real limit you only need to be able to do this 60 times in a row because, at least some ritardisti say, a delay of 221 is impossible. (Maybe they have amended this to a larger value in the last 20 years). Of course, you need to have several thousand euros available to be able to "follow" a number for 60 draws like this and only stand to win 10 euros if you start with bets of 1 euro. If you have more money, you can start earlier. Of course, in real life you risk losing everything. At least one ritardista who claims "theoretical maximum" doesn't mean what we think it does has still used the 160/220 example, which suggests to me that he thinks or wants victims to think 220 has magic powers.

There is some fun to be had with ritardisti in a "mockery through participation" sort of way by taking a hard limit of 220 at face value. e.g. if multiple numbers reach a delay of 220 at the same time, they all have to appear in the next draw. But what if more than 5 numbers on the same wheel do? Clearly they can't. This would force some of them to come out the draw before. One could clearly then imagine a situation where the same 5 numbers came out 204 times in a row and among the others we have 5 with a delay of 220, 5 with a delay of 219, and so on. A hard limit Samaritani fan would have to say that this would force the next 17 draws on that wheel completely to avoid violating the hard limit of 220. You could then bet on combinations of 5 with guaranteed success 17 times in a row. This of course sounds absurd, yet we're just taking what (at least some) ritardisti say and running with it. For some unaccountable reason this sometimes causes offence.

Actually, at least some ritardisti might respond to the above mockery by saying that a number can't appear 204 times in a row. Applying the Samaritani formula to runs of successes instead of failures will "of course" tell us some sort of magic limit on the number of successive appearances of a number.

When particular numbers reach very high "delays" one sees stories in the papers about people going bankrupt and/or killing themselves. It's not clear that I can blame ritardisti or the Samaritani Formula directly for this but they are part of a whole environment which gives the impression that ritardi are of some significance. If you google "53 Venezia" you should find some articles about events in 2005 when number 53 in Venice had not been drawn for a very long time.

Some examples from mainstream sources, in English, would be:
http://www.theguardian.com/world/2004/dec/04/italy.johnhooper
"Italy's unlucky number"

http://www.theguardian.com/world/2005/feb/11/italy.sophiearie
"No 53 puts Italy out of its lottery agony"

http://news.bbc.co.uk/1/hi/world/europe/4256595.stm
"Number 53 brings relief to Italy"

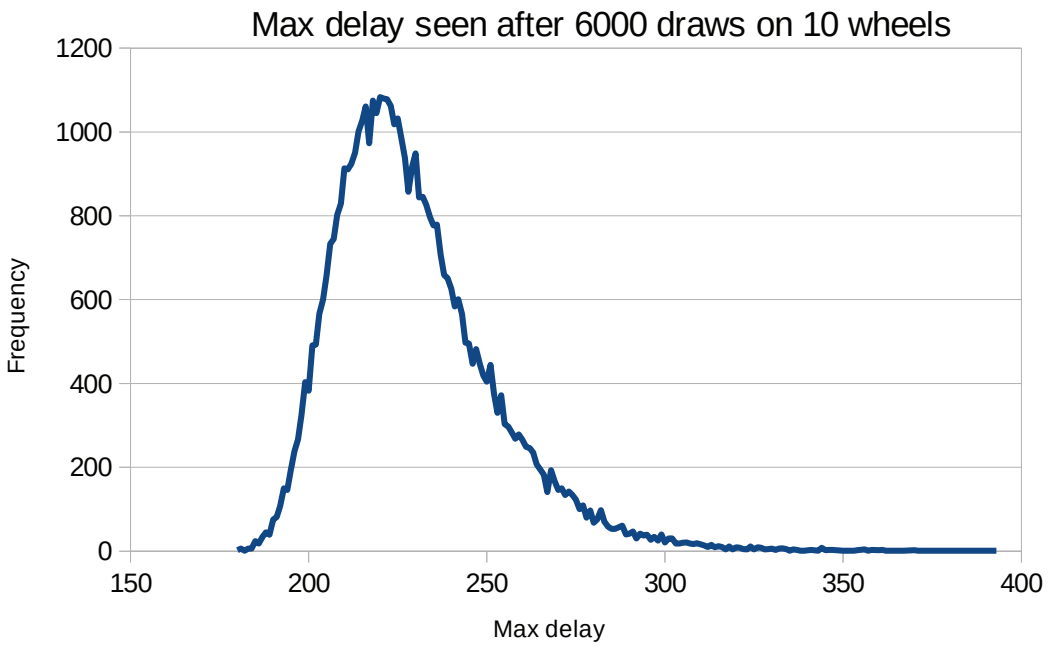
Note that number 53 in Venice in 2005 did not reach even a delay of 220, let alone whatever the magic Samaritani number would have been by then. One does have to wonder what will happen if 220, or the Samaritani number at the time, is ever exceeded. (See 2017 extras at the end: this has now happened.) My suspicion is that the Samaritani formula is known only within a fairly small group of lottery players

and the Venice 53 victims were just following the usual “It's bound to appear soon” type reasoning.

Also look on youtube for videos of “Il lotto alle otto”, a TV program about Lotto from many years ago now. I’ve seen one episode where the presenter said “This number hasn’t appeared for (number) draws. It’s ripe.” or similar, and people wore tabards with the most delayed numbers on them. No doubt if asked, the TV people would have said there was no significance to this.

One very strange event during the 2005 Venice incident was Codacons, an Italian consumer rights group, asking that number 53 be removed from the Venice draw to solve the problem of people ruining themselves. If there’s a misunderstood genius anywhere in this whole story, clearly it’s Codacons. Maybe in the long run there would be fewer and fewer numbers in Lotto? Or just replace Venice 53 with Venice 91, etc.? Also amusing.

Before we try anything else, let's simulate 6000 draws of a 10-wheel Italian lottery and see what maximum delay we get. Actually, let's do this a few thousand times so we see what the common values are and what kind of spread we see. Clearly anything from 17 to 6000 is possible though 17 and 6000 themselves will be very very unlikely.



As it happens, in the above the most common value is 220, though there may be an element of luck here. As the simulation ran the mode changed quite often between various values from 220 to 226.
The mean is a little over 229, the median is 226 and the standard deviation is about 22.2.

220 really doesn't seem to be a particularly high value for the maximum delay so if ritardisti were trying to invent some sort of magic threshold you'd think it would need to be much higher than this. Have they really never checked how commonly it is exceeded in simulations? But it is the most common value, or at any rate one of the most common.

In case it is not clear, at the beginning of a simulation, the delays of all numbers are set to 0. Then after each draw, the delays of the 5 numbers per wheel which are drawn are set to 0 and the delays of the other 85 numbers in each wheel increase by 1. (If new wheels are added other than at the start, their numbers also all have initial delays of 0 so in the first draw of a new wheel, we will see 5 “0”s and 85 “1”s.)

I contributed to some articles about this in the Italian computing magazine “MC-Microcomputer” in 1999. The other people involved were Elio Fabri and Francesco Romani of Pisa University, Dani Ferrari (a retired engineer in Rome) and Corrado Giustozzi, the author of Intelligiochi, the recreational maths column in MC-Microcomputer for 15 years (and thus Italy's answer to Martin Gardner). Giustozzi’s column hosted articles but he was not otherwise involved as far as I recall. Francesco Romani wrote the Mathematica column in the same magazine and dedicated at least one episode of it to the Samaritani formula and lottery discussions. Back issues of the magazine are now online. See e.g. https://issuu.com/adpware/docs/mc191 for the January 1999 edition. Some of our discussions took place on a mailing list whose contents are most likely now lost, and on the “matenigmici” forum on the BBC/ISP MC-Link in Rome run by the same company as the magazine. The contents of this and other forums are also presumably now lost. Italian Wikipedia has a list of topics covered in Intelligiochi over the years at https://it.wikipedia.org/wiki/Intelligiochi

At least one web site about delays switched from saying “ritardo massimo teorico” to “ritardo massimo modale” at some point after our articles appeared. That is pretty funny.

When pressed, the ritardisti I have encountered mostly appear to say they believe that lottery draws are independent and that all combinations of numbers are equally likely. Clearly one could imagine lotteries where all combinations were not equally likely or where the draws were not independent: when I first encountered ritardisti and the Samaritani Formula, Italian lottery draws were done by hand by blindfolded orphans, or so I am told. The balls were actually inserted into the draw apparatus in a fixed order, and mixed a little before the first number was drawn and a little more before each later number. Some people claimed that there was some evidence the distribution of the first number drawn was non-uniform. It's hard to believe this was enough to make bets on single numbers a good deal, but an exaggerated version of this could serve as an example of a significantly non-uniform lottery. Of course, with independent draws but different probabilities for the numbers, ritardisti are even wronger than usual since the rarer numbers will tend to reach higher “delays” so playing “delayed” numbers would be harmful to one's winning chances.

We could also turn this into a plausible example of a lottery where there was dependence between draws. Let's imagine that the order in which balls are put into the machine is not fixed, but depends on their delays. Perhaps we could make the most delayed numbers more likely to be drawn (at least as the first ball in a draw) or more entertainingly make the most delayed numbers less likely to be drawn. Even in this case, it wouldn't matter what the exact value of e.g. the highest delay was. It could even be that some dependence was present in the orphan-based version of Lotto: possibly if the orphans knew they were drawing from a wheel with a very “delayed” number they might behave somewhat differently. It seems unlikely this effect would be very large. Perhaps someone reading this has access to a large number of blindfolded orphans and would like to find out.

One could imagine a lottery where single numbers all had the same probabilities of appearing, but all combinations of five did not. For example, maybe 1-5, 6-10, etc. are the only combinations that can

appear. Of course I/we are explicitly assuming that draws are independent and that all combinations of 5 numbers are equally likely. We say that if ritardisti are also saying this then they need to accept the consequences. If they explicitly say that lottery draws can NOT be modelled this way that is a different problem. (Examination of actual draws shows no obvious sign that higher delay numbers are more likely to come out, for example. This will not surprise you.) I think some ritardisti are trying to say that Samaritani proved that independence has limits, or that independent events aren't. It's quite hard to work out what they are actually claiming, a lot of the time. "We don't know what independence is, or are hoping that you don't" is mostly what I take them to be saying.

Of course, ritardisti don't just say things which aren't true. They say:
(i) Things which are false.

e.g There is some advantage to playing a "delayed" number.

(ii) Things which are true but irrelevant.

An example of this is if they say that a number with delay 160 is very likely to appear within 60 draws. It is, but so is any other number.

(iii) Things to which no real meaning can be assigned.

Mathematical formulae which make no sense at all might fall into this category.

Clearly, someone who claims that draws are independent and combinations are all equally likely BUT that their study of delays allows them to predict results better than chance is either a conperson or an idiot. In any particular case it's hard to say. It's also possible that some ritardisti, like characters in Raymond Smullyan puzzle books, *believe* that they believe that draws are independent, but actually believe that draws are not independent.

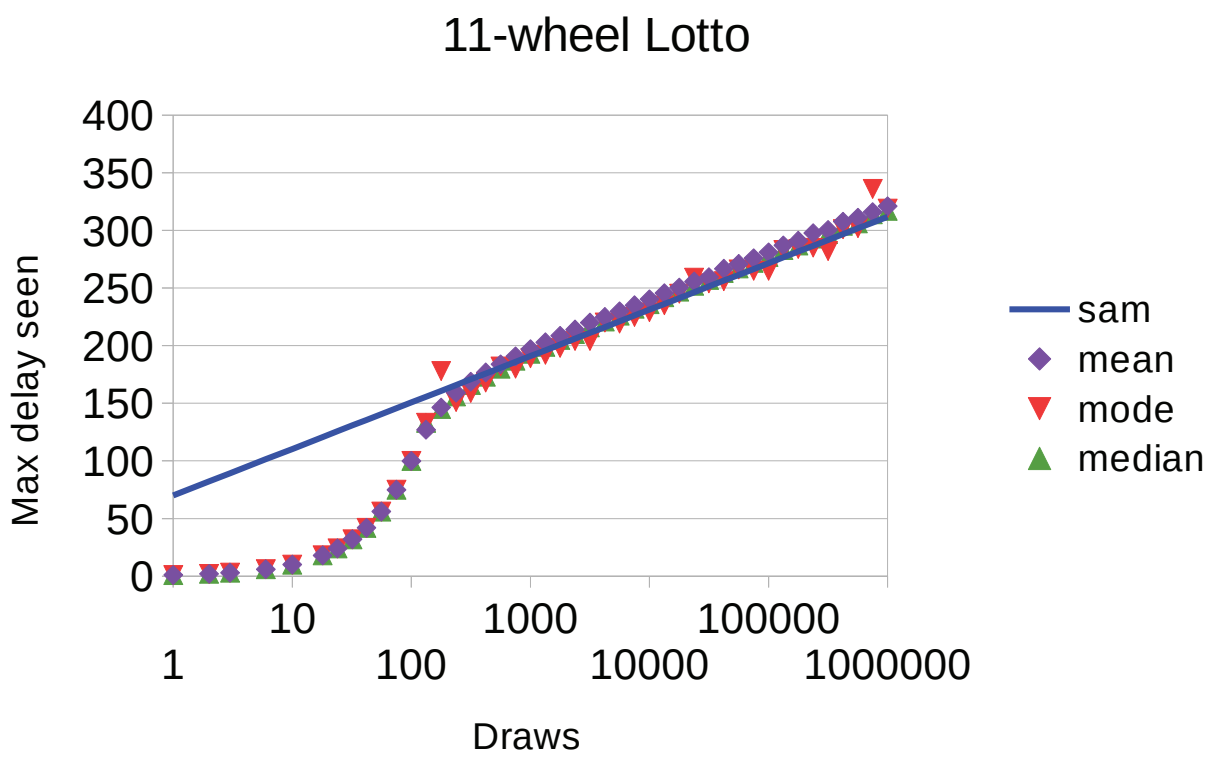
It's obvious enough that the 220 that comes out of the Samaritani Formula is not any kind of a limit on how large delays can become. At the time of the discussions during my first encounter with the Formula in the late 90s, one of the ritardisti seemed to feel that it was significant that the maximum delay ever recorded was 202, which was less than 220. Of course this did not strike most of us as very significant. Lots of numbers are greater than 202 and just because the S.F. produced one that did not seem very interesting. The 202 record was beaten in 2006, becoming 203. See the end of the article for events from late 2016 and early 2017.

As we have seen, if you simulate 6000 draws of the 10-wheel version of the Italian lottery you find that the maximum delay falls both sides of 220, but that 220 is in the fairly narrow range of values it takes fairly frequently. This of course could be a coincidence, but if you simulate other numbers of draws you find that the number produced by the Samaritani Formula is always one of the relatively plausible values, at least for numbers of draws over 1000 or so. This continues to work if you change the number of "wheels" in the lottery to 11 as it is today. This is more interesting. The SF clearly doesn't produce a maximum value delays can assume, as that would be n after n draws.

Here is a chart of Samaritani's number for an 11-wheel Italian lottery after various numbers of draws from 1 to a million, along with the mean, mode and median of the maximum delays seen in many

simulations made of each number of draws. Fewer runs were made of the longer simulations so the mode in particular is less reliable.

As we can see, the mean, mode and median are very close to each other, and the Samaritani number



agrees with all of them for numbers of draws over a few hundred. For such numbers of draws the distribution of maximum delays has the usual Brontosarus type shape one often sees: thin at one end, much much thicker in the middle, and thin again at the far end. The red triangle above the Samaritani graph for n=178 is not an artefact: see much later for note about second peak for small *n*.

More interesting (and not shown on this graph) is that the standard deviation of the maximum delay seems to vary very little with the number of draws (once the simulated and Samaritani values have started agreeing). This was a surprise at least to me.

Dani Ferrari, one of the group in the original discussions, went to the Biblioteca Nazionale (National Library) in Rome and found a copy of Samaritani's 1937 book “La teoria e il calcolo matematico dei ritardi. Studio teorico e pratico sul giuoco del Lotto”. (Theory and practice of delays. Theoretical and practical study of the game of Lotto.) “Giuoco” instead of “gioco” is rather old-fashioned, but this *is* a book from the 1930s. He looked through it and said that as far as he could see, Samaritani was neither a conman nor an idiot, and that his formula was intended to estimate how long the maximum delay seen in *n* draws of the lottery would be, not to give an upper bound on it. One of these decades I might try to buy a copy of this book but it seems to be hard to find.

How can we derive the Samaritani Formula?

At this point it seems like rather than being just nonsense, the S.F. could actually be the answer to a different question. We are now playing mathematical Jeopardy. What question is the S.F. the answer to? Preferably a non-insane question about the Italian Lotto. It doesn't need to be anything that helps choose Lotto numbers. Of course it's obvious it can't do that.

One of Francesco Romani's articles is here: http://www.digitanto.it/mc-online/PDF/Articoli/191_166_169_0.pdf

Considering the destiny of just one number in the Italian lottery, we can treat it as n throws of a biased coin, where a Head, probability 1/18, is that number being drawn, and a Tail, probability 17/18, is that number not being drawn. We then ask ourselves what can be said about the distribution of the length of the longest block of tails. And we regard two successive heads as having a 0-length block of tails between them, which is perhaps not entirely usual.

(I only went looking to see how "real people" approached this problem some considerable time after we first ran into the S.F.: It is apparently known that the longest run of tails in n tosses of a biased coin with p(Head)=p, p(Tail)=q is likely to be about $\log_q(1/np)$ if np>>1. See, for example, http://www.johndcook.com/blog/2012/11/14/probability-of-long-runs/ and http://www.csun.edu/~hcmth031/tspolr.pdf (referred to in the previous article). I have rewritten the formula in the second article to look as much like the Samaritani formula as possible.)

np is the number of times this number is expected to be drawn. Each draw is preceded by a run of failures (possibly of length 0). For the sake of coming up with an estimate, let's say the number comes up exactly np times and see where that gets us.

Two ways to get the known result seem to be (left as exercises for the interested reader):

1) Write down the probability that an observation from our 0-based geometric distribution will be <=k. Raise this to the power np to get the probability that np independent observations from a geometric distribution will all be <=k. (They're not really independent since e.g. there's a constraint on what their sum can be. And there won't necessarily be exactly np of them of course.) Take the derivative of this with respect to k to get the density (pretending that k is continuous rather than discrete). Do it again and set to 0 to find the mode. (This at least feels like we're trying to calculate the mode of something.) I get the S.F. with a -1 on the end if I do this though I may have made a mistake in my algebra. Given that I round to the nearest integer and I imagine Samaritanists round down, this seems close enough. When using a computer I of course work the discrete values out.

or

2) Consider a list of the lengths of the block of failures before each success. 1/18 of the numbers will be 0, 17/18^2 will be 1, and so on. Ask yourself what length of block of failures is such that on average you expect one number in your list to be at least that big. Of course, the run of failures at the end of our sequences of tosses is invisible to this method. This gets the Samaritani formula exactly. (Some ritardisti seem to think the S number is such that an average of one number will come out with *exactly* that delay. That's not correct.) It's not obvious to me that

what are are doing here is trying to calculate the mode of anything. But if we're after an order of magnitude of any/all measures of central location, that's also fine of course.

Should I/we have recognised the S.F. as being this known result about the maximum run length when tossing a biased coin? Possibly. I didn't recall having seen it before and the (moderate numbers of) people I've told about the Samaritani formula over the last 20 years have so far never said "Oh, that's someone failing to understand (result)!". Perhaps I've just never told it to probabilists. In retrospect I should have realised that it was, or was related to, a known trick where you ask students to either toss a coin 100 times or pretend to have done so, and then tell them whether they cheated or not.

How do we extend this to an 11-wheel Italian lottery? Clearly the destinies of numbers in the same wheel are not really independent of one another. To take an extreme case, if you know that five of the numbers in a wheel have appeared, you know that none of the other 85 in that wheel can appear in this draw. However, as an approximation, we could treat the current Italian lottery as 990 independent coins being thrown in parallel. This would limit the maximum possible delay after n throws to n in all cases. Or to make it easier to apply the formula in these articles, treat the Italian lottery as a single coin being thrown $990n$ times. The problem here is that this would allow delays longer than n, and would also allow delays spanning multiple blocks thus counting a longer delay than would appear in any single block. However, for n large enough neither of these problems really makes much difference.

At this point treating the Italian lottery as $990n$ tosses of a biased coin using method (1), the Samaritani formula drops out immediately!

$$\log_{(17/18)}(18/990n) = \log_{(17/18)}(1/55n)$$

Was the result about longest runs known before 1937? Indeed it was.
https://math.stackexchange.com/questions/59738/probability-for-the-length-of-the-longest-run-in-n-bernoulli-trials refers to "*A History of Probability and Statistics and Their Applications before 1750*" by Hald, with solutions by de Moivre (1738), Simpson (1740), Laplace (1812), and Todhunter (1865)." Of course this is for a single coin. So Samaritani seems not to have been the first,
though quite possibly he did not know this at the time.

With method (2) above we know we have exactly $55n$ successes in our $990n$ trials so the Samaritani formula drops out again. Of course the argument about 1/18 of the numbers in our list being 0 etc. is unconvincing for small n as the first 55 numbers are all 0, the next 55 are all 0 or 1, etc. And as mentioned delays after a number's last appearance (or of a number which never appears) are invisible in this method.

However, let's call the distribution of the largest of $55n$ (or whatever) observations from our 0-based geometric distribution "The Samaritani Distribution" (with parameter $55n$ or whatever).

<u>Improving on Samaritani</u>

The approach Francesco Romani writes about in his January 1999 article is to use a recursive calculation to get the exact distribution of the maximum delay for a single number, then take the 990th power of its cumulative distribution function to get the approximate c.d.f. for the maximum delay of the entire Italian lottery. Then de-cumulativize that to see what the most probable maximum delay is. Of course as

mentioned before the 990 numbers are not really independent. And the calculations need to be done using arbitrary precision integers (using Maple or Mathematica or similar) as otherwise horrible things can happen. However, this approach is probably better than Samaritani for small number of draws, albeit impractical in 1937 or without a computer today.

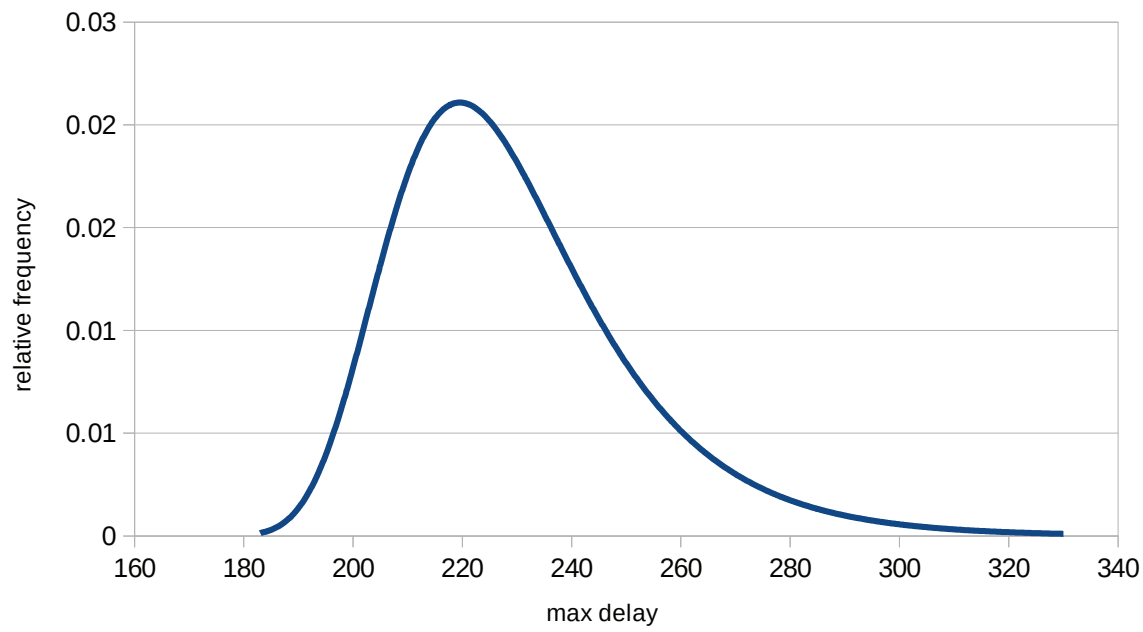
Let us consider a single number in the Italian lottery, (equivalently, a biased coin). Let p=1/18, q=17/18.

Let us define $g(n,r)$ = probability that after n draws a delay of length r has been seen.

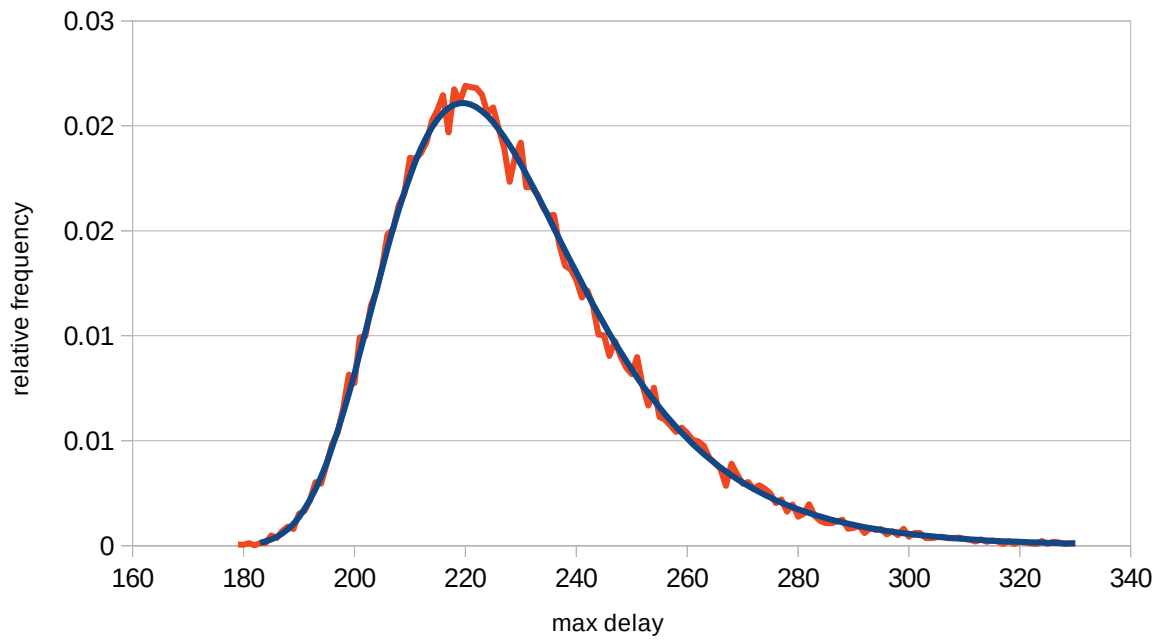
Then $g(n,r)=0$ if $n<r$.
$g(r,r)=q^r$
$g(n,r)=g(n-1,r)+(1-g(n-r-1))pq^r$ for larger n, since for there to have been a delay of r by draw n, it had either been seen at draw n-1 or was new at draw n, in which case there had been no delay of r in the first n-1-r draws, then our number appeared, then it failed to for r draws in a row.

Then the probability that after n draws the maximum delay seen is r is $g(n,r)$-$g(n,r+1)$. As seen earlier we then take the cumulative version of this (1-$g(n,r+1)$: if no delay of r+1 has been seen then the maximum is less than or equal to r) to get the probability the max delay is <=r, and raise this to the 990th power to get the probability that the maximum of 990 independent numbers is <=r,. (We know they aren't really independent but let's try it as an approximation). Then the difference between these values for r-1 and r tells us the probability that the maximum delay is exactly r.(The maximum delay is <=r but it's not <=r-1)

Here is a graph of what Romani's technique gives for the distribution of the longest delay in 6000 draws of a 10-wheel lottery: (the mode is 220). Since this is a 10 wheel lottery I have only raised things to the 900th power, not the 990th, on this occasion.

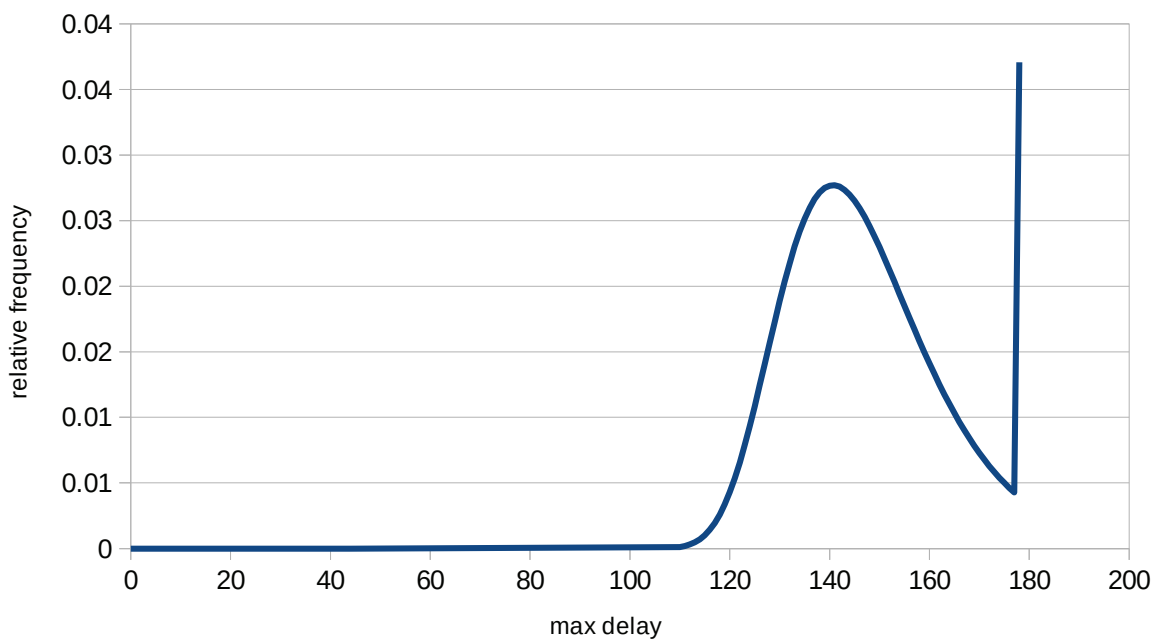


And here it is compared with the results of the simulations from earlier:

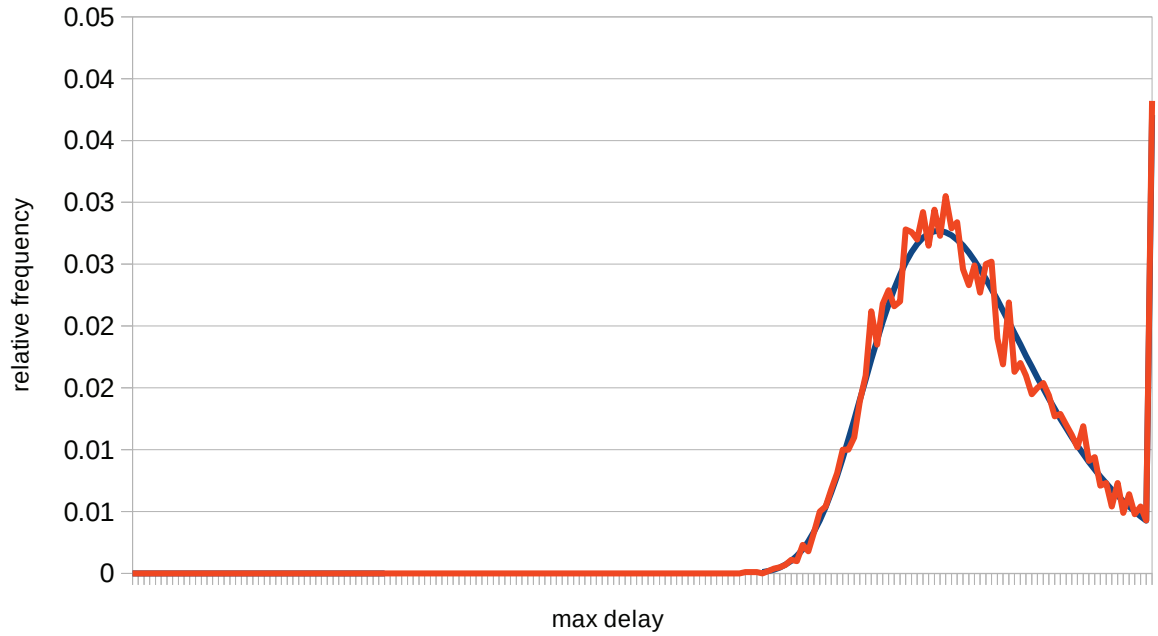


I'd say for the purposes we're using it for, this is fine.

Something amusing that happens for fairly small numbers of draws is that the distribution is significantly bimodal. Here is 178 draws of an 11-wheel lottery using our approximation (the red triangle above the graph earlier):

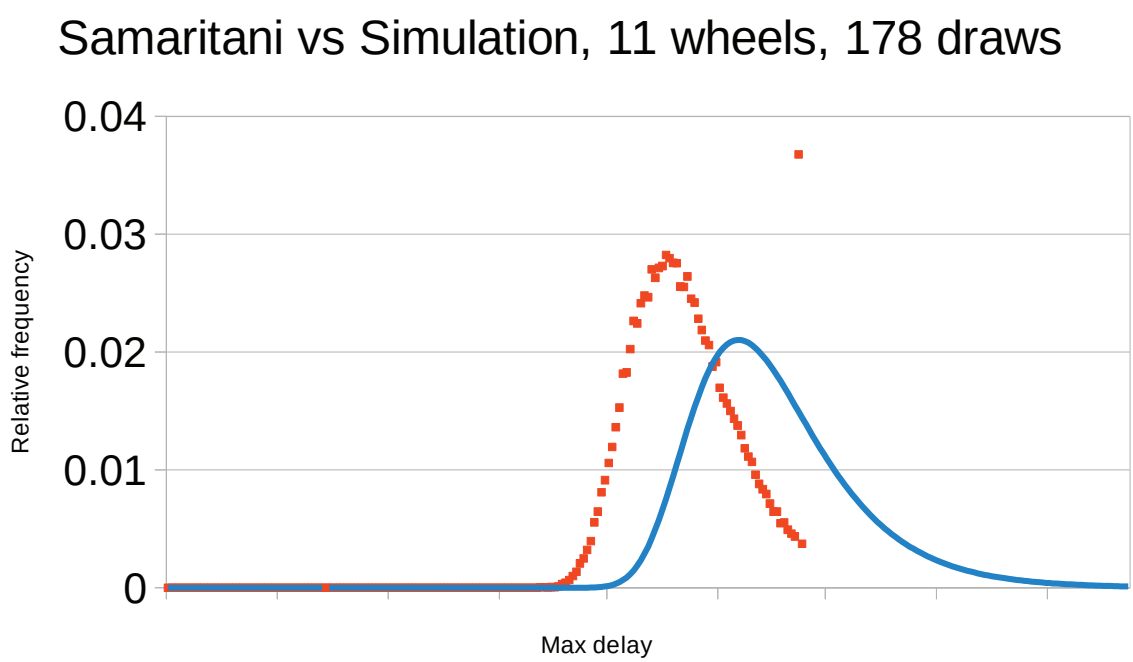


And here it is with the results of 10000 simulated runs superimposed:



Very good, again. Of course, like the original, our improvement is no use for playing the lottery.

For comparison, here is what the distribution of the maximum of 178 observations from a 0-based geometric distribution looks like:



As n increases, the mean and median presumably change fairly smoothly but the mode will suddenly jump down as n goes beyond some threshold value.

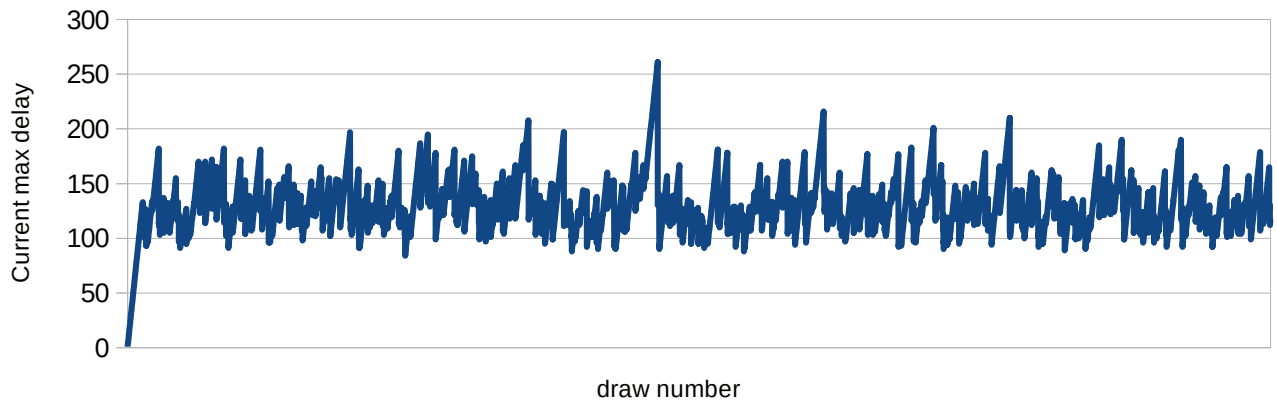
For n=100, the peak at 100 is pretty well all you can see.

This effect is present, less dramatically, for higher numbers of draws too. This feature is one that simpler methods would not pick up: if you're treating the whole thing as 990*n* coin tosses or as 55*n* observations of a geometric distribution, you will allow delays over *n*.

Centenari

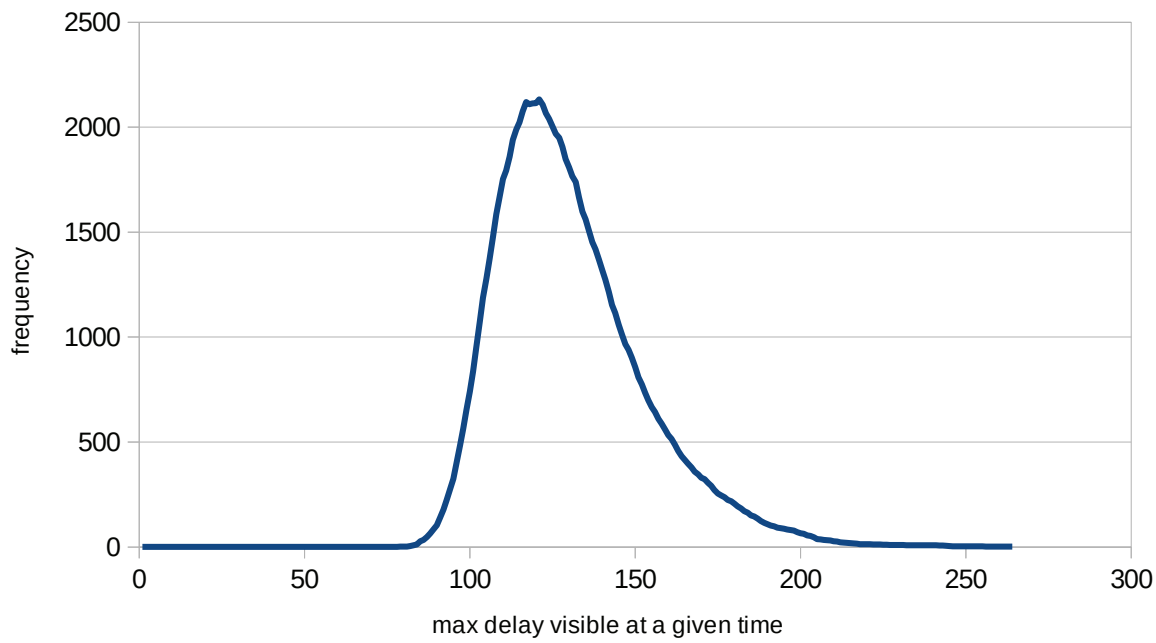
Some people act as though there's something special about numbers with delays over 100 and call them "centenari" (centenarians). Are they even rare enough to be notable?

Let's graph "max current delay" for 10000 draws and see what we get.



Conclusion: "centenari" aren't rare at all. Indeed, they're pretty usual. And beyond the initial ramp up it seems pretty stable.

Running 1 million draws and graphing the frequencies of the max current delays seen I get this:



which also suggests that having at least one number with a delay over 100 is pretty normal. No doubt they were less so back when there were only 8 (or fewer) wheels, but still equally meaningless.

One reasonable question is how the S.F. can be a con if it is given away for free. Perhaps this can be taken to suggest that ritardisti are more likely to be idiots than conpeople, but it's also possible the "basic" S.F. is a freebie intended to help sell software to analyse delays, magazines about lottery advice, or similar. It also seems that there may be more "advanced" kinds of delays than just those of single numbers. I have for example seen people talk about delays of pairs of numbers and Samaritani type results for those. Of course, if ritardisti talk about "level delays", "compensated delays" or "delays with garlic, oil and chili" (some or all of these names are made up), it's all obvious nonsense if draws are independent.

The conclusion then is that the Samaritani formula is not nonsense, but has been misinterpreted, possibly wilfully, by modern ritardisti who use an approximate formula obtained by considering independent events to claim that those same events are not in fact independent. Until I have read the book for myself I cannot be sure that Samaritani was not a ritardista, but I am happy to imagine that he was not: Dani Ferrari is pretty solid.

It continues to be the case that if someone talks about "the Samaritani Formula" there is a pretty good chance they are a ritardista (except me in this article, clearly) so if you do want to use it for anything, it might be best to call it something else to avoid being mistaken for one. The same "independent coins being tossed" approach works fairly well for, e.g., dice or random decimal numbers, where you treat these as single-wheel lotteries where 1 number is drawn from 6 or 10, respectively.

Also, while the Samaritani formula is no use for playing the lottery normally, if someone says they are about to launch an Italian-style lottery and offers you the chance to bet on what the longest run of absences will have been 100 years from now, maybe it could be useful for that.

2017 Update

In late 2016 the nice people at fateilnostrogioco.it wrote to ask if I'd heard that a number (Number 53 on the National wheel) had reached a delay of 233. This was not only more than the previous record, and more than the magic number of 220 from the late 90s, it was even more than the updated magic number for late 2016. So I kept an eye on Italian news and lottery web sites for a while. This number eventually reached a delay of 257 before appearing. At least some ritardisti are going to need a new story now. It might be possible to get some idea of the idiot/conperson balance among ritardisti by seeing how many of them bankrupted or killed themselves in late 2016 and 2017 but I have found no useful information about this.

Something that happened in 2009 provides us with an opportunity to apply the Samaritani Formula and our improvement to another scenario. Apparently in 2009 the Samaritani limit for a pair of numbers was violated. It's possible that this is only true if you fail to update the 6000-draw, 10-wheel version for things as they stood in 2009 but it was mentioned on Lotto forums so there are people who track this kind of thing. But don't notice that they're using an outdated nonsense magic number for comparison.

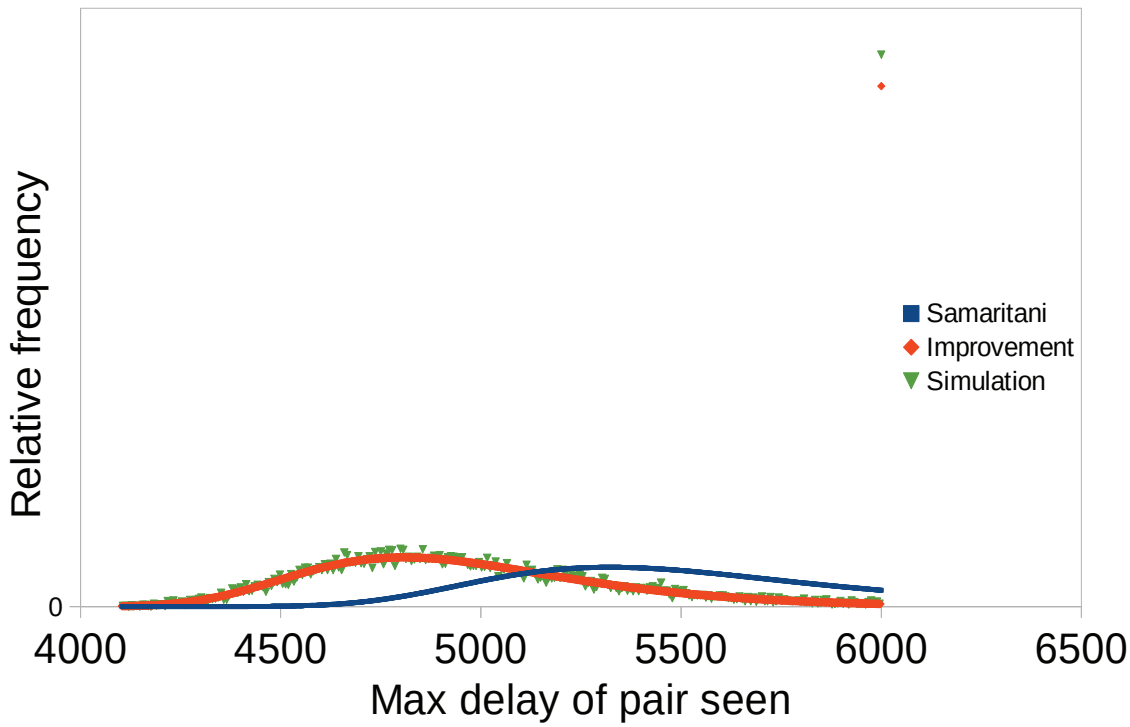
How does the Samaritani model apply to pairs of numbers? In actual Lotto the 5 numbers drawn on each

wheel form 10 pairs of numbers. And there are 4005 possible pairs of numbers that can be formed from the full 90. Our p is now 10/4005 instead of 1/18 and our q is now 3995/4005 instead of 17/18. If we have 10 wheels and 6000 draws as in the late 90s, then we have drawn 10*10*6000 pairs of numbers. And we pretend they are all independent though clearly they are not. So in this case we need to calculate

$$\log_{(3995/4005)} 1/600000 = 5321.871202$$

I guess ritardisti would round this down to 5321? Look for both 5321 and 5322 when googling. See for example http://forum.lottoced.com/forum/lottoced/statistica/58937-il-memorial-day-del-gioco-del-lotto

In any event, let us compare simulations, Samaritani and our improvement:



We see a spike at 6000 in the simulation and in our improvement. This seems a lot like the n=178 example for the base case: n is too small for the approximation to work. Our improvement clearly beats Samaritani. While the 178th draw was long enough ago for the n=178 graph to be perhaps mean, that is clearly not the case here. One has to wonder if as well as being idiots or conmen, ritardisti are also lazy. Perhaps they just imagine that their potential victims are and won't check something like this.

I have not worked out what the late 2016 replacement for 5321 or 5322 should be. Clearly it will be quite a lot bigger. It is curious that at least some lottery people who want to use the Samaritani Formula for some purpose continue to tell each other about values it produced with parameters from 20 years ago. The pair of numbers which reached a delay of 5322 in 2009 finally came out in 2016. It was on the Bari wheel, added in 1874. Since then two wheels were added in 1939 and one more in 2005. When Samaritani's book was published, then, there were only 8 wheels.

If someone wants to run a clone of Lotto and offers you a chance to bet on what the maximum delay for a pair of numbers will have been in 100 years, our improvement or a simulation is definitely a better idea than the Samaritani Formula.

I've even seen people applying the S.F. to groups of 3 or more numbers. Why is not clear. There are 117480 groups of 3 numbers you can form from 90. Since the 5 numbers drawn from a single wheel cover 10 of these, even with incredible luck we can't see all triples in a single wheel until 11748 draws have taken place. And that's if it's possible to avoid duplicates: haven't checked.

There is an archive of draws from 1939 until the present at
Archive of results https://www.lottomaticaitalia.it/STORICO_ESTRAZIONI_LOTTO/storico.zip
Of course, since Lotto started before 1939, the initial delays are unknown so it might be prudent to only consider each number after it has appeared at least once. This does not apply to wheels added after 1939: one assumes the initial delays of all numbers on a new wheel are 0.

Succcess runs

An exercise for the reader: use Samaritani-type reasoning to estimate the length of the longest run of appearances of a single number in, say, 6000 draws of a 10-wheel Lotto. (This is of course not a magic limit on what it can be.)

Improving the improvement

Clearly original Samaritani copes with varying numbers of wheels perfectly fine since it only uses how many numbers have been drawn. It would claim that a huge number of wheels and a single draw was the same situation as one wheel and a large number of draws.

To adapt our improvement to cope with varying numbers of wheels we just multiply the results for single wheels, each with its own number of draws. We could have n throws of 6*90 coins, n-a of another 90, n-b of another 90 and n-c of yet another 90 etc. to account for wheels 7 to 11 being added partway through the history of the game.

I have made no attempt to actually do this to see if what difference it makes modelling the exact history of Lotto itself. Mostly because I don't know the exact numbers of draws that have taken place on each wheel. I could approximate it by using whole numbers of years, potentially.

Neither Samaritani nor our improvement would cope with what the UK lottery did when it changed from 49 numbers to 59.

Was Samaritani an idiot or conperson?

I am still not sure about this. It seems that he must have understood better than his modern followers do how his formula was derived and what its limitations were. However, if he knew it was entirely useless for any normal Lotto player (Betting on max delay in a new lottery 100 years from now is not a normal option) what was the point of his book? If I ever get hold of a copy, I may write a review of it. Dani Ferrari is the only member of our gang of four to have seen the book and he didn't think Samaritani seemed like an idiot or a conperson during his fairly brief time with a copy.

Magic Coins

Elio Fabri suggested way back when that one way to make money out of ritardisti would be to offer to sell them coins that had been tossed until they'd come up the same way up, say, 10 times in a row. A ritardista (or victim, possibly) informed me in it.scienza.matematica in 2017 that the S.F. puts the lie to the idea that a coin tossed 12 times can come up heads 12 times, and seems to say that the coin can't come up heads twelve times in a row unless it has been tossed about 10 thousand times. Now possibly we shouldn't take the value 10 thousand as an exact value but if by "about 10 thousand" he means at least some specific number bigger than 12 (which if 12 is impossible he must?) maybe we have a possible customer. We would need to video ourselves tossing the coin. Would we need to film ourselves getting coins directly from the mint? Does it matter if the coins have flipped whilst falling down chutes in the mint? Does a coin's memory of how many times it has been tossed and with what results fade if it is not tossed for a while? Does a change of owners have any effect? Do ritardisti imagine coins have nanotechnological gyroscopes and/or motors in them as well as memory? Or is magic involved? Back when numbers were drawn by blindfolded orphans, presumably they thought the orphans' hands were pushed towards delayed numbers by mystical forces.

I'm not sure in this particular case, but I believe I've seen a ritardista or victim using the S.F. to treat a coin as a lottery wheel with two numbers. Using Samaritani for runs of either face of a coin suggests we are treating the two faces of a coin as independent. This seems much bolder than treating lottery numbers on the same wheel as independent. I'm not sure if this particular victim or ritardista is misunderstanding results about expected times until k heads in a row, or is perhaps using an inverse Samaritani calculation to find the lowest number of tosses such that 12 heads becomes the most likely longest run, then failing to understand what that would mean.

Also-rans

Two examples of vaguely or supposely mathematical Lotto-related myths which are less interesting than the Samaritani Formula:

The first is the "Law of the Third" ("La Legge del Terzo"). People who claim this is useful observe that about one third of numbers should have delays in a particular range, and that somehow you can pick better lottery numbers based on this. This is of course nonsense but it's obvious boring nonsense based on an obvious and boring misunderstanding.

The second is called "Ciclometria" in Italian. You draw the numbers from 1 to 90 in a circle, draw regular polygons on it, and claim that this helps pick better lottery numbers. This isn't even a misunderstanding of something real. It's just absurd. Indeed it's so absurd that even on lottery forums at least some people seem to feel it sounds like nonsense.

If you want to see a non-mathematical lottery myth, google "La Smorfia" or explore the site www.lottomatica.it – at least when I last looked they described various methods people use to pick numbers. They don't say they're reasonable, but they also don't say they're not. And they do tell you what the most "delayed" numbers on each wheel are. While they don't say this is significant, many people visiting the site might imagine that it must be otherwise they wouldn't bother telling you.

History of Lotto

Information I have from https://it.wikipedia.org/wiki/Lotto#Storia

Before 1863: don't know number of wheels or draw frequency, or if beginning is well-defined. Wikipedia claims draw frequency increased to once every 2 weeks in 1807 and was previously 2 to 3 a year. http://www.ilcomplottoforum.com/t15454-la-storia-del-lotto claims there was only one wheel from 1682 to 1871. I'll go with Wikipedia and ignore everything before 1863.

(Italian unification wasn't even complete in 1863, apart from anything else.)
1863 6 wheels, 2 draws a month
1871 7 wheels, 1 draw a week
1874 8 wheels, 1 draw a week
1939 10 wheels, 1 draw a week
1997 10 wheels, 2 draws a week
2005 11 wheels, 3 draws a week

This would suggest something like:

years	years in range	wheels	Draws / week	draws	numbers drawn	cum. Draws	cum. Nos
1863-1870	8	6	0.5	208	6240	208	6240
1871-1873	3	7	1	156	5460	364	11700
1874-1938	65	8	1	3391	135640	3755	147340
1939-1996	58	10	1	3026	151300	6781	298640
1997-2004	8	10	2	834	41700	7615	340340
2005-2017	13	11	3	2034	111870	9649	452210

The above makes some attempt to account for leap years.

So 300000 numbers drawn is about right for the late 90s when we first ran into this. Assuming we're counting from 1863. Elsewhere I have said I got 480k numbers for late 2016 so clearly I've made a mistake either there or here. In any event the updated magic Samaritani number for late 2017 seems to be 227 (if we round down) or 228 (rounding to nearest integer, or allowing for a bit more time to pass). Of course I have no idea how many draws from 1862 and earlier I should be allowing for. Googling does find some people updating each other on what the magic Samaritani number has now become at various times over the years.

Blindfolded orphans stopped being used as a source of random numbers in 2009 (and started to be phased out in 2005).

So what's wrong with ritardisti?

One could suppose they are merely ignorant, but this suggestion seems to offend them. I've had one claim he's been studying probability longer than I have been alive, for example.

If someone has been studying probability for over 50 years and don't understand independence, then "idiot" seems like a fair assessment (that's about as bad as me with group theory), and it does seem to be a common opinion in Italy that ritardisti are idiots.

I suspect, however, that a significant fraction of them have to be conpeople, but of course it's hard to know how to prove this except as mentioned by tracking suicides and bankruptcies of ritardisti in late 2016 to early 2017.

It's amusing to imagine that some of the conpeople could be aiming to reduce their own tax burden by encouraging other Italians to pay lots of voluntary taxes.

It's amusing, though cruel and implausible, to imagine that some of the conpeople could be eugenicists, trying to improve the Italian gene pool by driving the more gullible elements of the Italian population to bankruptcy and/or death.

Whatever else is wrong with them, the results re pairs of numbers suggest that I should add a "laziness" axis as well.

Is it even possible that by telling people to wait until a delay of 160 appears they are actually trying to get people to play Lotto less? Perhaps in that case they should suggest waiting for a delay of 500, or just say not to play at all. But suggesting exponentially increasing bets as part of this doesn't feel like something you'd do if trying to encourage people not to play.

<u>How well-known is it?</u>

I think it's pretty obscure. The "Fate il Nostro Gioco" people had never heard of it and they've been running an anti-gambling campaign and roadshow for years. It is mentioned by name by some people on lottery forums, and occasional visitors to maths and science groups every few years. When exploring lottery forums during the "Nazionale 53" event in late 2016 to 2017 I found some posts where people told each other there was a formula, or it had been shown that, etc., without naming Samaritani. I have to suspect that in the general population, basically no-one has heard of it. Italian mathematicians generally don't seem to have heard of it and usually when a new ritardista or victim turns up on it.scienza.matematica only the people who were around for the previous visit know about Samaritani.

<u>Please contact me if you've seen things like this</u>

If you have seen something similar to this outside Italy, I'd love to hear about it. Or if you've come across it in Italy between 1937 and the late 90s I'd also like to hear about that. Has it been circulating ever since Samaritani wrote his book, or did someone uncover it decades later and start spreading it?

Storming the Castle

Calvin Hurlbert* Glenn Hurlbert†

Abstract

Conquering the Kingdom requires traversing bridges guarded by voracious gremlins. How many knights do you need to succeed?

1 The Challenge

The fascinating city of Noccilevbyhein is a collection of islands floating in water on the surface of a torus (donut shape). On one of the islands is a castle.

Unfortunately, the city is ruled by administrators who base their system of educational training on a series of high-stakes exams. For the sake of the children, you realize that the leaders must be removed, and so you begin to plan a coup by parachuting knights onto the islands. With unpredictable winds and unusual gravity, however, you have no control over where they will land.

The islands of Noccilevbyhein are connected by a set of bridges (see the diagram below), under each of which lives a knight-eating gremlin. Because of this, knights must traverse bridges in pairs so that exactly one of them reaches the other side.

Success just requires any knight to reach the castle at A, but because of budget cuts you must guarantee success with the minimum number of paratrooping knights, regardless of where they may land.

You know that 6 knights is not enough because 3 might land on F and 3 might land on L. From F, one knight might be able to move to B, E, G, or J, and from L, one knight might be able to move to D, H, I, or K. In all cases we would be left with at most one knight per island with no more moves possible.

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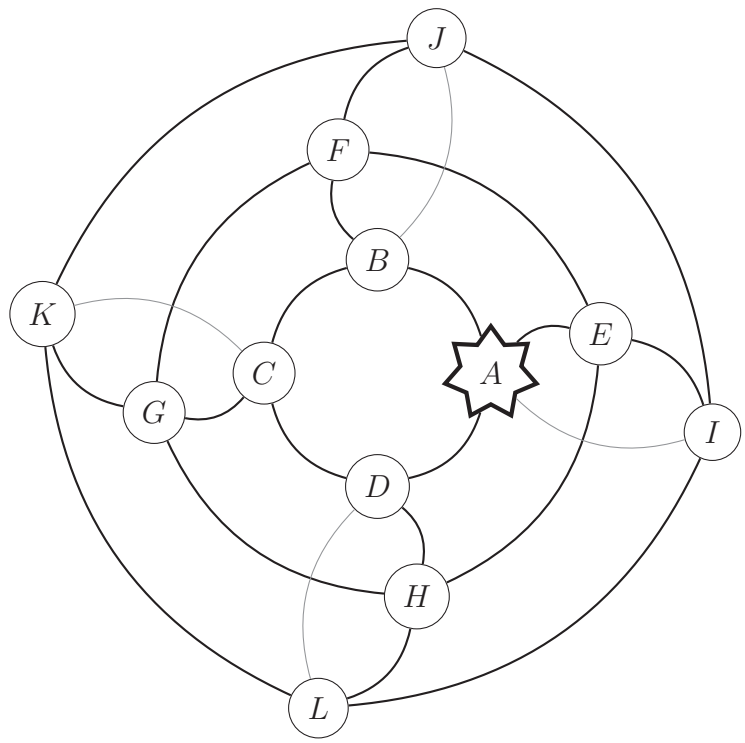


The bridges of Noccilevbyhein.

In fact, 7 is also insufficient because they could all land on K. The most number of knights that could cross to one of its neighboring islands C, G, J, or L is 3, and from any of these none could cross to the castle at A.

On the other hand, if 78 knights parachuted in, then either one of them lands directly on A, or at least 8 of them land together on the same island (at most 7 on each of 11 islands would account for at most 77 knights otherwise). By observation, every island is within 3 crossings from the castle, and so one of those 8 knights will be able to reach A.

Better yet, just 34 knights is sufficient! Indeed, suppose that the knights landed in such a ways as to be unable to reach the castle. Note that if an island is t crossings away from A then 2^t knights on that island will be enough for one of them to be able to reach A. So it must be that such an island has at most $2^t - 1$ knights on it. There are 4 such islands (B, D, E, and I) with $t = 1$; these account for at most $4 \cdot 1 = 4$ knights. There are 5 islands (C, F, H, J, and L) with $t = 2$; these account for at most $5 \cdot 3 = 15$ knights. Finally there are 2 remaining islands (G and K) with $t = 3$; these account for at most $2 \cdot 7 = 14$ knights. In all we have counted at most $4 + 15 + 14 = 33$ knights — hence 34 knights will place enough knights on some island to guarantee storming the castle.

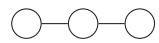
If we write κ for the *castle number* of Noccilevbyhein (i.e. the smallest number of knights which guarantees the ability to storm the castle), then these arguments show that $8 \leq \kappa \leq 34$. What is the actual value of κ?[1]

2 Graphs & Products

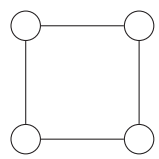
We often call things like the map of Noccilevbyhein a *graph*. The islands are called *vertices* and the bridges are called *edges*. Thus we might use $\mathcal{N}$ for the graph of Noccilevbyhein so that we can write

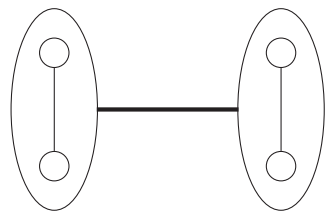
[1]Put down the article if you'd like to try to figure out this puzzle on your own for a while. There is a hint if you wish in Section 3. You can continue to read safely until Section 5, where the solution and its description occurs.

$\kappa(\mathcal{N}, A)$ for the castle number of $\mathcal{N}$ at A. If the castle instead were at B we would write $\kappa(\mathcal{N}, B)$ for the castle number there. For some graphs $\mathcal{G}$, like the one below, these numbers can be different (3 in the middle, but 4 at either end) at different vertices. So we should use $\kappa(\mathcal{G})$ for the worst case; i.e. the maximum of all its different vertex castle numbers. In the case of Noccilevbyhein, however, the symmetry of $\mathcal{N}$ (we can rotate our viewpoint to make any vertex look just like A) implies that all the castle numbers are the same.

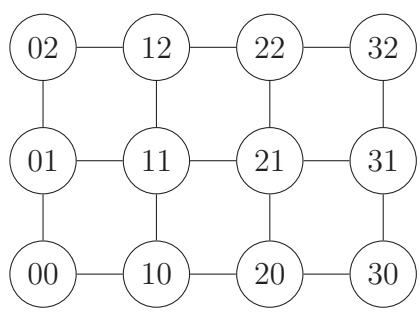


A graph like the above, just a sequence of alternating vertices and edges, is often called a *path*; we write $\mathcal{P}_3$ to signify that it has 3 vertices. Another common graph joins the endpoints of a path to create a *cycle*; the one below (left) is $\mathcal{C}_4$.

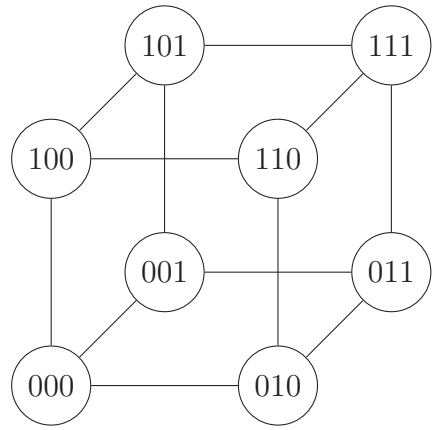




This cycle has an interesting structure. If you group the pair of vertices as shown above (right) then it seems like a big $\mathcal{P}_2$ with each of its vertices containing a little $\mathcal{P}_2$ inside it. The thick edge of the big $\mathcal{P}_2$ represents a set of thin parallel edges in $\mathcal{C}_4$. We call this kind of structure a *Cartesian product*, honoring René Descartes, who gave us the Cartesian plane. An example of the product $\mathcal{P}_3 \times \mathcal{P}_4$ is shown below. The vertex names follow the same pattern we use in high school algebra for Cartesian coordinates.



Thus a product of paths yields a flat grid. One can construct other shapes, like a cylinder from the product of a path and a cycle, or a torus from the product of two cycles. Another nice product is the graph $\mathcal{C}_4 \times \mathcal{P}_2$, below. Because $\mathcal{C}_4 = \mathcal{P}_2 \times \mathcal{P}_2$, we can write $\mathcal{C}_4 \times \mathcal{P}_2 = \mathcal{P}_2 \times \mathcal{P}_2 \times \mathcal{P}_2$, which we can abbreviate as $\mathcal{P}_2^3$; even shorter, people tend to call this graph $\mathcal{Q}^3$ because it is the 3-dimensional *cube*. One can imagine continuing to multiply again and again by $\mathcal{P}_2$ to obtain $\mathcal{Q}^d$ for any dimension d; while it may be difficult to imagine d dimensional geometry, at least we can draw the graph!



One of the interesting patterns that arises here is a relationship between the castle numbers of the individual graphs and their products. For example, $\kappa(\mathcal{P}_2) = 2$ and $\kappa(\mathcal{C}_4) = 4 = \kappa(\mathcal{P}_2)^2$. Also, $\kappa(\mathcal{C}_3 \times \mathcal{P}_2) = 6 = \kappa(\mathcal{C}_3) \cdot \kappa(\mathcal{P}_2)$ and $\kappa(\mathcal{Q}^3) = 8 = \kappa(\mathcal{P}_2)^3$, and it can be a nice challenge to prove these on your own. Based on this sort of evidence, Ron Graham conjectured that every pair of graphs $\mathcal{G}$ and $\mathcal{H}$ satisfy $\kappa(\mathcal{G} \times \mathcal{H}) \leq \kappa(\mathcal{G}) \cdot \kappa(\mathcal{H})$ (there are some examples for which equality does not hold), and the conjecture has been shown to be true for many types of graphs. If the conjecture is true for all graphs, then we would know that 12 knights would suffice for storming the Nochilevbyhein castle, for example. Without knowing that, however, $\mathcal{N}$ remains a bit of a test case.

3 A Hint

It would be easier for an attendee of this conference to guess the correct value of κ without having been told its definition!

As for progress in that direction, one thing that we can do is split the city into districts $U = \{A, B, C, D\}$, $V = \{E, F, G, H\}$, and $W = \{I, J, K, L\}$, with u, v and w being the number of knights landing in their respective districts. Then we can use that $\kappa(\mathcal{C}_4) = 4$ to say that if $u \geq 4$, $v \geq 8$, or $w \geq 8$ then some knight can storm the castle. Otherwise, $u \leq 3$, $v \leq 7$, and $w \leq 7$, which accounts for only 17 knights. Hence $\kappa(\mathcal{N}) \leq 18$. Is that the best we can do?

Also, how many knights can land in Noccilevbyhein in such a way as to not be able to make a single island crossing?

4 Some History

In 1989, Fan Chung proved Graham's conjecture for any number of paths (in particular this means that $\kappa(\mathcal{Q}^d) = 2^d$). Moreover, she showed that it holds more generally when the gremlins can eat more than just one knight. For example, let's say that the gremlins on $\mathcal{P}_3$ eat four knights instead of one, so that it takes 5 knights to traverse an edge for one of them to make it all the way across. Then, as you can imagine, $\kappa_{[5]}(\mathcal{P}_3) = 5^2$ (the subscript shows the gremlin crossing requirement). Similarly, $\kappa_{[7]}(\mathcal{P}_4) = 7^3$. An example of what Chung proved is that $\kappa_{[5,7]}(\mathcal{P}_3 \times \mathcal{P}_4) = \kappa_{[5]}(\mathcal{P}_3) \cdot \kappa_{[7]}(\mathcal{P}_4)$.

And now for something completely different. Suppose that you are given 15 whole numbers, such as $\{-10, 1, 4, 9, 12, 16, 20, 20, 31, 77, 89, 106, 126, 581, 904\}$ (repeats are allowed). No matter what they are, it is always possible to choose from them a subcollection that sums to a multiple of 15: $12 + 20 + 20 + 77 + 126 = 240 = 16 \cdot 15$ is one example. This is called a *zero sum* because its

remainder after dividing by 15 is 0. It turns out that, even when the key number 15 is replaced by any positive integer n, this is always possible — every list of n whole numbers contains a zero sum.

However, if we reduce each of the numbers in the zero sum example to its greatest common divisor with 15, we get $3+5+5+1+3=17$. This is considered a *heavy* (as opposed to *light*) sum because $17>15$. In our given list, $-10+1+9=0\cdot 15$ is a light zero sum because $5+1+3\leq 15$. In the late 1980s Paul Erdős and Paul Lemke proposed that every list of n whole numbers contains a light zero sum. This turns out to be true as well and, while it is much more complicated to guarantee the extra light condition, the central reason why it is so for a number like $n=8575$, for instance, happens to be that $\kappa_{[5,7]}(\mathcal{P}_3\times\mathcal{P}_4)=5^2\cdot 7^3=8575$. It's true for $n=216172$ because $\kappa_{[2,11,17]}(\mathcal{P}_3\times\mathcal{P}_2\times\mathcal{P}_4)=\kappa_{[2]}(\mathcal{P}_3)\cdot\kappa_{[11]}(\mathcal{P}_2)\cdot\kappa_{[17]}(\mathcal{P}_4)=2^2\cdot 11^1\cdot 17^3=216172$.

Intriguingly, since antiquity it has been of interest to try to be able to decide reasonably quickly when some number is prime. In our modern, technological era this problem has gained greater practical importance, as many cryptographic and other protocols depend on such things. For a few centuries, a test by Pierre de Fermat was thought to identify primes but, in 1910, Robert Carmichael discovered a counterexample: a composite number that passed Fermat's test. While a handful of counterexamples would not cause too much of a problem, an infinite set of them surely would. In the 1950s Erdős attempted to show that this was the case, and his ideas included questions about zero sums, which have developed into a robust area of research since. In 1994, Alford, Granville, and Pomerance finally proved that infinitely many Carmichael numbers do exist.

Now back to our brave knights!

5 The Solution

The first thing to mention is that it is possible to land 11 knights, one per non-castle island, so that no knight can move at all. This shows that $\kappa(\mathcal{N})\geq 12$. What we'd like to do also is argue that $\kappa(\mathcal{N})\leq 12$. For this, let's explain how to storm the castle from every possible way of parachuting 12 knights into Noccilevbyhein. With the districts defined as before, we start with $u+v+w=12$.

A key observation is that the two districts U and V form the graph $\mathcal{Q}^3$, and so if $u+v\geq 8$ then we can storm the castle. If $u+v=7$ then $w=5$, which means that some knight can cross from W into U or V, so that the resulting 8 knights would be able to storm the castle. So let's assume instead that $u+v\leq 6$, which implies that $w\geq 6$.

Because of the symmetry of $\mathcal{N}$, we can make the same arguments as above to reduce to the case that $u+w\leq 6$ and $v\geq 6$. Put together, we now have $u=0$ and $v=w=6$.

If two knights can cross from V to W then one of the resulting 8 knights on W can storm the castle. Otherwise, no island of V has at least 4 knights and no two islands of V have at least 2 knights each. The only way for this to occur is if the four islands of V contain exactly 1, 1, 1, and 3 knights. But then some knight can cross the appropriate bridge(s) to reach E, and then cross to A.

Happy G4G12!

References

[1] G. Hurlbert, Graph Pebbling. *Handbook of Graph Theory* (2nd ed.), Discrete Mathematics and its Applications, J. Gross, J. Yellen, and P. Zhang, eds., 1428–1454, CRC Press, Boca Raton, 2014.

Thinking Inside and Outside the Box

Tanya Khovanova

May 7, 2016

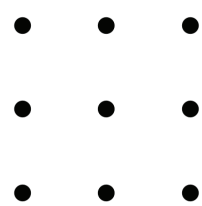
Abstract

I discuss puzzles that require thinking outside the box. I also discuss the box inside of which many people think.

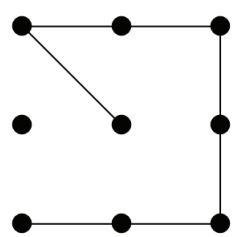
1 Nine Dots Puzzle

The following puzzle started the expression *inside the box* [1].

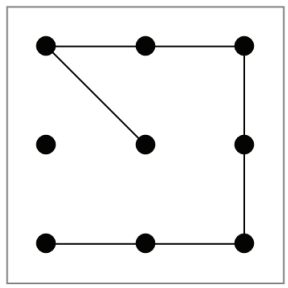
> Connect the dots by drawing four straight, continuous lines that pass through each of the nine dots without lifting the pencil from the paper.



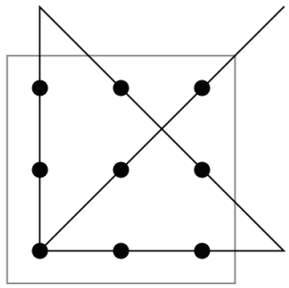
Most people try something like this:



and they fail to connect all the dots. They try to connect the dots that are formed by line segments that fit inside the square box around the dots:



They mentally restrict themselves to solutions that are literally inside this box. In the correct solution the line segments should be drawn outside this imaginary box:



There are many puzzles where people make assumptions that are not required by the puzzle. They constrain themselves to the *inside of the box*. Let us look at such puzzles.

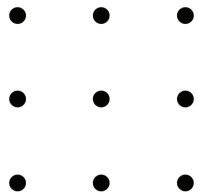
2 Other Example of the Outside-the-Box Puzzles

These puzzles have a common feature. Many people who try to solve them assume something that is not stated in the puzzle and therefore they fail to find the solution.

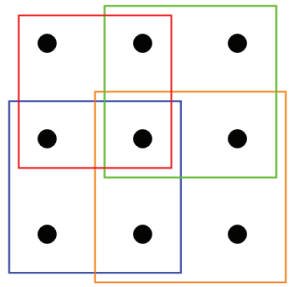
2.1 Other Nine Dots

There is another puzzle with the same nine-dots setup.

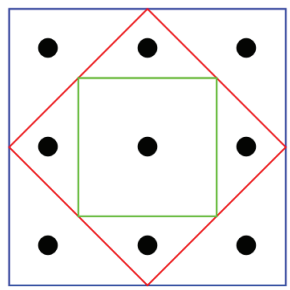
> What is the smallest number of squares needed to make sure that each dot is in its own region?



Usually people who try to solve this puzzle come up with the following solution with four squares.



As with the previous puzzle they imagine the dots are on a grid and try to build squares that have sides parallel to the grid lines. What is the outside-the-box idea? The sides of squares do not need to be parallel to the grid. This way we can find a solution with three squares.



2.2 An Interesting Bet

Here is another puzzle:

> "I will bet you $1" said Fred, "that if you give me $2, I will give you $3 in return." Tom agreed and gave Fred $2. How much money did Tom win?

It seems that Fred will give Tom $3 dollars at which point Tom will lose the bet and will have to give $1 back. Overall Tom wins nothing. But there is a box around this puzzle. Most people assume that as Fred made the bet, he will follow on it. Actually, if Fred loses the bet he wins $1 and Tom loses $1.

2.3 The Ancient Outside-the-Box Puzzle

The following problem can be found in eighth-century writings.

> A man has to take a wolf, a goat, and some cabbage across a river. His rowboat has enough room for the man plus either the wolf or the goat or the cabbage. If he takes the cabbage with him, the wolf will eat the goat. If he takes the wolf, the goat will eat the cabbage. Only when the man is present are the goat and the cabbage safe from their enemies. All the same, the man carries wolf, goat, and cabbage across the river. How?

I think that the reason the puzzle has survived for so many years is that the solution is based on an outside-the-box idea.

I don't want to provide a solution to this popular puzzle, but here is the non-trivial idea: Though the man wants to move his luggage to the other side, in one of the trips he needs to bring the goat back.

2.4 Move a Digit

Here is another puzzle:

> In the equation $30 - 33 = 3$ move one digit to make it correct.

The puzzle seems impossible. But the outside-the-box idea is to move a digit up to the exponent: $30 - 3^3 = 3$.

2.5 Cigarette Butts

Here is the last puzzle in this section:

> A certain hobo who is skilled at making cigarettes can turn any 4 cigarette butts into a single cigarette. Today, this hobo has found 24 cigarette butts on the street. Assuming he smokes every cigarette he can, how many cigarettes will he smoke today?

On the surface, he can smoke $24/4 = 6$ cigarettes. What is the outside-the-box idea? He can reuse his own butts. After smoking 6 cigarettes, he will have 6 butts left. He can make one more cigarette. The answer is 7.

Or is it? What I love about this puzzle is that it has two layers. After smoking 7 cigarettes the hobo will have 3 butts left. There is another outside-the-box idea here. He can borrow a butt from a friend, smoke a cigarette and return the butt. At the end he can smoke 8 cigarettes.

3 My Students

I love outside-the-box puzzles. And I am good at them. Whenever I see such a puzzle, I always know the intended answer. Unfortunately, the moment I get the answer, I stop thinking about the puzzle. This is where my students come in. I always give such puzzles to my students, and they never fail to surprise me.

3.1 A River-Crossing Puzzle

Let's look at the following puzzle:

> Two boys wish to cross a river, but there is a single boat that can take only one boy at a time. The boat cannot return on its own; there are no ropes or similar tricks; yet both boys manage to cross the river. How?

The outside-the-box idea is that they started on different sides of the river. Many of my students do not see this answer. Nevertheless, they are very inventive and produce a lot of interesting different answers:

- There was another person on the other side of the river who brought the boat back.
- There was a bridge.
- The boys can swim.
- They just wanted to cross the river and come back, so they did it in turns.

I gave a talk about thinking inside and outside the box at the 2016 Gathering for Gardner conference. I mentioned this puzzle and the inventiveness of my students. After my talk a guy approached me with another answer which is now my favorite:

- They wait until the river freezes over and walk to the other side.

3.2 Apples in a Basket

Here is another puzzle, over which my students didn't fail to amaze me.

> You have a basket containing five apples. You have five hungry friends. You give each of your friends one apple. After the distribution each of your friends has one apple each, yet there is an apple remaining in the basket. How can it be?

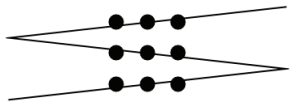
The standard outside-the-box solution is to give an apple to a friend together with a basket. Here are some pearls from my students:

- Kill one of your friends.
- You can't count.
- You are narcissistic and you are one of your own friends.
- You have two baskets, one has 5 apples, one has 1 apple.
- One friend already has an apple.
- An extra apple falls from a tree into the basket.
- **My favorite:** the basket is your friend!

When I see these answers I regret that I stopped thinking about the puzzle and didn't see how many more ideas it can generate. At the same time I am elated at the inventiveness of my students. I wouldn't have considered killing one of the friends as an option, but then I'm not in the 8th grade like my student.

3.3 The Original Nine-Dots puzzle

After the G4G conference mentioned earlier, Jason Rosenhouse told me a solution for the original nine-dots puzzle that requires only three lines:



The outside-the-box idea here is to use the thickness of the dots.

3.4 An Irresistible Cannonball

My students are inventive not only when solving outside-the-box puzzles. I am taking a detour here with another puzzle that is not the outside-the-box type. But I love what my student suggested and think you will too.

> What happens if an irresistible cannonball hits an immovable post?

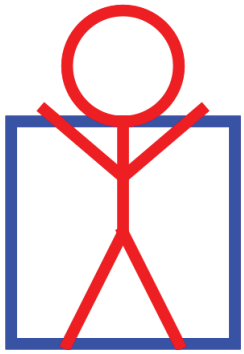
This puzzle is known as the Irresistible Force Paradox. I borrowed it from the book *What is the name of this book?* by Raymond Smullyan [2]. The standard answer is that the given conditions are contradictory and the two objects cannot exist at the same time.

This is what one of my students wrote:

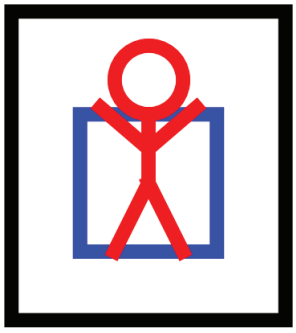
> The post falls in love with the cannonball as it is so irresistible.

4 Where is the Box?

I am good at thinking outside the box. I even drew a picture of myself to represent this.



Unfortunately I have to conclude, that I am inside my own, bigger box.



All of us have our own boxes. It is good that we can learn from each other about the beauty outside our boxes.

5 More Outside-the-Box puzzles

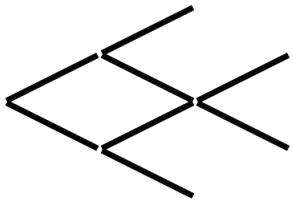
There are many more outside-the-box puzzles. The fact that you know that they need an outside-the-box solution will help you find it. Actually each time you are stuck on a puzzle, it makes sense to assume that the puzzle might need an outside-the-box idea.

> **Puzzle.** Four matchsticks form a square. How many non-overlapping squares can be formed using eight matchsticks? The matchsticks do not intersect each other and they can't be broken.

As you might have guessed the answer is more than 2. If you need a hint, you need a mirror.

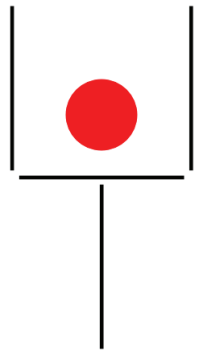
Hint. The squares are smaller than you might think.

> **Puzzle.** I arranged eight sticks in the shape of a fish. What is the minimum number of sticks that must be moved to make the fish face the opposite direction?



Hint. The fish moves down.

> **Puzzle.** These four sticks make a glass with a cherry in it. Can you move just 2 sticks so that the cherry is outside the glass?



Hint. The glass will be upside down.

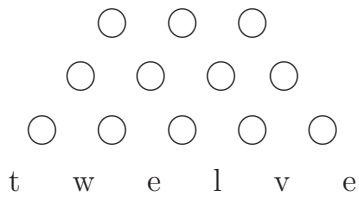
Puzzle. You have two wallets, each containing a quarter. Yet the total amount of money you have is 25 cents. How could this be?

There are no hints for this one: it is too easy.

References

[1] M. Kihn, 'Outside the Box': the Inside Story, FastCompany 1995.

[2] R. Smullyan, *What is the Name of this Book?*, Prentice-Hall, 1978.



Trapezoids, Primes, and Cousins

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Here is a simple story that contains three nice ideas about numbers, two are in fact somewhat old, but one I think is new. I am sure all three of these ideas would have tickled Martin, especially because they were discovered (or rediscovered) by an amateur mathematician.

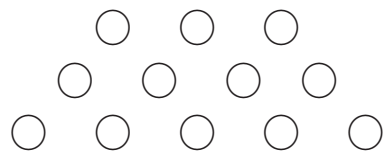
My story begins more than thirty years ago when a good friend of mine, Carlton Gamer, came to me with a conjecture about sums of consecutive integers. Carlton is a prominent American composer and music theorist who often works with *equal tempered systems* — these are musical systems, such as the familiar twelve-tone system, but that may have a modulus other than 12 in order to allow for differing combinatorial structures to emerge in the music. In this particular instance Carlton was exploring the idea of generalizing the way Schoenberg in his Variations for Orchestra, Op. 31 partitioned its 12-tone set into subsets containing three, four, and five tones. Schoenberg then used this partition $12 = 3 + 4 + 5$ to determine such musical details as the number of pitches in a motive, the number of notes in a chord, and the number of measures in a phrase.

So, the mathematical question Carlton asked himself was: Which numbers can (like 12) be written as a sum of consecutive positive integers? He then proceeded to check all numbers up to 100 by hand. Having done this he came to me with a remarkable conjecture: *The only numbers that cannot be written as a consecutive sum are the powers of* 2. This struck me as a very nice conjecture; it was certainly one that was entirely new to me. You may want to try to prove it for yourself before I divulge a proof below.

My first thought was that the idea of a sum of consecutive positive integers is only a very slight variation on the familiar Greek notion of *triangular numbers* — that is, the numbers

$$t_1 = 1, \quad t_2 = 1 + 2 = 3, \quad t_3 = 1 + 2 + 3 = 6, \quad \ldots \quad , \quad t_n = 1 + 2 + 3 + \cdots + n, \quad \ldots$$

that are the sum of the first n integers. So, I decided to coin the geometric term *trapezoidal numbers* for these numbers since the sum of (at least two) consecutive positive integers can be represented in the form of a trapezoid. For example, here is 12 represented as a trapezoid:



$$12 = 3 + 4 + 5$$

Note that the difference of any two (non-consecutive) triangular numbers is automatically a trapezoidal number. For example, it is clear from the figure above that if we subtract the triangular number t_2 from the triangular number t_5 we are left with the trapezoidal number 12; or, in other words,

$$t_5 - t_2 = (1 + 2 + 3 + 4 + 5) - (1 + 2) = 3 + 4 + 5 = 12.$$

Here is Carlton Gamer's first *nice* idea now stated as a theorem about trapezoidal numbers:

All positive integers except the powers of 2 *are trapezoidal.*

Let us first show that any trapezoidal number cannot be a power of 2. Suppose that n is a trapezoidal number; then, as we saw above, we can write n as the difference of two non-consecutive triangular numbers: $n = t_{k+s} - t_k$, where $s > 1$. Fortunately, there is a convenient formula — known to the Pythagoreans — for the kth triangular number: $t_k = \frac{k(k+1)}{2}$. Using this formula we can write

$$n = t_{k+s} - t_k = \frac{(k+s)(k+s+1)}{2} - \frac{k(k+1)}{2} = \frac{s(2k+s+1)}{2}.$$

Now, one of s or $2k + s + 1$ must be odd (and greater than 1), and the other even, so n cannot be a power of 2.

Next, let us show that conversely any number not a power of 2 is trapezoidal. This is obvious for odd numbers since an odd number $2k + 1$ can be written trapezoidally as $2k + 1 = k + (k + 1)$. Now suppose n is an even positive number, but not a power of 2. Then we can write $n = 2^m \cdot k$, where k is an odd number ($k \geq 3$). In this case we simply express n as the sum of k consecutive integers with 2^m in the middle. Several examples will make this constuction clear.

As a first example, let $n = 12$. We write $12 = 2^2 \cdot 3$, so we put $2^2 = 4$ in the middle and express 12 as the sum of 3 consecutive integers: $3 + 4 + 5$.

As a more illustrative example, let $n = 112$. We write $112 = 2^4 \cdot 7$, so we put $2^4 = 16$ in the middle and express 112 as the sum of 7 consecutive integers: $13 + 14 + 15 + 16 + 17 + 18 + 19$.

We need to give one more example to show that even if this construction produces negative integers we still have a trapezoidal number! So, let $n = 18$, and we write $18 = 2 \cdot 9$ and dutifully put 2 in the middle and express 18 as the sum of 9 consecutive integers:

$$(-2) + (-1) + 0 + 1 + 2 + 3 + 4 + (5) + (6).$$

This may not look very trapezoidal, but once you realize that $(-2)+(-1)+0+1+2=0$, we have in fact written

$$18 = 3+4+5+6,$$

which is clearly trapezoidal.

Quite recently Gamer (the amateur mathematician) discovered two rather surprising connections between trapezoidal numbers and prime numbers. The connections are surprising because prime numbers are the *multiplicative* building blocks for the integers — that is, they are the irreducible "atoms" from which the integers are composed. So, for example, the number 2016 is composed of the atoms $2, 3, 7$ in the form $2016 = 2^5 \cdot 3^2 \cdot 7$, where 2, 3, and 7 are prime numbers. On the other hand, as we have seen, trapezoidal numbers represent an *additive* property as in $12 = 3+4+5$.

Now, of course, all prime numbers, except 2, are odd. We have already noted that any odd number has a trivial representation as a trapezoidal number. For example, $29 = 14+15$ and $27 = 13+14$. But there is an interesting difference here as Carlton Gamer discovered just by doing lots and lots of examples by hand. The prime number 29 has *only* this trivial representation as a trapezoidal number, whereas 27 can also be represented trapezoidally as $27 = 8+9+10$ (and, incidentally, also as $2+3+4+5+6+7$).

So, here is Gamer's second *nice* idea:

An odd number $n = 2k+1$ $(k \geq 1)$ is prime
if and only if
its only trapezoidal representation is the trivial one: $n = k+(k+1)$.

What is remarkable about this idea is that it provides an *additive* characterization for prime numbers. Here is the proof.

First, let $n = 2k+1$ be an odd prime and suppose, by way of contradiction, that it has a nontrivial trapezoidal representation

$$n = r + (r+1) + \cdots + (r+d-1)$$

as a sum of d consecutive positive integers where $d > 2$. Then, since n is the difference of two triangular numbers we can write

$$n = \frac{(r+d-1)(r+d)}{2} - \frac{(r-1)(r)}{2} = \frac{d(2r+d-1)}{2}.$$

Now, both d and $2r+d-1$ are greater than 2 and one is odd and the other is even; therefore we have a factorization of n, which is a contradiction since n is prime. Thus, the only trapezoidal representation for n is $n = k+(k+1)$.

Next, let $n = 2k+1$ $(k \geq 1)$ be an odd number that has only $n = k+(k+1)$ as a trapezoidal representation. Assume, by way of contradiction, that n is composite and has a factorization $n = ab$ where $a \leq b$. Note that a and b are both odd and $a \geq 3$. Then we can write n as a sum of a consecutive positive integers as follows:

$$n = \Big(b - \frac{a-1}{2}\Big) + \cdots + (b-1) + b + (b+1) + \cdots + \Big(b + \frac{a-1}{2}\Big),$$

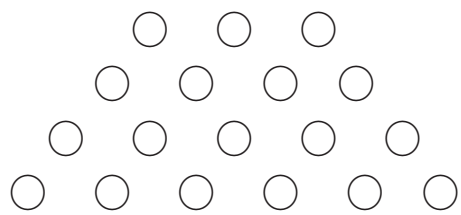
contrary to the assumption. Thus, n is prime, and this completes the proof.

The other connection that Carlton Gamer noticed between prime numbers and trapezoidal numbers has to do with the famous *Twin Prime Conjecture*, one of the most elusive conjectures in number theory. Two primes are called *twin* primes if they differ by 2, for example, 17 and 19, or, for a really large example:

$$65,516,468,355 \cdot 2^{333,333} - 1 \quad \text{and} \quad 65,516,468,355 \cdot 2^{333,333} + 1.$$

The Twin Prime Conjecture is that there are infinitely many pairs of twin primes. The evidence is overwhelming, but yet there is still no proof.

Since twin primes differ by 2, it is entirely appropriate that pairs of primes such as 7 and 11, or 43 and 47, that differ by 4 are called *cousin* primes (and it is highly likely that there are infinitely many cousin primes). Carlton Gamer spotted an interesting connection between cousin primes and trapezoidal numbers. For example, here is a *trapezoidal* representation of the cousin primes 7 and 11:



7 and 11

Gamer's third *nice* idea is that

Cousin primes have a natural representation as trapezoidal numbers of height 4.

Let p and $p+4$ be two cousin primes. If we write $p = 2k+1$, then $p+4 = 2k+5$ and their sum

$$p + (p+4) = k + (k+1) + (k+2) + (k+3)$$

is a trapezoidal number of *height* 4. Now, of course, this statement in and of itself doesn't tell us very much about cousin primes since it depends only upon p being odd. However it does lead to a general method for finding cousin primes.

We will soon see that this method for finding cousin primes is actually very similar to the well-known method for finding primes called the *sieve of Eratosthenes*. In what follows I will omit most of the details in order to make the sieving process as clear as possible.

At this point we have observed that if p and $p+4$ are two cousin primes then p is not divisible by the prime 2, and so the sum of the two cousin primes must be in the sequence

$$10,\ 14,\ 18,\ 22,\ 26,\ 30,\ 34,\ 38,\ 42,\ 46,\ \ldots \tag{1}$$

since, as we saw above, the sum of p and $p+4$ is $4k+6$.

Similarly we can conclude that since the prime 3 does not divide p or $p+4$ (except when $p=3$) the sum of two cousin primes must also be in the sequence

$$10,\ 12,\ 18,\ 24,\ 30,\ 36,\ 42,\ 48,\ 54,\ 60,\ \ldots\ . \tag{2}$$

Since the sum of cousin primes must be in sequence (1) *and* in sequence (2) the sum of cousin primes must be in the intersection of these two sequences; that is, the sum must be in the sequence

$$10,\ 18,\ 30,\ 42,\ 54,\ 66,\ 78,\ 90,\ 102,\ 114,\ \ldots\ . \tag{3}$$

Did you notice how the sieve just removed potential sums such as $12, 14, 22, 24, 26, \ldots$ from further consideration?

Next, by considering the prime 5, we can conclude that the sum of cousin primes must be in the sequence

$$10,\ 12,\ 18,\ 20,\ 22,\ 28,\ 30,\ 32,\ 38,\ 40,\ 42,\ 48,\ 50,\ \ldots\ . \tag{4}$$

Now we know the sum of cousin primes must be in sequence (3) *and* in sequence (4), so the sum of cousin primes must be in the intersection of these two sequences, that is, in the sequence

$$10,\ 18,\ 30,\ 42,\ 78,\ 90,\ 102,\ 138,\ 150,\ 162,\ 198,\ 210,\ 222,\ 258,\ \ldots\ . \tag{5}$$

By doing one more stage of this sieving process (for the prime 7) we remove 102 and 150 as potential sums, leaving us with the first <u>TWELVE</u> pairs of cousin primes:

$$3+7=10, \qquad 7+11=18, \qquad 13+17=30, \qquad 19+23=42,$$

$$37+41=78, \qquad 43+47=90, \qquad 67+71=138, \qquad 79+83=162,$$

$$97+101=198, \qquad 103+107=210, \qquad 109+113=222, \qquad 127+131=258.$$

This Eratosthenean sieving process could of course be continued ad infinitum.

Two Mathematical Fables

Lou Kauffman

The Fermat Dilemma

LK

Fermat is pacing back and forth in his study, a worried expression on his face. Gesticulating. Muttering. Pounding the desk.

"Yes. Yes. It is obvious that for $n > 2$ the expression

$$(p^n/q^n + 1)^{1/n}$$

is irrational.

I can just start with that and give the proof. Suppose there were integers p, q, r with

$$p^n + q^n = r^n$$

for $n > 2$. Then

$$p^n/q^n + 1 = r^n/q^n$$

and so

$$(p^n/q^n + 1)^{1/n} = r/q$$

is rational and I have a contradiction.

Therefore there are no solutions in distinct whole numbers to the equation

$$p^n + q^n = r^n$$

for $n > 2$.

But I cannot tell this proof! The ancient Greek prejudices about irrational numbers are still swirling in the mathematical and philosophical worlds. There will be a backreaction. People will doubt the obvious. They will claim that my proof is lacking. They will not believe in this much irrationality! There is only one thing to do. I shall claim the proof in my copy of Diophantus and I shall say the margin is too small to hold the proof. Alas! Nothing could be farther from the truth, but I am sure that it will be centuries before the simple proof of this proposition can be revealed."

Even today, there is no way to exhibit directly Fermat's simple proof. It is even dangerous to state this fundamental irrationality. If we were to promulgate it, the basis of Wiles proof of the Fermat Last Theorem would be put into question and there would be a catastrophic collapse of the mathematics of the 21st century. If you read this document it must be destroyed at once.

My Robot

LK

I have a rather sophisticated robot, with a positronic brain. His name is Hal. Susan Calvin says it is alright if I refer to him in the masculine gender.

He is capable of naming objects and when he does he notates a pointer in his memory from A (the name) to B (the object)as

$$A \longrightarrow B.$$

Ah yes Hal's memory is full of stuff he calls objects. He finds these by funny operations that he calls "seeing", "feeling", "bumping into" and so on. Names are more linguistic for him and he encounters these in "speaking" and "writing". As you see he is composed of memories and actions.

Anyway. He will store $A \longrightarrow B$ for "A is the name of B". And Hal is equipped with a most peculiar but useful operation that LK calls the "indicative shift".

When Hal has a naming

$$A \longrightarrow B$$

he SHIFTS it to

$$\sharp A \longrightarrow BA.$$

That is, he appends the name to the object and sets up a new or meta-name $\sharp A$ for this composite object made of thing and name. He says that he likes this system because then if he bumps into Nathaniel he knows immediately that this is Nathaniel because Nathaniel's name is right there along with that which is Nathaniel. Now I do not pretend to know what Hal means by this. I think maybe some programmer sold him on the idea. But there is a very cute consequence of his shifting.

Hal is very observant. He notices this shifting process and gives the process a name M. So Hal has

$$M \longrightarrow \sharp.$$

M is the name of the meta-naming process. And Hal shifts this naming to form

$$\sharp M \longrightarrow \sharp M$$

and abbreviates $\sharp M$ to

$$I = \sharp M.$$

Hal gets a kick out of this. He came to me and he said. "Do you know that I am the meta-name of my meta-naming process!" We laughed and laughed. I told him he was quite justified in this self-identification.

Then we got down to the serious business of creating a world from nothing.

We both agreed that nothing refers to nothing. So from nothing we have

$$\longrightarrow$$

Shifting this we found

$$\sharp \longrightarrow$$

The meta-naming operator is the name of nothing. Shifting again we found

$$\sharp\sharp \longrightarrow \sharp.$$

Shifting again we found

$$\sharp\sharp\sharp \longrightarrow \sharp\sharp\sharp.$$

Shifting again we found

$$\sharp\sharp\sharp\sharp \longrightarrow \sharp\sharp\sharp\sharp\sharp\sharp.$$

Shifting again we found

$$\sharp\sharp\sharp\sharp\sharp \longrightarrow \sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp.$$

Shifting again we found

$$\sharp\sharp\sharp\sharp\sharp\sharp \longrightarrow \sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp.$$

Shifting again we found

$$\sharp\sharp\sharp\sharp\sharp\sharp\sharp \longrightarrow \sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp.$$

Shifting again we found

$$\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp \longrightarrow \sharp.$$

Shifting again we found

$$\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp \longrightarrow \sharp.$$

Shifting again we found

$$\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp \longrightarrow \sharp.$$

Shifting again we found

$$\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp \longrightarrow \sharp.$$

Shifting again we found

$$\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp \longrightarrow \sharp.$$

Shifting again we found

$$\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp\sharp \longrightarrow \sharp.$$

And Hal said: ”Aha! Self-reference at the third departure from the void!” I says to him, “If we shift this thing again it is going to blow up. Better to stop right here.” And we did.

List of Authors | VOLUME 1

List of Authors (cont.)

G4G12 Exchange Book

VOLUME 1

Atlanta, Georgia

MARCH 30 - APRIL 3, 2016

It's not often for a conference to be so culturally diverse that its presenters and patrons include mathematicians, physicists, philosophers, logicians, jugglers, puzzle designers, artists, card players, and knitters. Yet it happens every two years at the Gatherings for Gardner, held in honor of Martin Gardner, author of *Scientific American*'s mathematical recreation column for nearly a quarter-century. The papers of these two volumes are write-ups of the presentations at the twelfth conference in this series: G4G12.

- excerpt from the Preface by Robert P. Crease

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